Girraween High School

Year 12 Mathematics Extension 2

Task 1, December 2024

General Instructions

- Reading Time - 5 minutes
- Working Time - 90 minutes
- Write using black pen only
- Calculators approved by NESA may be used
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Total Marks: 66

Section I - 5 marks

- Attempt questions 1-5
- Allow about 10 minutes for this section

Section II - 61 marks

- Attempt questions 6-11
- Allow about 80 minutes for this section

Section I

5 marks
Attempt questions 1 - 5
Allow about 10 minutes for this section

Use the multiple choice answer sheet for Questions 1 to 5.

Question 1

A quadratic equation with zeros of $4+i$ and $4-i$ is

(A) $x^2 - 8x + 17 = 0$ 4

(B) $x^2 + 8x + 17 = 0$

(C) $x^2 - 8x - 17 = 0$

(D) $x^2 + 8x - 17 = 0$

Question 2

The complex number w is a root of the equation $z^3 + 1 = 0$.

Which of the following is FALSE?

(A) $\overline{w}$ is also a root

(B) $w^2 + 1 - w = 0$

(C) $\frac{1}{w}$ is also a root

(D) $(w-1)^3 = -1$

Question 3

$P(z)$ is a polynomial in z of degree 4 with real coefficients.

Which one of the following statements must be **false**?

(A) $P(z)$ has four real roots.

(B) $P(z)$ has two real roots and two non-real roots.

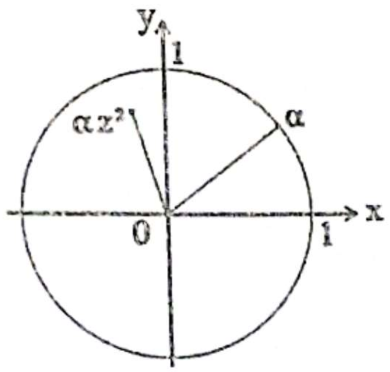
(C) $P(z)$ has one real root and three non-real roots.

(D) $P(z)$ has no real roots.

Examination continues next page...

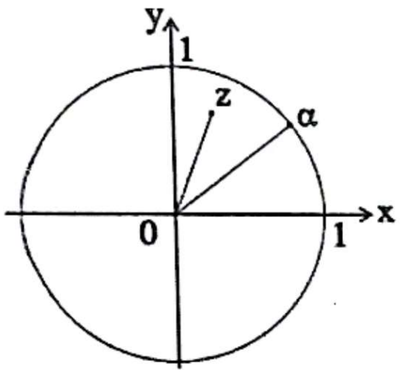
Question 4

The Argand diagram below shows the complex numbers α and αz^2.

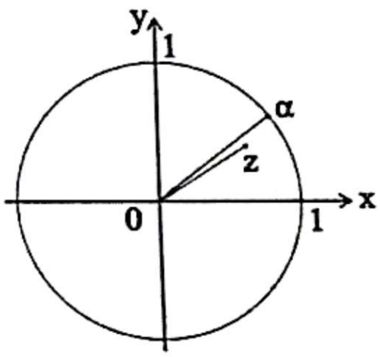


Which of the following best represents the positions of α and z?

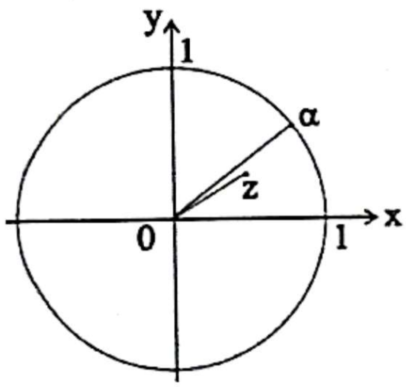
A)



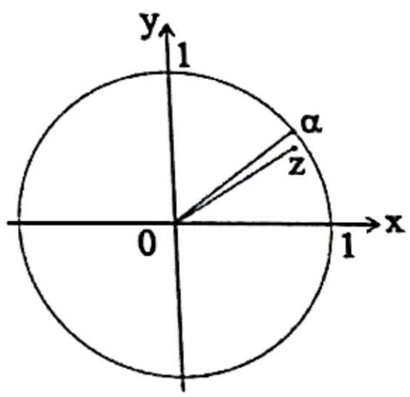
B)



C)



D)



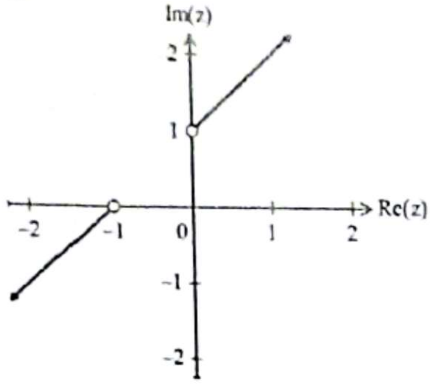
Examination continues next page...

Question 5

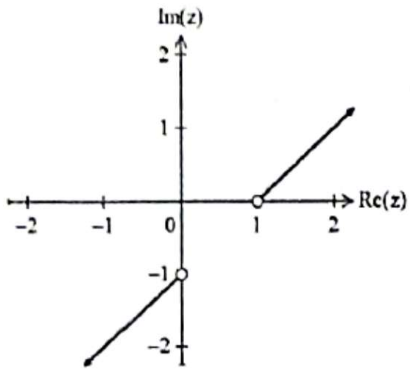
Which of the following Argand diagrams describes the locus defined by

$$\arg(z-i)=\arg(z+1)?$$

A)

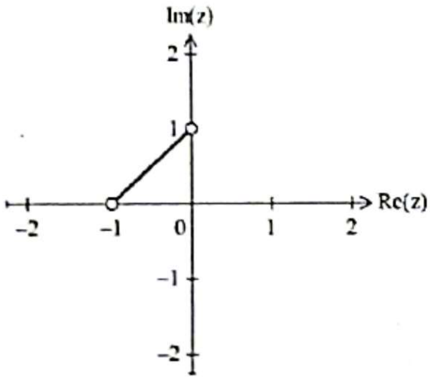


B)

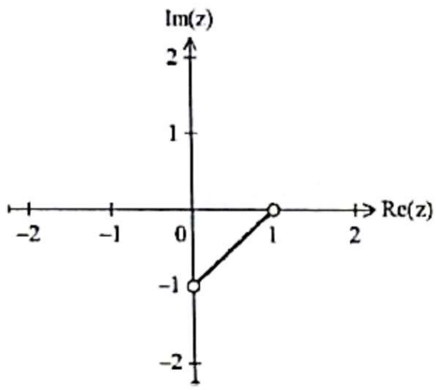


$(z-i)-(z+1)$

C)



D)



End of Section 1

Examination continues next page…

Section II

61 marks

Attempt questions 6 - 11
Allow about 80 minutes for this section

Start each question on a new page.

In Questions 6 - 11, your responses should include relevant mathematical reasoning and/or calculations.

Question 6 **(11 marks)** **Start a new answer page** **Mark**

a) Given that $z_1 = 3 - i,\ z_2 = 2 + 5i,$ express in the form, $a + ib,$ where a and b are real

i) $(\overline{z_1})^2$ 1

ii) $\frac{z_1}{z_2}$ 1

iii) $\overline{\left|\frac{z_1}{z_2}\right|}$ 1

b) i) Express $1 - i\sqrt{3}$ in modulus-argument form. 1

ii) Hence evaluate $(1 - i\sqrt{3})^6$. 1

c) If w is one of the complex roots of $z^3 - 1 = 0$, find the value of $\frac{1}{1+w} + \frac{1}{1+w^2}$. 2

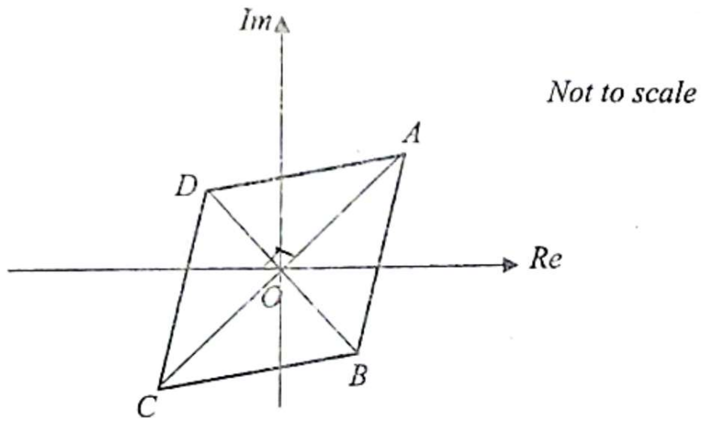
d) i) Find the complex square roots of $-18i$, expressing your answer in the form $a + bi$, where a and b are real numbers. 2

ii) Hence, solve the quadratic equation $z^2 - 3(1 + i)z + 9i = 0$ 2

Examination continues next page...

Question 7 (9 marks) Start a new answer page **Mark**

a) In the diagram below, $ABCD$ is a rhombus whose diagonals meet at the origin, O.



If A represents the complex number $z_1 = 3 + 4i$ and $OD = 2$ units, find the complex number (in Cartesian form) represented by

i) D (let this point be z_2) 2

ii) C (let this point be z_3) 1

b) The complex number z and its conjugate $\bar{z}$ satisfies $(z - \bar{z})^2 + 8(z + \bar{z}) = 16$. 2

Find the Cartesian equation of the locus.

c) i) Find the complex sixth roots of unity in modulus-argument form. 2

ii) Sketch the roots on an Argand diagram. 2

Examination continues next page...

Question 8 (12 marks) Start a new answer page **Mark**

a) i) Given $\sqrt{3} + i = 2cis\frac{\pi}{6}$. state $\sqrt{3} - i$ in modulus argument form. 1

ii) Hence, find the smallest positive integer m, such that $(\sqrt{3} + i)^m = (\sqrt{3} - i)^m$. 3

b) Consider the polynomial $P(x) = x^4 - 4x^3 + 11x^2 - 14x + 12$. 4

Given $x = 1 - i\sqrt{2}$ is a root of $P(x) = 0$, solve the equation $P(x) = 0$.

c) Let $z^n = \cos n\theta + i\sin n\theta$

i) Show that $\sin n\theta = \frac{z^n - z^{-n}}{2i}$ and $\cos n\theta = \frac{z^n + z^{-n}}{2}$. 2

ii) Hence, or otherwise prove the identity 2

$$32\sin^4\theta\cos^2\theta = \cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2.$$

$\sin^4 \ (1-\cos^2\theta)^2$

Question 9 (10 marks) Start a new answer page **Mark**

a) (i) Draw a sketch of the locus of z given by the equation 2

$$\arg(z-1) - \arg(z+1) = \frac{\pi}{4}.$$

(ii) Find the centre of the circle on which this locus lies. 2

b) i) By considering $(cos\theta + isin\theta)^5$ using De Moivre's Theorem prove that 3

$$sin\,5\theta = sin\theta(5 - 20sin^2\theta + 16\,sin^4\theta)$$

ii) By letting $\theta = \frac{\pi}{5}$, show that the exact value of $sin^2(\frac{\pi}{5})$ is $\frac{1}{8}(5 - \sqrt{5})$. 3

Examination continues next page...

Question 10 (11 marks) **Start a new answer page** **Mark**

a) The complex number z satisfies both $|z-1| \leq |z-i|$ and $|z-2-2i| \leq 1$.

i) In an Argand diagram, indicate the region which contains the point P representing z. 3

ii) If P moves on the boundary of this region and $arg(z-1) = \frac{\pi}{4}$, find the value of z in the form $x+iy$ where x and y are real. 3

b) Let $z = e^{i\theta}$. Consider the expression $S_n = 1 + z + z^2 + \ldots + z^n$.

i) Show that $S_n = \frac{e^{i(n+1)\theta}-1}{e^{i\theta}-1}$ 1

ii) Show that $S_n = e^{\frac{in\theta}{2}} \times \frac{(\sin\frac{n+1}{2}\theta)}{\sin(\frac{\theta}{2})}$ 2

iii) Deduce that $\sin\theta + \sin 2\theta + \sin 3\theta + \ldots + \sin n\theta = \frac{\sin\frac{n\theta}{2}\sin\frac{(n+1)\theta}{2}}{\sin\frac{\theta}{2}}$ 2

Question 11 (8 marks) **Start a new answer page** **Mark**

Consider the polynomial $z^5 - i = 0$.

i) Find the roots of $z^5 = i$ 2

ii) Show that $(z-i)\left(z^2 - 2i\sin\frac{\pi}{10}z - 1\right)\left(z^2 + 2i\sin\frac{3\pi}{10}z - 1\right) = 0$ 4

iii) Given $z^5 - i = (z-i)(z^4 + iz^3 - z^2 - iz + 1)$, show that $\sin\frac{\pi}{10}\sin\frac{3\pi}{10} = \frac{1}{4}$. 2

End of examination.

Maths Ext 2 Solutions
2024-25 Task 1

Multichoice

Q1 $(x-4-i)(x-4+i)=0$

$[(x-4)-i][(x-4)+i]=0$

$(x-4)^2-i^2=0$

$(x-4)^2+1=0$

$x^2-8x+16+1=0$

$x^2-8x+17=0$ (A)

Q2 The roots of $z^3+1=0$

are ω, $\frac{1}{\omega}$ and -1.

sum of roots $\omega+\frac{1}{\omega}-1=0$

which is equivalent to

$\omega^2+1-\omega=0$

(D) is false

Q3 A and D are possible.

however non real roots occur in conjugate pairs since coefficients are real.

$\therefore$ (C) is false.

Q4 Since $|z| < 1$

then $|z|^2 < |z| < 1$

and $\arg(\alpha z^2) = 2\arg(z) + \arg(\alpha)$

$\therefore$ D

Q5 A

Q6

a) i) $(\bar{z})^2 = (3+i)^2$

$= 9 + 6i + i^2$

$= 8 + 6i$ 1

Very well done

ii) $\frac{z_1}{z_2} = \frac{3-i}{2+5i} \times \frac{2-5i}{2-5i}$

$= \frac{6-15i-2i-5}{4-25i^2}$

$= \frac{1}{29} - \frac{17i}{29}$ 1

Very well done

iii) $\left|\overline{\frac{z_1}{z_2}}\right| = \left|\frac{1}{29} + \frac{17i}{29}\right|$

$= \sqrt{\left(\frac{1}{29}\right)^2 + \left(\frac{17}{29}\right)^2}$

$= \sqrt{\frac{10}{29}}$ 1

most students have done it well
some students forgot to find modulus.

OR $\left|\frac{z_1}{z_2}\right| = \left|\frac{3-i}{2+5i}\right| = \frac{\sqrt{3^2+1^2}}{\sqrt{2^2+5^2}}$

$= \sqrt{\frac{10}{29}}$

4

Q6 b) i) $1-\sqrt{3}i$

Now $|1-\sqrt{3}i| = \sqrt{1^2+\sqrt{3}^2} = 2$

$\arg(1-\sqrt{3}i) = \tan^{-1}(-\sqrt{3})$

$= \frac{-\pi}{3}$

Very well done

$\therefore 1-\sqrt{3}i = 2\,\text{cis}\left(\frac{-\pi}{3}\right)$ 1

ii) $(1-\sqrt{3}i)^6 = \left[2\,\text{cis}\left(\frac{-\pi}{3}\right)\right]^6$

$= 2^6\,\text{cis}\left(\frac{-6\pi}{3}\right)$

Very well done

$= 64$ 1

Most students have shown excellent understanding of Q6 except for some calculation mistakes.

c)

$$\omega^3 - 1 = 0$$

$$(\omega - 1)(\omega^2 + \omega + 1) = 0$$

$$\therefore \omega^3 = 1$$
$$\omega^2 + \omega + 1 = 0$$
(since ω is complex) 1

$$\frac{1}{1+\omega} + \frac{1}{1+\omega^2} = \frac{1+\omega^2+1+\omega}{(1+\omega)(1+\omega^2)}$$

$$= \frac{1+\omega^2+1+\omega}{1+\omega^2+\omega+\omega^3}$$

$$= \frac{1+0}{0+1}$$ since $\omega^2+\omega+1=0$, $\omega^3=1$

$$= 1$$ 1

Students have shown good understanding of this concept. Most students used $\omega^3=1$ and $\omega^2+\omega+1=0$ but a few number of students found the roots and substituted them.

Q6d) $z^2 = -18i$

let $z = a + ib$

$(a+ib)^2 = -18i$

$a^2 + 2abi - b^2 = -18i$

equate real & imaginary

$a^2 - b^2 = 0$ (1)

$2ab = -18$

$a = \frac{-9}{b}$ (2)

1

Very well done
Almost all students have shown under-standing of finding square roots

Subs (2) into (1)

$\left(\frac{-9}{b}\right)^2 - b^2 = 0$

$\frac{81}{b^2} - b^2 = 0$

$b^4 = 81$

$b = \pm 3$

$\therefore a = \frac{-9}{\pm 3} = \mp 3$

1

$\therefore$ The square roots are $3 - 3i$ and $-3 + 3i$

or $\pm 3 \mp 3i$

Q6 d(ii) $z^2 - 3(1+i)z + 9i = 0$

$$z = \frac{3(1+i) \pm \sqrt{9(1+i)^2 - 4(1)(9i)}}{2.1}$$

$$= \frac{3+3i \pm \sqrt{18i - 36i}}{2}$$

$$= \frac{3+3i \pm \sqrt{-18i}}{2}$$ 1

$$= \frac{3+3i \pm (3-3i)}{2}$$ from (i)

$$= \frac{6}{2} \text{ or } \frac{6i}{2}$$

$$= 3 \text{ or } 3i$$ 1

Very well done apart from a few calculation mistakes.

(19 a)(i) $|OA| = \sqrt{3^2+4^2} = 5$

z_2 is a $\frac{\pi}{2}$ rotation from z_1 and $\frac{2}{5}$ the length of z_1

Almost everyone could see the direction was $-3+4i$. However, many students didn't take into account the modulus of OD was 2 & didn't make $OD = \frac{2}{5}OA$.

$\therefore z_2 = z_1 \times i \times \frac{2}{5}$

$= (3+4i) \times i \times \frac{2}{5}$

$z_2 = -\frac{8}{5} + \frac{6}{5}i$

1 multiplying z_1 by i

1 multiplying z_1 by $\frac{2}{5}$

(ii) z_3 is a rotation of π from z_1

$\therefore z_3 = -1 \times (3+4i)$

$= -3-4i$ 1

Very simple.
Almost everyone was correct.

Q7 b)

$(z-\bar{z})^2 + 8(z+\bar{z}) = 16$

let $z = x+iy$

$[x+iy-(x-iy)]^2 + 8[x+iy+(x-iy)] = 16$ ①

Now $z+\bar{z} = 2\,\mathrm{Re}(z) = 2x$

$z-\bar{z} = 2i\,\mathrm{Im}(z) = 2iy$

$(2iy)^2 + 8(2x) = 16$

$-4y^2 = 16-16x$

$y^2 = 4(x-1)$ ①

Most students realised to let $z = x+iy$ and also realised $z-\bar{z} = 2iy$ & $z+\bar{z} = 2x$.

Some students forgot that $(2iy)^2 = -4y^2$; NOT $4y^2$.

Q7(c)

i) $z^6 = 1$

let $z = \text{cis}\,\theta$

$z^6 = (\text{cis}\,\theta)^6 = \text{cis}\,6\theta$

Now $1 = \text{cis}\,0$

equating arguments

$6\theta = 0 + 2k\pi$

$\theta = \frac{k\pi}{3}$ for $k = 0, \pm 1, \pm 2, 3$

1 progress

Almost everyone got (i) right.

$z_1 = \text{cis}\,0 = 1$

$z_2 = \text{cis}\,\frac{\pi}{3}$

$z_3 = \text{cis}\,-\frac{\pi}{3}$

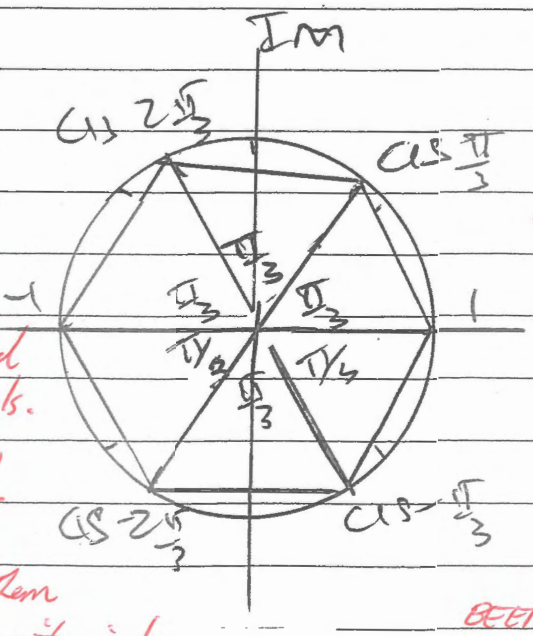
$z_4 = \text{cis}\,\frac{2\pi}{3}$

$z_5 = \text{cis}\,-\frac{2\pi}{3}$

$z_6 = \text{cis}\,\pi = -1$

1

(ii)



1 shape

1 labels

Marking note:

(1) If they labelled cis $\frac{\pi}{3}$ etc. I counted that as angle labels.

(2) If they labelled all of the moduli as 1

I didn't penalise them for not having the unit circle. [This was very rare].

A simple question done with a LOT of careless ommissions.

Students often didn't either (1) show the angles were $\frac{\pi}{3}$ or (2) show the moduli were all 1

[which SHOULD HAVE BEEN DONE BY PUTTING THEM ALL ON A UNIT CIRCLE!

Q 8(a) (i) $z = 2\text{cis}\,\frac{\pi}{6}$

$\therefore \bar{z} = 2\text{cis}\left(-\frac{\pi}{6}\right)$ 1

well done

(ii) $(\sqrt{3}+i)^n = (\sqrt{3}-i)^n$

$\left(2\text{cis}\,\frac{\pi}{6}\right)^n = \left(2\text{cis}\,-\frac{\pi}{6}\right)^n$

$2^n\,\text{cis}\,\frac{n\pi}{6} = 2^n\,\text{cis}\,-\frac{n\pi}{6}$ 1

by De Moivre's

$\cos\frac{n\pi}{6} + i\sin\frac{n\pi}{6} = \cos\left(-\frac{n\pi}{6}\right) + i\sin\left(-\frac{n\pi}{6}\right)$

Now, $\cos\frac{n\pi}{6} = \cos\left(-\frac{n\pi}{6}\right)$

and $\sin\left(-\frac{n\pi}{6}\right) = -\sin\frac{n\pi}{6}$

$\therefore\ 2i\sin\frac{n\pi}{6} = 0$ 1

$\frac{n\pi}{6} = \pi$

$\therefore$ Smallest positive integer n is $\underline{6}$ 1

well done

Q 8b)

Since the coefficients of the polynomial are real

Must mention

(1)

$z = 1 + \sqrt{2}\,i$ is also a root.

$[z-(1+\sqrt{2}i)][z+(1+\sqrt{2}i)]$

Now $z^2 - 2\,\mathrm{Re}(z) + |z|^2$

$= z^2 - 2z + (1^2 + \sqrt{2}^2)$

$= z^2 - 2z + 3$ is a factor

(1)

$$
\begin{array}{r}
z^2 - 2z + 4 \\
z^2-2z+3 \overline{\smash{)}\, z^4 - 4z^3 + 11z^2 - 14z + 12} \\
\underline{z^2 - 2z^3 + 3z^2} \\
-2z^3 + 8z^2 - 14z \\
\underline{-2z^3 + 4z^2 - 6z} \\
4z^2 - 8z + 12 \\
\underline{4z^2 - 8z + 12} \\
0
\end{array}
$$

(1)

$z^2 - 2z + 4 = 0$

$$z = \frac{--2 \pm \sqrt{(-2)^2 - 4(1)(4)}}{2(1)}$$

well done

$$= \frac{2 \pm 2\sqrt{3}i}{2} = 1 \pm \sqrt{3}i$$ (1)

$\therefore$ solutions are $x = 1 \pm \sqrt{2}i$

$x = 1 \pm \sqrt{3}i$

Q 8 c) i)

Given $z^n = \cos n\theta + i \sin n\theta$ (1)

$z^{-n} = \cos(-n\theta) + i\sin(-n\theta)$

1

since $\cos(-n\theta) = \cos n\theta$

and $\sin(-n\theta) = -\sin n\theta$

$\therefore \quad z^{-n} = \cos n\theta - i\sin n\theta$ (2)

$z^n + z^{-n} = 2\cos n\theta$

$\therefore \quad \cos n\theta = \frac{1}{2}(z^n + z^{-n})$

1

$z^n - z^{-n} = 2i\sin n\theta$

$\therefore \quad \sin n\theta = \frac{1}{2i}(z^n - z^{-n})$

well done.
students penalised if
$\sin(-n\theta) = -\sin(n\theta)$
not shown.

Q8(c)(ii)

$RHS = 32\sin^4\theta\cos^2\theta$

$= 32\sin^4\theta(1-\sin^2\theta)$

$= 32\sin^4\theta - 32\sin^6\theta$

1 $= 32\left[\frac{1}{2i}\left(z-\frac{1}{z}\right)\right]^4 - 32\left[\frac{1}{2i}\left(z-\frac{1}{z}\right)\right]^6$

$= 2(z^4 - 4z^2 + 6 - 4z^{-2} + z^{-4})$

1 $+\frac{1}{2}(z^6 - 6z^4 + 15z^2 - 20 + 15z^{-2} - 6z^{-4} + z^{-6})$

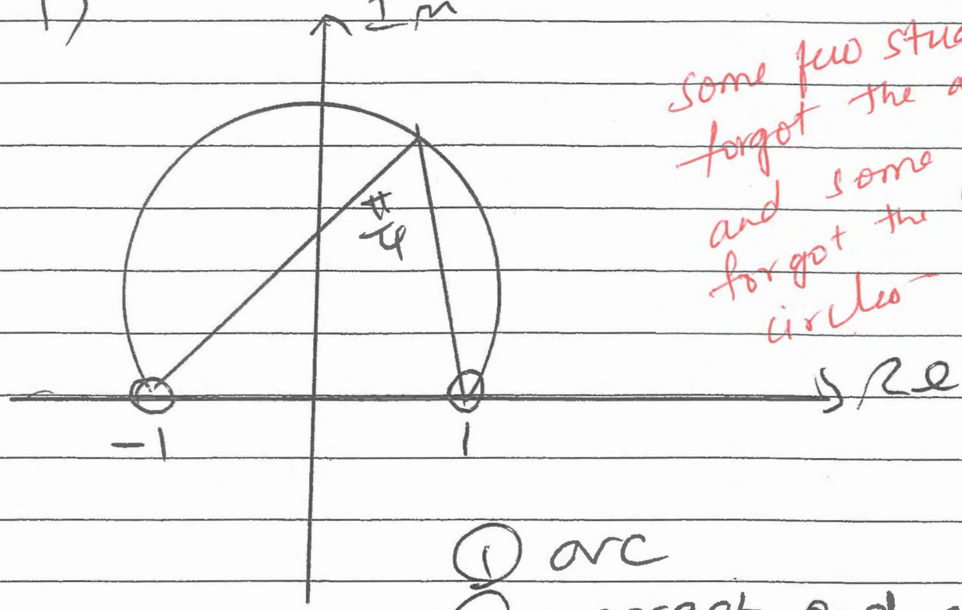
$= \frac{1}{2}\left(z^6+\frac{1}{z^6}\right) - \left(z^4+\frac{1}{z^4}\right) - 2\left(z^2+\frac{1}{z^2}\right) + 2$

$= \cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2$

$= LHS$

poorly done

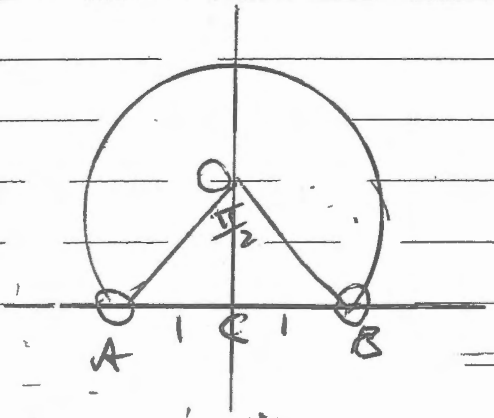
Q9 a) i)



some few students forgot the arc and some students forgot the open circles

(1) arc
(1) correct end pts

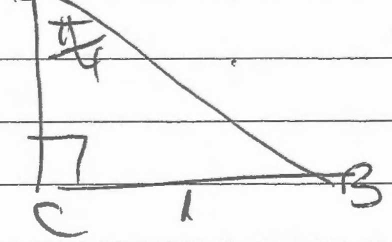
ii) centre lies on the vertical axis.



(1) reasoning
(1) centre

$\angle AOB = \frac{\pi}{2}$
Angle at the centre is twice the angle at the circumference

$\triangle AOB$ is also isosceles



$\tan \frac{\pi}{4} = \frac{1}{OC}$

$\therefore OC = 1$

$\therefore$ centre $(0, 1)$

very well done
Students to write $C(0,1)$ however not penalised

9(b) i)

$$(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$$

by De Moivre's Theorem

1 5 10 10 5 1

$$(\cos\theta + i\sin\theta)^5$$
$$= \cos^5\theta + 5\cos^4\theta\, i\sin\theta + 10\cos^3\theta\, i^2\sin^2\theta$$
$$+ 10\cos^2\theta\, i^3\sin^3\theta + 5\cos\theta\, i^4\sin^4\theta + i^5\sin^5\theta$$

1 expanding

$$= \cos^5\theta + 5\cos^4\theta\, i\sin\theta - 10\cos^3\theta\sin^2\theta$$
$$- 5\cos\theta\, i\sin^3\theta + i\sin^5\theta$$

equating imaginary parts

$$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$$

1

$$= \sin\theta\,(5\cos^4\theta - 10\cos^2\theta\sin^2\theta + \sin^4\theta)$$

apply $\cos^2\theta = 1 - \sin^2\theta$

$$= \sin\theta\,[5(1-\sin^2\theta)^2 - 10(1-\sin^2\theta)\sin^2\theta + \sin^4\theta$$

$$= \sin\theta\,[5(1 - 2\sin^2\theta + \sin^4\theta) - 10\sin^2\theta + 10\sin^4\theta + \sin^4\theta$$

18

8)

$$\sin^2\frac{\pi}{5} = \frac{5+\sqrt{5}}{8} \text{ or } \frac{5-\sqrt{5}}{8}$$

Now $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$$\sin^2\frac{\pi}{4} = \frac{1}{2}$$

$$\sin^2\frac{\pi}{5} < \sin^2\frac{\pi}{4} < \frac{1}{2}$$

justification

$$\therefore \sin^2\frac{\pi}{5} = \frac{5-\sqrt{5}}{8}$$

1

Very well done
apart from justification
Most students who lost a mark
as they did not give a convincing
justification.

Q10

a) let $z = x + iy$

$$|z-1| \le |z-i|$$

$$|x+iy-1| \le |x+iy-i|$$

$$|x-1+iy| \le |x+i(y-1)|$$

$$\sqrt{(x-1)^2+y^2} \le \sqrt{x^2+(y-1)^2}$$

$$x^2-2x+1 \le x^2+y^2-2y+1$$

$$-2x \le -2y$$

The vast majority of students did (10)(a)(i) well.

$$y \le x$$

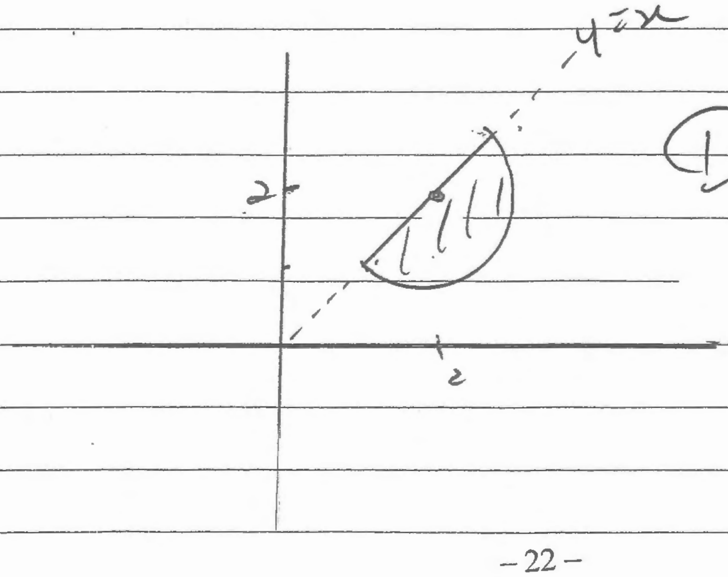
1 or correct graph

$$|z-2-2i| \le 1$$

$$|z-(2+2i)| \le 1$$

This is a circle with centre $(2,2)$ and radius 1

1 or correct graph

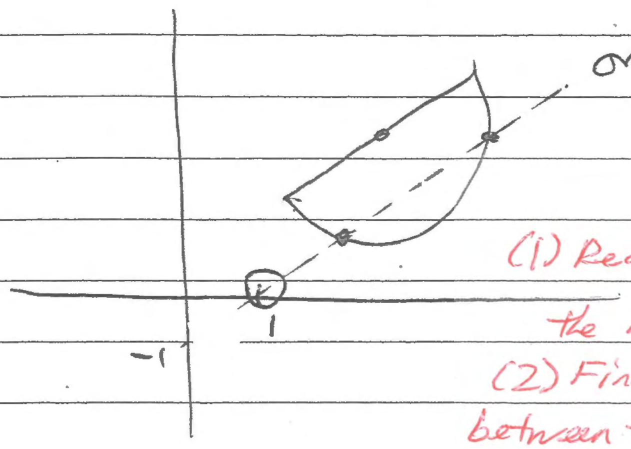


1 correct region

(no penalty if $y=x$ is all solid)

ii) If $\arg(z-1)=\frac{\pi}{4}$

then z lies on the line $y=x-1$ (1)



Many students had trouble
(1) Realising ~~eg~~ $Arg(z-1)=\frac{\pi}{4}$ is the line $y=x-1, x>1$
(2) Finding BOTH points of intersection between the line $y=x-1$ & the circle.

Solve $y=x-1$ (1)

& $(x-2)^2+(y-2)^2=1$ (2)

simultaneously

Subs (1) into (2)

$$(x-2)^2+(x-1-2)^2=1$$

$$x^2-4x+4+x^2-6x+4=1$$

$$2x^2-10x+12=0$$

$$x^2-5x+6=0$$

$$(x-2)(x-3)=0$$

(1) progress

$x=2$ $\quad$ $x=3$

$y=1$ $\quad$ $y=2$

$\therefore$ $z=2+i$ or $3+2i$ (1)

Q(10 b)

i) $S_n = 1 + e^{i\theta} + e^{i2\theta} + e^{i3\theta} + \dots + e^{in\theta}$

This is a GP with

$a = 1 \quad r = e^{i\theta} \quad n = n+1$

$$S_n = \frac{a(r^n - 1)}{r-1}$$

$$= \frac{1(e^{i\theta(n+1)} - 1)}{e^{i\theta} - 1}$$

$$= \frac{e^{i(n+1)\theta} - 1}{e^{i\theta} - 1}$$

1

Almost everyone realised it was the sum of a GP but I needed SOME WORKING → either the sum of a GP formula or $a=$ $r=$ and $n=$ Otherwise I could NOT award the mark.

ii) Take out $e^{i\left(\frac{n+1}{2}\right)\theta}$ and $e^{\frac{i\theta}{2}}$ as common factors

$$S_n = \frac{e^{i\left(\frac{n+1}{2}\right)\theta}\left(e^{i\left(\frac{n+1}{2}\right)\theta} - e^{-i\left(\frac{n+1}{2}\right)\theta}\right)}{e^{\frac{i\theta}{2}}\left(e^{\frac{i\theta}{2}} - e^{-\frac{i\theta}{2}}\right)}$$

1

Very poorly done. Some students went to De Moivre (only 1 succeeded this way)

Now $z + \frac{1}{z} = 2i\sin\theta$

$$= \frac{e^{i\left(\frac{n+1}{2}\right)\theta}\left[2i\sin\left(\frac{n+1}{2}\right)\theta\right]}{e^{\frac{i\theta}{2}}\left(2i\sin\frac{\theta}{2}\right)}$$

1

$$\therefore S_n = \frac{e^{i\left(\frac{n+1}{2}\right)\theta} \sin\left(\frac{n+1}{2}\right)\theta}{e^{i\frac{\theta}{2}} \sin\frac{\theta}{2}}$$

(iii) $\operatorname{Im}\left(1 + z + z^2 + \ldots + z^n\right)$

$= \sin\theta + \sin 2\theta + \ldots \sin n\theta$ ①

About 15 or 20 students realised to equate imaginary parts & they all succeeded with this question.

from (ii)

$$S_n = \left(\cos\frac{n\theta}{2} + i\sin\frac{n\theta}{2}\right) \times \frac{\sin\left(\frac{n+1}{2}\right)\theta}{\sin\frac{\theta}{2}}$$

expand

$$① = \frac{\cos\frac{n\theta}{2}\sin\frac{(n+1)\theta}{2}}{\sin\frac{\theta}{2}} + \frac{i\sin\frac{n\theta}{2}\sin\frac{(n+1)\theta}{2}}{\sin\frac{\theta}{2}}$$

$$\operatorname{Im}(S_n) = \frac{\sin\frac{n\theta}{2}\sin\left(\frac{n+1}{2}\right)\theta}{\sin\frac{\theta}{2}}$$

$$\therefore \sin\theta + \sin 2\theta + \ldots + \sin n\theta = \frac{\sin\frac{n\theta}{2}\sin\left(\frac{n+1}{2}\right)\theta}{\sin\frac{\theta}{2}}$$

Q11 i) let $z^5 = r^5 \operatorname{cis} 5\theta$

$i = 1 \operatorname{cis} \frac{\pi}{2}$

$r^5 = 1$ $\quad 5\theta = \frac{\pi}{2} + 2k\pi$

$\therefore r = 1$

$\theta = \frac{\pi}{10} + \frac{2k\pi}{5}$

$= \frac{\pi + 4k\pi}{10}$

where $k = 0, \pm 1, \pm 2$ ①

$z_1 = \operatorname{cis} \frac{\pi}{10}$

$z_2 = \operatorname{cis} \frac{\pi}{2} = i$

$z_3 = \operatorname{cis} \frac{9\pi}{10}$

$z_4 = \operatorname{cis} \left(-\frac{3\pi}{10}\right)$

$z_5 = \operatorname{cis} \left(-\frac{7\pi}{10}\right)$ ①

Well done.

ii) $z^5 - i = 0$ can be expressed as

$(z - i)\left(z - \text{cis}\frac{\pi}{10}\right)\left(z - \text{cis}\frac{9\pi}{10}\right)$ (1)

$\left(z - \text{cis}\frac{-3\pi}{10}\right)\left(z - \text{cis}\frac{-7\pi}{10}\right)$

Now, notice: $\alpha = \text{cis}\frac{\pi}{10}$ and $-\bar{\alpha} = \text{cis}\frac{9\pi}{10}$ (1)

many students did not realise this

$\therefore \beta = \text{cis}\frac{-3\pi}{10}$ and $-\bar{\beta} = \text{cis}\frac{-7\pi}{10}$

$\therefore (z - i)(z - \alpha)(z + \bar{\alpha})(z - \beta)(z + \bar{\beta}) = 0$

$(z - i)\left[z^2 + (\bar{\alpha} - \alpha)z - \alpha\bar{\alpha}\right]$

$\left[z^2 + (\bar{\beta} - \beta)z - \beta\bar{\beta}\right] = 0$

Now $\alpha\bar{\alpha} = 1$ $\beta\bar{\beta} = 1$ (1)

and

$\bar{\alpha} - \alpha = \cos\frac{\pi}{10} - i\sin\frac{\pi}{10} - \cos\frac{\pi}{10} - i\sin\frac{\pi}{10}$

$= -2i\sin\frac{\pi}{10}$ The more successful students wrote out the expansion. (1)

Similarly

$\bar{\beta} - \beta = -2i\sin\frac{-3\pi}{10}$

$= 2i\sin\frac{3\pi}{10}$

$\therefore (z-i)\left(z^2 - 2\sin\frac{\pi}{10} z - 1\right)$

$\left(z^2 + 2\sin\frac{3\pi}{10} z - 1\right)$

Poorly done

as required

All steps must be shown to obtain full marks.

iii) $(z-i)(z^4 + iz^3 - z^2 - iz + 1)$

$= (z-i)\left(z^2 - 2i\sin\frac{\pi}{10} z - 1\right)$

$\left(z^2 + 2i\sin\frac{3\pi}{10} z - 1\right)$ 1

equate coefficents of z^2

$-1 = -2i\sin\frac{\pi}{10} \cdot 2i\sin\frac{3\pi}{10} - 1 - 1$

$1 = -4i^2 \sin\frac{\pi}{10} \sin\frac{3\pi}{10}$ 1

$\sin\frac{\pi}{10} \sin\frac{3\pi}{10} = \frac{1}{4}$

Generally well done

Please show working explicitly!