HURLSTONE AGRICULTURAL HIGH SCHOOL

YEAR 12

MATHEMATICS EXTENSION 2

2003

HSC COURSE

HALF YEARLY ~ ASSESSMENT TASK 2

EXAMINERS ~ J. DILLON, P. BICZO, S. GEE

GENERAL INSTRUCTIONS

- READING TIME – 5 MINUTES.
- WORKING TIME – TWO HOURS.
- ATTEMPT ALL QUESTIONS.
- QUESTIONS ARE OF EQUAL VALUE.
- ALL NECESSARY WORKING SHOULD BE SHOWN IN EVERY QUESTION.
- THIS PAPER CONTAINS SIX (6) QUESTIONS.
- TOTAL MARKS – 90 MARKS
- MARKS MAY NOT BE AWARDED FOR CARELESS OR BADLY ARRANGED WORK.
- BOARD APPROVED CALCULATORS MAY BE USED.
- EACH QUESTION IS TO BE STARTED ON A NEW PIECE OF PAPER.
- THIS ASSESSMENT TASK MUST NOT BE REMOVED FROM THE EXAM CENTRE.

STUDENT NAME: ______________________________

TEACHER: ______________________________

Question 1:

Marks

(a) If $A = 3 + 4i$ and $B = 2 - i$ express the following in the form $x + iy$ where $x, y \in \Re$ $x, y \in$ □ .

(i) AB **1**

(ii) $\sqrt{A}$ **2**

(iii) $\dfrac{A}{B}$ **2**

(b) If $z = \sqrt{3} + i$

(i) Find the exact values of $|z|$ and $\arg(z)$ **2**

(ii) By using your answers to (i) and De Moivre’s Theorem, write z^5 in the form $x + iy$ where $x, y \in \Re$. **2**

(c) On an Argand diagram, shade the region containing all the points representing the complex numbers z such that: **2**

$$1 \le |z - 1| \le 2 \text{ and } \frac{\pi}{4} < \arg(z - 1) < \frac{\pi}{2}$$

(d) Explain algebraically, or geometrically, why the locus described by $\arg\left(\dfrac{z}{z-1}\right) = \dfrac{\pi}{2}$ is a circle. **2**

(e) Given that z and w represent two complex numbers, explain why $|z| + |w| \ge |z - w|$. **2**

QUESTION 2: **(START A NEW PIECE OF PAPER)**

MARKS

a) The equation $x^3 - 5x + 7 = 0$ has roots α, β, γ. **2**

Find $\alpha^3 + \beta^3 + \gamma^3$.

b) If $ax^3 + bx^2 + d = 0$ has a double root, show that $27a^2d + 4b^3 = 0$. **2**

c) Solve $2x^4 + 9x^3 + 6x^2 - 20x - 24 = 0$, if it has a root of multiplicity 3. **3**

d) The equation $x^4 + 4x^3 - 3x^2 - 4x + 2 = 0$ has roots α, β, γ, δ.

Find the equations with roots

i) $2\alpha,\ 2\beta,\ 2\gamma,\ 2\delta$. **2**

ii) $\dfrac{1}{\alpha^2}, \dfrac{1}{\beta^2}, \dfrac{1}{\gamma^2}, \dfrac{1}{\delta^2}$. **2**

e) Given that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

(i) Solve the equation $16x^5 - 20x^3 + 5x - 1 = 0$ **2**

(ii) Hence, or otherwise, show that $\cos\dfrac{2\pi}{5} + \cos\dfrac{4\pi}{5} = \dfrac{1}{2}$ **2**

QUESTION 3: **(START A NEW PIECE OF PAPER)**

MARKS

(a) A line through $P(a\sec\theta, b\tan\theta)$, perpendicular to the x - axis, meets an asymptote of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at Q. The normal at P meets the x - axis at G. **4**

Show that GQ is at right angles to the asymptote.

(You may assume that the equation of the normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point $P(a\sec\theta, b\tan\theta)$ is $ax\tan\theta + by\sec\theta = (a^2+b^2)\sec\theta\tan\theta$.)

(b) Consider the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ where $a > b > 0$. **5**

R and R' are the feet of the perpendiculars from the foci S and S' on to the tangent at $P(a\cos\theta, b\sin\theta)$.

Show that $SR \times S'R' = b^2$.

(You may assume the tangent to the ellipse at the point $P(a\cos\theta, b\sin\theta)$ has equation $bx\cos\theta + ay\sin\theta - ab = 0$.)

(c) T is a variable point $(ct, \frac{c}{t})$ on the hyperbola $xy = c^2$. The perpendicular from the centre O to the tangent at T meets it in N.

(i) Find the coordinates of N. **1**

(ii) Show that N lies on the curve $(x^2+y^2)^2 = 4c^2xy$. **2**

(d) Consider the curve $xy(x+y)+16=0$. **3**
Find the equation of the tangent to this curve at the point $P(-2,-2)$.

QUESTION 4: (START A NEW PIECE OF PAPER)

MARKS

(a) Consider the function $f(x)=\dfrac{1-x}{x}$. On separate diagrams, sketch the graphs of the following functions / relations.

For each graph label any asymptotes.

(i) $y=f(x)$ **2**

(ii) $y^2=f(x)$ **3**

(b) Let $f(x)=-x^2+6x-8$.

On separate diagrams, and without using calculus, sketch the following graphs.

Indicate clearly any asymptotes and intercepts with the axes.

(i) $y=f(x)$ **2**

(ii) $y=\left|f(x)\right|$ **2**

(iii) $y^2=f(x)$ **2**

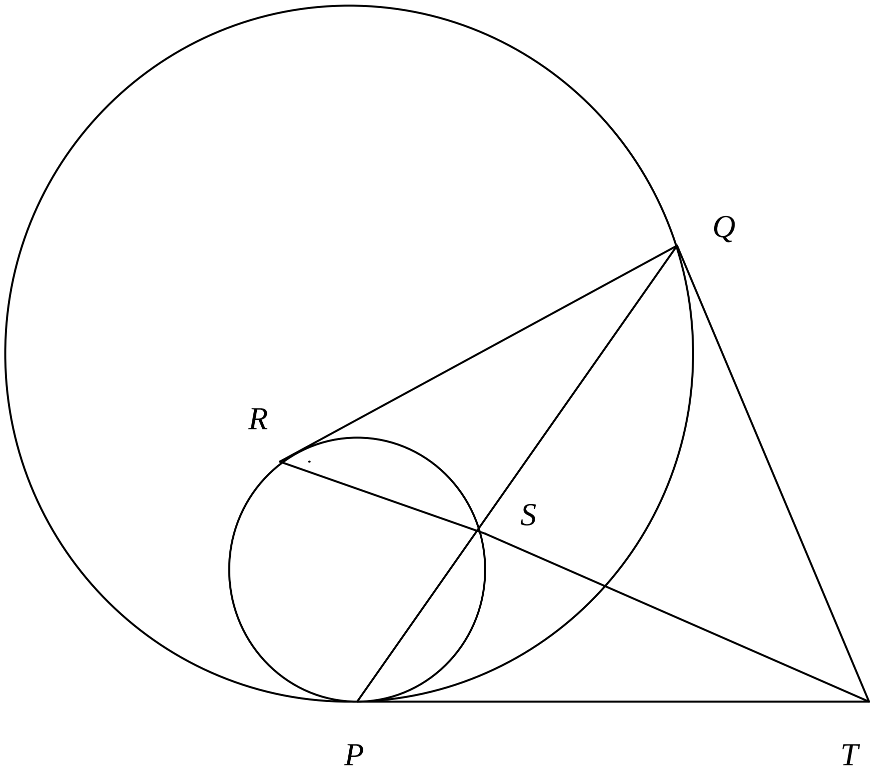
(iv) $y=\dfrac{1}{f(x)}$ **2**

(v) $y=e^{f(x)}$ **2**

QUESTION 5: **(START A NEW PIECE OF PAPER)**

MARKS

(a)



PT is the common tangent to the two circles at P.
QT is a tangent to the large circle at Q.
RQ is a tangent to the small circle at R.
RST and PSQ are straight.

(i) Show that $PRQT$ is cyclic. **3**

(ii) Deduce RS bisects $\angle PRQ$. **3**

(b) (i) By rationalising the numerator, prove that $\sqrt{n+1}-\sqrt{n} > \dfrac{1}{2\sqrt{n+1}}$. **2**

(ii) Hence prove by induction that $1+\dfrac{1}{2}+\dfrac{1}{3}+...+\dfrac{1}{n} < \sqrt{n}$, for n $\geq$ 7. **3**

(c) (i) Find any stationary points on the curve $y = e^x - x - 1$ and determine their nature. **2**

(ii) Sketch the curve. **1**

(iii) Use the graph to show that $e^x \geq x+1$ **1**

QUESTION 6: (START A NEW PIECE OF PAPER)

MARKS

(a) Given that the process of differentiation from first principles can be performed using **3**

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

and Newton's expansion for e^x is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ...$$

where $n! = n(n-1)(n-2)(n-3).......3.2.1$,

show that $\frac{d}{dx}\left(e^x\right) = e^x$.

(b) (i) Sketch $y = \frac{1}{x+1}$ for $x > -1$, showing where it cuts the y-axis and its asymptotes. **1**

(ii) Show that the area under the curve from $x = 0$ to $x = h$ is given by $\log_e(1+h)$. **1**

(iii) Show, with reasons, that $\frac{h}{1+h} < \log_e(1+h) < h$. **2**

(iv) Prove that $\lim_{h \to 0} \frac{\log_e(1+h)}{h} = 1$. **2**

(v) Using part (d) for $x > 0$, show $\lim_{x \to \infty} x \log_e(1 + \frac{1}{x}) = 1$ **2**

(c) (i) Given that $a > 0$ and $b > 0$, prove that $\frac{a+b}{2} \geq \sqrt{ab}$ **2**

(ii) Hence, or otherwise, given $a > 0$, $b > 0$, $c > 0$, $d > 0$ prove **2**

$\frac{a+b+c+d}{4} \geq \sqrt[4]{abcd}$

X2 Half Yearly — Answers

Question 1

(a) (i) $AB = (3+4i)(2-i)$
$= 6-3i+8i-4i^2$
$= 10+5i$

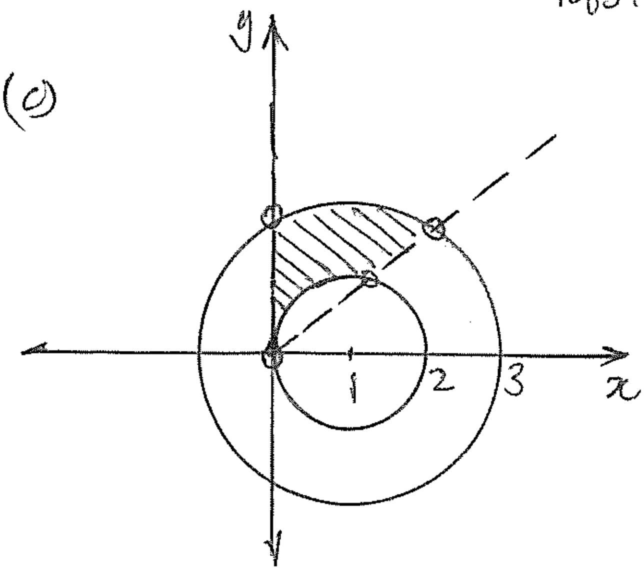
(ii) $\sqrt{A} = \sqrt{3+4i} = x+iy$
$3+4i = x^2-y^2+2xyi$
$\therefore x^2-y^2 = 3 \quad \text{.....(1)}$
$xy = 2 \quad \text{.....(2)}$
$(2) \Rightarrow y = \frac{2}{x} \quad \text{.....(3)}$
$(3) \rightarrow (1) \quad x^2 - \frac{4}{x^2} = 3$
$x^4 - 4 = 3x^2$
$\therefore (x^2-4)(x^2+1) = 0$
$\therefore x = \pm 2, \; y = \pm 1$
$\therefore \sqrt{A} = \pm(2+i)$

(iii) $\frac{A}{B} = \frac{3+4i}{2-i} \times \frac{2+i}{2+i}$
$= \frac{2+11i}{5}$
$= \frac{2}{5} + \frac{11}{5}i$

(b) (i) $|z| = 2$ & $\arg(z) = \frac{\pi}{6}$

(ii) $z^5 = \left(2\,\text{cis}\frac{\pi}{6}\right)^5 = 2^5\,\text{cis}\frac{5\pi}{6}$
$= 32\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$
$= 32\left(-\frac{\sqrt{3}}{2} + i\cdot\frac{1}{2}\right)$
$= -16\sqrt{3} + 16i$

(c)



(d) $\arg\left(\frac{z}{z-1}\right) = \frac{\pi}{2}$
$\arg z - \arg(z-1) = \frac{\pi}{2}$
$\tan^{-1}\frac{y}{x} - \tan^{-1}\frac{y}{x-1} = \frac{\pi}{2}$
$$\frac{\frac{y}{x} - \frac{y}{x-1}}{1 + \frac{y}{x}\cdot\frac{y}{x-1}} = \tan\frac{\pi}{2}$$
$\therefore 1 + \frac{y}{x}\cdot\frac{y}{x-1} = 0$
$x(x-1) + y^2 = 0$
$x^2 - x + y^2 = 0$

Circle, centre $(\frac{1}{2}, 0)$, radius $= \frac{1}{2}$ $\longrightarrow$ $\left(x-\frac{1}{2}\right)^2 + y^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$

(e) Consider $(|z|+|w|)^2$

$= |z|^2+|w|^2+2|z||w|$

From the cosine rule

$|z-w|^2 = |z|^2+|w|^2-2|z||w|\cos\theta$

$\leq |z|^2+|w|^2$

$\therefore |z|^2+|w|^2-2|z||w|\cos\theta \leq |z|^2+|w|^2+2|z||w|$

$\therefore |z-w|^2 \leq |z|^2+|w|^2+2|z||w|$

$\therefore |z-w| \leq |z|+|w|.$

Question 2

(a) $x^3-5x+7=0$

$\alpha^3-5\alpha+7=0$

$\beta^3-5\beta+7=0$

$\gamma^3-5\gamma+7=0$

$\therefore \alpha^3+\beta^3+\gamma^3-5(\alpha+\beta+\gamma)+21=0$

$\alpha^3+\beta^3+\gamma^3-5\,0+21=0$

$\alpha^3+\beta^3+\gamma^3=-21$

(b) $P(x) = ax^3+bx^2+d$

$P'(x) = 3ax^2+2bx$

$= x(3ax+2b)$

$P'(x)=0 \Rightarrow x=0$ or $x=-\frac{2b}{3a}$

NB $x=0$ is not a solution to $P(x)=0$

$\therefore P\left(-\frac{2b}{3a}\right)=0$

$\therefore a\left(\frac{-2b}{3a}\right)^3+b\left(\frac{-2b}{3a}\right)^2+d=0$

$-\frac{8b^3}{27a^2}+\frac{4b^3}{9a^2}+d=0$

$-8b^3+12b^3+27a^2d=0$

$\therefore 27a^2d+4b^3=0$

(c) $P(x)=2x^4+9x^3+6x^2-20x-24$

$P'(x)=8x^3+27x^2+12x-20$

$P''(x)=24x^2+54x+12$

$P''(x)=0 \quad 24x^2+54x+12=0$

$4x^2+9x+2=0$

$(4x+1)(x+2)=0$

$x=-2$ or $-\frac{1}{4}$

$P'(-2)=8(-2)^3+27(-2)^2+12(-2)-20=0$

$P'(-\frac{1}{4})=8(-\frac{1}{4})^3+27(-\frac{1}{4})^2+12(-\frac{1}{4})-20=-\frac{343}{16}$

& $P(-2)=2(-2)^4+9(-2)^3+6(-2)^2-20(-2)-24=0$

$\therefore$ Triple root is $x=-2$

Roots are $-2,-2,-2,\alpha$

& $\alpha-6=-4$

$\alpha=2$

$\therefore$ Roots are

$x=-2,-2,-2,2.$

(d) (i) Let $y=2x \Rightarrow x=\frac{y}{2}$

The equation is $\left(\frac{y}{2}\right)^4+4\left(\frac{y}{2}\right)^3-3\left(\frac{y}{2}\right)^2-4\left(\frac{y}{2}\right)+2=0$

$\therefore y^4+8y^3-12y^2-32y+32=0$ #

(II) let $y=\frac{1}{x^2} \Rightarrow x=\frac{1}{\sqrt{y}}$

The equation is $\left(\frac{1}{\sqrt{y}}\right)^4+4\left(\frac{1}{\sqrt{y}}\right)^3-3\left(\frac{1}{\sqrt{y}}\right)^2-4\left(\frac{1}{\sqrt{y}}\right)+2=0$

$$\frac{1}{y^2}+\frac{4}{y\sqrt{y}}-\frac{3}{y}-\frac{4}{\sqrt{y}}+2=0$$

$$1+4\sqrt{y}-3y-4y\sqrt{y}+2y^2=0$$

$$(1-3y+2y^2)=4\sqrt{y}(y-1)$$

$$\therefore (1-3y+2y^2)^2=16y(y-1)^2$$

$$\therefore 1-6y+13y^2-12y^3+4y^4=16y^3-32y^2+16y$$

$$\therefore 4y^4-28y^3+45y^2-22y+1=0 \ \#$$

(e) (i) let $x=\cos\theta$

$\cos 5\theta=16x^5-20x^3+5x$

$\therefore 16x^5-20x^3+5x-1=0$

$\Rightarrow 16x^5-20x^3+5x=1$

$\Rightarrow \cos 5\theta=1$

$5\theta=0, 2\pi, 4\pi, 6\pi, 8\pi$

$\theta=0, \frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}, \frac{8\pi}{5}$

$\therefore x=\cos 0, \cos\frac{2\pi}{5}, \cos\frac{4\pi}{5}, \cos\frac{6\pi}{5}, \cos\frac{8\pi}{5}$

$=1, \cos\frac{2\pi}{5}, \cos\frac{4\pi}{5}, \cos\frac{4\pi}{5}, \cos\frac{2\pi}{5}$

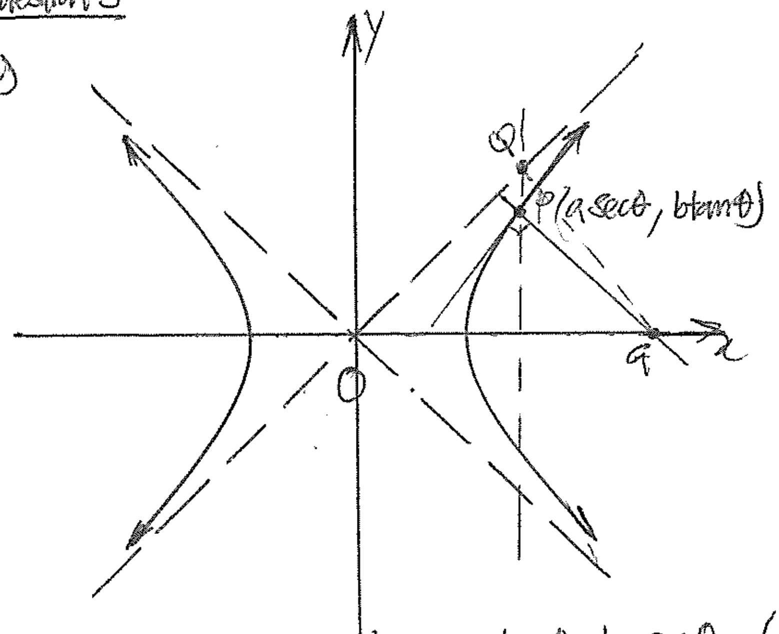
(II) $1+\cos\frac{2\pi}{5}+\cos\frac{4\pi}{5}+\cos\frac{4\pi}{5}+\cos\frac{2\pi}{5}=0$

$2\left(\cos\frac{2\pi}{5}+\cos\frac{4\pi}{5}\right)=-1$

$\therefore \cos\frac{2\pi}{5}+\cos\frac{4\pi}{5}=-\frac{1}{2}$ #

Question 3

(a)



Normal to hyperbola is $ax\tan\theta + by\sec\theta = (a^2+b^2)\sec\theta\tan\theta$

When $y=0$, $ax\tan\theta = (a^2+b^2)\sec\theta\tan\theta$

$$ax = (a^2+b^2)\sec\theta$$

$$x = \frac{(a^2+b^2)\sec\theta}{a}$$

$\therefore G\left(\frac{(a^2+b^2)\sec\theta}{a}, 0\right)$

Asymptote has equation $y = \frac{b}{a}x$

When $x = a\sec\theta$, $y = \frac{b}{a}\cdot a\sec\theta = b\sec\theta$

$\therefore$ Q is $(a\sec\theta, b\sec\theta)$

$m_{OQ} = \frac{b}{a}$ $\qquad m_{GQ} = \dfrac{b\sec\theta - 0}{a\sec\theta - \frac{(a^2+b^2)\sec\theta}{a}}$

$$= \frac{ab\sec\theta}{\cancel{a^2\sec\theta} - \cancel{a^2\sec\theta} - b^2\sec\theta}$$

$$= \frac{ab\sec\theta}{-b^2\sec\theta}$$

$$= -\frac{a}{b}$$

$\therefore m_{OQ} \times m_{GQ} = \frac{b}{a} \times -\frac{a}{b} = -1 \Rightarrow$ GQ is at right angles to the asymptote.

(9)(i) $y = \frac{c}{x} \Rightarrow y' = -\frac{c^2}{x^2}$

$m_{tangent} = \frac{-c^2}{(ct)^2} = -\frac{1}{t^2}$

Equation of tangent is $y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$

i.e. $x + t^2 y = 2ct$

Gradient of ON is t^2 & has equation $y = t^2 x$

N: $x + t^2(t^2 x) = 2ct$

$x(1+t^4) = 2ct$

$x = \frac{2ct}{1+t^4}$ & $y = \frac{2ct^3}{1+t^4}$

$\therefore$ N is $\left(\frac{2ct}{1+t^4}, \frac{2ct^3}{1+t^4}\right)$

(ii) Show that the point lies on $(x^2+y^2)^2 = 4c^2xy$

LHS $= (x^2+y^2)^2$

$= \left(\left(\frac{2ct}{1+t^4}\right)^2 + \left(\frac{2ct^3}{1+t^4}\right)^2\right)^2$

$= \left(\frac{4c^2t^2 + 4c^2t^6}{(1+t^4)^2}\right)^2$

$= \left[\frac{4c^2t^2(1+t^4)}{(1+t^4)^2}\right]^2$

$= \left[\frac{4c^2t^2}{(1+t^4)}\right]^2$

$= \frac{16c^4t^4}{(1+t^4)^2}$

RHS $= 4c^2xy$

$= 4c^2 \cdot \frac{2ct}{(1+t^4)} \cdot \frac{2ct^3}{(1+t^4)}$

$= \frac{16c^4t^4}{(1+t^4)^2}$

$=$ LHS

$\therefore$ N lies on the curve $(x^2+y^2)^2 = 4c^2xy$ ※

(d) $xy(x+y)+16=0$

$x^2y+xy^2+16=0$

$x^2 \cdot \frac{dy}{dx}+y \cdot 2x+x \cdot 2y\frac{dy}{dx}+y^2 \cdot 1=0$

$\frac{dy}{dx}(x^2+2xy)+2xy+y^2=0$

$\frac{dy}{dx}=-\frac{2xy+y^2}{x^2+2xy}$

At $(-2,-2)$

$\frac{dy}{dx}=m_{tangent}=-\frac{2 \cdot -2 \cdot -2+(-2)^2}{(-2)^2+2 \cdot 2 \cdot -2}=-\frac{12}{12}=-1$

Equation of tangent is $y+2=-1(x+2)$

$y=-x-4$ #

Question 4

(a) & (b) — check answers using GeoGebra.

Question 5

(a) (i) Let $S\hat{P}T=\alpha°$

$P\hat{R}S=\alpha°$ (alternate segment theorem)

$PT=QT$ (intercepts on tangents from an external point are equal)

$\therefore P\hat{Q}T=\alpha°$ (angles opposite equal sides are equal)

$\therefore$ PT subtends equal angles at R & Q on the same side of it

$\therefore$ PRQT is a cyclic quadrilateral.

(ii) $Q\hat{R}T=Q\hat{P}T$ (angles at circumference standing on arc QT are equal)

$=\alpha°$

$\therefore Q\hat{R}S=P\hat{R}S$

$\therefore$ RS bisects $Q\hat{R}P$.

(b) (i) $\sqrt{n+1}-\sqrt{n}=\sqrt{n+1}-\sqrt{n} \times \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}=\frac{1}{\sqrt{n+1}+\sqrt{n}}$

Consider $\sqrt{n+1}-\sqrt{n}-\frac{1}{2\sqrt{n+1}} = \frac{1}{\sqrt{n+1}+\sqrt{n}} - \frac{1}{2\sqrt{n+1}}$

$$= \frac{2\sqrt{n+1}-(\sqrt{n+1}+\sqrt{n})}{2\sqrt{n+1}\left[\sqrt{n+1}+\sqrt{n}\right]}$$

$$= \frac{\sqrt{n+1}-\sqrt{n}}{2\sqrt{n+1}\left[\sqrt{n+1}+\sqrt{n}\right]}$$

$\sqrt{n+1} > \sqrt{n}$ for $n>0$

$\sqrt{n+1}>0$, $\sqrt{n}>0$

$\therefore \sqrt{n+1}+\sqrt{n} > 0$

> 0

$$\therefore \sqrt{n+1}-\sqrt{n} > \frac{1}{2\sqrt{n+1}}$$

(II) Prove true for $n=7$

$LHS = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7} = \frac{363}{140} \doteqdot 2.592857143$

$RHS = \sqrt{7} \doteqdot 2.645751311$

$\therefore 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7} < \sqrt{7}$ $\therefore$ True for $n=7$

Assume true for $n=k$

i.e. $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots+\frac{1}{k} < \sqrt{k}$

Prove true for $n=k+1$

$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots+\frac{1}{k}+\frac{1}{k+1} < \sqrt{k}+\frac{1}{k+1}$

Consider $\sqrt{k+1}-\sqrt{k}$.

We know $\sqrt{k+1}-\sqrt{k} > \frac{1}{2\sqrt{k+1}}$

Consider $\frac{1}{2\sqrt{k+1}} - \frac{1}{k+1} = \frac{k+1-2\sqrt{k+1}}{2\sqrt{k+1}(k+1)} = \frac{\sqrt{k+1}\left[\sqrt{k+1}-2\right]}{2(k+1)(\sqrt{k+1})} > 0$

Since $k \geq 7$ $\sqrt{k+1}-2 > 0$ $\therefore \frac{1}{2\sqrt{k+1}} > \frac{1}{k+1}$

& $\sqrt{k+1} > 0$ $(k+1) > 0$

Back to the question...

$\therefore \sqrt{k} + \frac{1}{k+1} < \sqrt{k+1}$

$\therefore 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{k+1} < \sqrt{k+1}$

$\therefore$ True for $n = k+1$

$\therefore$ True by Mathematical Induction for $n \geq 7$.

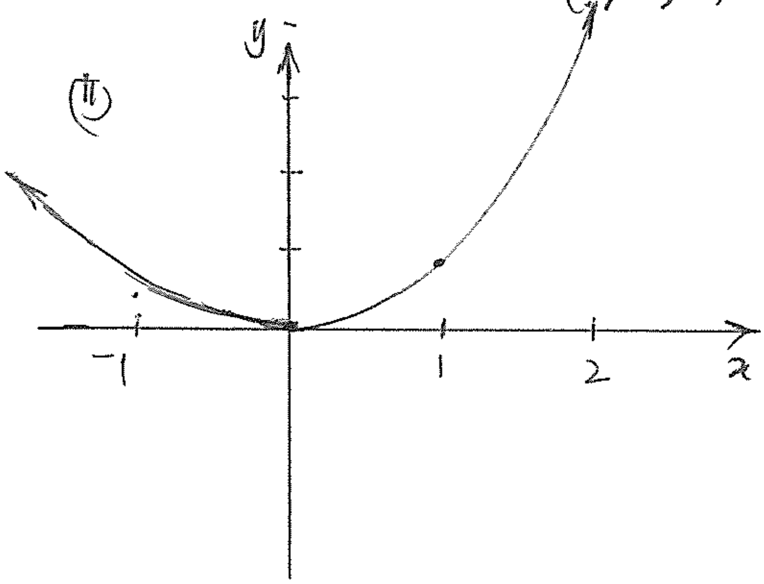
(c) (i) $y = e^x - x - 1$ Stationary points $y' = 0 \Rightarrow e^x - 1 = 0$

$y' = e^x - 1$ $\qquad e^x = 1$

$y'' = e^x$ $\qquad x = 0$

$\therefore$ At $(0,0)$, $y'' = e^0 = 1 > 0$ $\therefore$ Relative minimum

(ii)



(iii) From the graph

$e^x - x - 1 \geq 0$ for all x

$\therefore e^x \geq x + 1$.

Question 6

(a) $\frac{dy}{dx} = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} = e^x \lim_{h \to 0} \frac{e^h - 1}{h}$

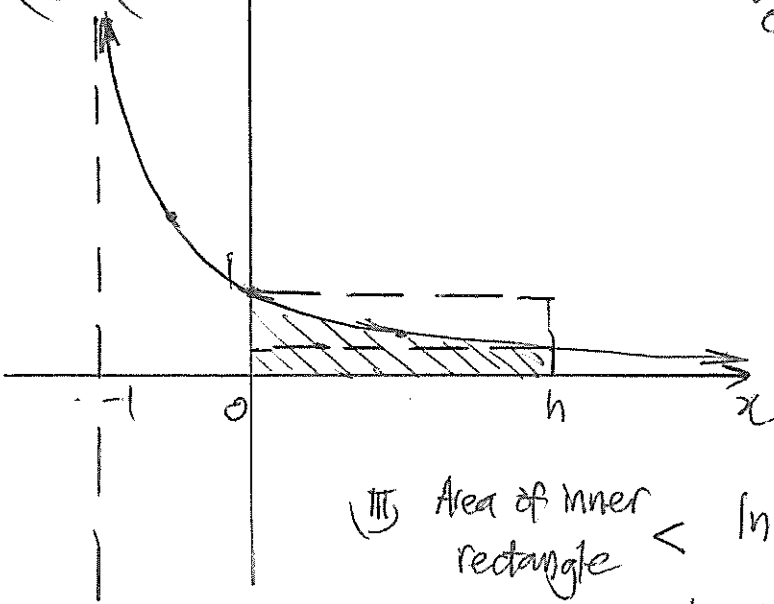
$= e^x \cdot \lim_{h \to 0} \frac{1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + \cdots - 1}{h}$

$= e^x \cdot \lim_{h \to 0} \frac{h + \frac{h^2}{2!} + \frac{h^3}{3!} + \cdots}{h}$

$= e^x \cdot \lim_{h \to 0} \left(1 + \frac{h}{2!} + \frac{h^2}{3!} + \cdots\right)$

$= e^x \cdot 1 \quad = e^x$

(b) (i)



(ii) Area $= \int_0^h \frac{1}{1+x}\,dx = \left[\ln(1+x)\right]_0^h$

$= \ln(1+h) - \ln(1)$

$= \ln(1+h)$

(iii) Area of inner rectangle $<$ $\ln(1+h)$ $<$ Area of outer rectangle

$h \times \frac{1}{1+h} < \ln(1+h) < 1 \cdot h$

$\therefore \frac{h}{1+h} < \ln(1+h) < h$

(iv) Divide the result of (iii) by h

$\frac{1}{1+h} < \frac{\log_e(1+h)}{h} < 1$

As $h \to 0$ $\frac{1}{1+h} \to 1$ $\therefore$ $\lim_{h\to 0} \frac{\log_e(1+h)}{h} = 1$

($\frac{\log_e(1+h)}{h}$ is sandwiched between two values both approaching 1)

(v) $\lim_{h\to 0} \frac{\log_e(1+h)}{h} = 1$

Let $h = \frac{1}{x}$ $\therefore$ $\lim_{x\to\infty} \frac{\log_e\left(1+\frac{1}{x}\right)}{\frac{1}{x}} = 1$

As $h \to 0$, $x \to \infty$

$\therefore \lim_{x\to\infty} x \log_e\left(1+\frac{1}{x}\right) = 1$

(e) (i) Consider $\frac{a+b}{2} - \sqrt{ab}$

$$= \frac{a+b-2\sqrt{ab}}{2}$$

$$= \frac{(\sqrt{a}-\sqrt{b})^2}{2}$$

$$\geq 0$$

$$\therefore \frac{a+b}{2} \geq \sqrt{ab}$$

(ii) $$\frac{a+b+c+d}{4} = \frac{\frac{a+b}{2} + \frac{c+d}{2}}{2} \geq \sqrt{\left(\frac{a+b}{2}\right)\left(\frac{c+d}{2}\right)}$$

$$\geq \sqrt{\sqrt{ab} \cdot \sqrt{cd}}$$

$$= \sqrt{\sqrt{abcd}}$$

$$= \sqrt[4]{abcd}$$

$\frac{a+b}{2} \geq \sqrt{ab}$

$\frac{c+d}{2} \geq \sqrt{cd}$