HURLSTONE AGRICULTURAL HIGH SCHOOL

PRO PATRIA

MATHEMATICS – EXTENSION 2

2005

YEAR 12

HALF YEARLY EXAMINATION

Examiners ~ J Dillon, G Rawson

GENERAL INSTRUCTIONS

- Reading Time – 5 minutes.
- Working Time – 2 HOURS.
- Attempt **all** questions.
- **All** necessary working should be shown in every question.
- This paper contains five (5) questions.
- Marks may not be awarded for careless or badly arranged work.
- Board approved calculators may be used.
- **Each question is to be started on a new piece of paper.**
- This examination paper must **NOT** be removed from the examination room.

STUDENT NAME: ______________________________

TEACHER: ______________________________

Question 1 (15 marks) Start a NEW answer booklet

Marks

(a) Let $u = 4 + 2i$ and $v = 3 - i$

(i) Find $\bar{u}v$ **1**

(ii) Find v^2 **1**

(b) The complex number z satisfies both equations $|z-1| = \frac{1}{2}|z|$ and $\arg(z-1) - \arg z = \frac{\pi}{3}$

(i) Show that $\frac{z-2}{z} = \frac{1}{2}\text{cis}\left(\frac{\pi}{3}\right)$. **2**

(ii) Hence, show that $z = \frac{3 + i\sqrt{3}}{3}$. **2**

(c) (i) In the **same** Argand diagram, sketch the locus of $|z-3| = 1$ and the locus of $|z| = |z-2i|$. **2**

(ii) Write down the complex number represented by the point of intersection of the two loci. **1**

(d) (i) Find all solutions of the equation $z^3 = 1$. Leave your solutions in the form $z = \text{cis}\,\theta$. **1**

(ii) Let $\omega = \text{cis}\frac{2\pi}{3}$. Show that $\omega^2 + \omega + 1 = 0$. **1**

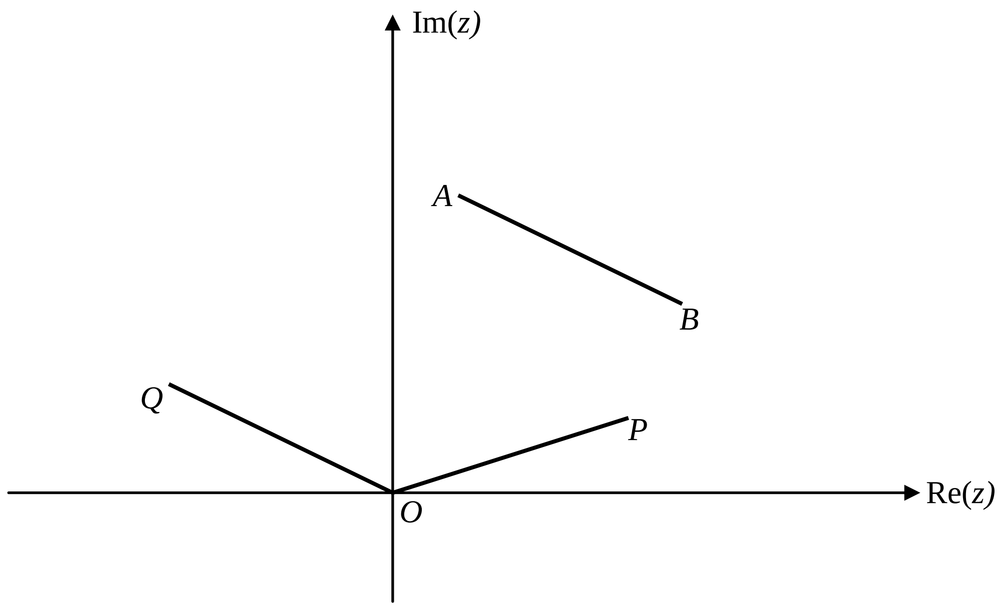
(iii) By expanding $(z+2)(z^2+z+1)$, or otherwise, solve the equation $z^3 + 3z^2 + 3z + 2 = 0$. **2**

Question 1 continued on page 3

Question 1 *(continued)*

Marks

(e) **2**



In the Argand diagram above, the intervals AB, OP and OQ are equal in length. Also, $QO \parallel AB$ and $\angle POQ = \dfrac{2\pi}{3}$, where P has been rotated $\dfrac{2\pi}{3}$ anticlockwise about O to the point Q.

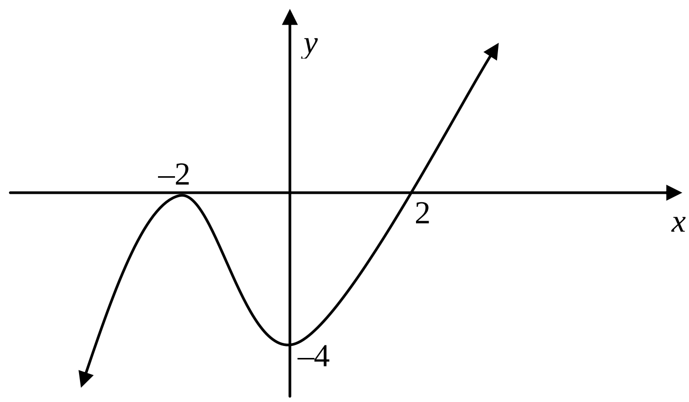
Let A and B represent the complex numbers u and v respectively.

Find in terms of u and v, the complex number represented by P.

Question 2 (15 marks) Start a NEW answer booklet

Marks

(a)



The curve shown in the diagram has equation $y = f(x)$.
There is a minimum turning point at (0, –4) and a maximum turning point at (–2, 0).
The curve crosses the x-axis at (2, 0).

Sketch on separate diagrams:

(i) $y = |f(x)|$ **1**

(ii) $y = \dfrac{1}{f(x)}$ **2**

(iii) $y^2 = f(x)$ **2**

(b) Let $f(x) = x - \dfrac{4}{x}$.
Provide separate half-page sketches of the graphs of the following functions.

(i) $y = f(x)$ **2**

(ii) $y = \sqrt{f(x)}$ **2**

(iii) $y = e^{f(x)}$ **2**

Label each graph with its equation

(c) Find the equation of the tangent to the curve $xy(x+y)+16=0$ at the point on the curve where the gradient is –1. **4**

Question 3 (15 marks) Start a NEW answer booklet

Marks

(a) (i) Find all the zeros of $P(x) = x^4 - 5x^2 + 6$ over the rational numbers **2**

(ii) Find $P(x)$, given that $P(x)$ is monic, of degree 4, with -1 as a single zero and 3 as a zero of multiplicity 3. **2**

(b) Find the remainder when $P(x) = x^3 + 2x^2 + 1$ is divided by $x + i$ **2**

(c) The equation $x^3 + x^2 - 2x - 3 = 0$ has roots α, β and γ. Find the equations with roots

(i) 2α, 2β and 2γ **2**

(ii) $\frac{\alpha}{2}$, $\frac{\beta}{2}$ and $\frac{\gamma}{2}$ **2**

(iii) $\alpha - 2$, $\beta - 2$ and $\gamma - 2$. **2**

(d) Consider $P(x) = x^4 + 3x^3 + 6x^2 + 12x + 8$,

(i) Show that $x = 2i$ is a root of $P(x)$ **1**

(ii) Find all the roots of $P(x)$ in the complex field. **2**

Question 4 (15 marks) Start a NEW answer booklet

Marks

Consider the ellipse $\dfrac{x^2}{16}+\dfrac{y^2}{12}=1$.

(i)	Find the eccentricity of the ellipse.	**1**
(ii)	Find the coordinates of the foci and the equations of the directrices of the ellipse.	**2**
(iii)	Sketch the graph of the ellipse showing clearly all of the above features and the intercepts on the coordinate axes.	**2**
(iv)	Use differentiation to derive the equations of the tangent and normal to the ellipse at the point (2, 3).	**3**
(v)	The tangent and normal to the ellipse at P cut the y – axis at A and B respectively. Find the coordinates of A and B.	**1**
(vi)	Show that AB subtends a right angle at the focus S of the ellipse.	**2**
(vii)	Show that the points A, P, S and B are concyclic.	**1**
(viii)	Find the centre and radius of the circle which passes through the points A, P, S and B.	**3**

Question 5 (15 marks) Start a NEW answer booklet

Marks

(a) Find real constants A and B such that $\frac{7x-4}{2x^2-3x-2}=\frac{A}{2x+1}+\frac{B}{x-2}$ **3**

(b) Show that $(1+\cos 2\theta+i\sin 2\theta)^n=2^n\cos^n\theta(\cos n\theta+i\sin n\theta)$ **3**

(c) Let $f(x)=\dfrac{x^2-x}{x-2}$

(i) Show that $f(x)=x+1+\dfrac{2}{x-2}$ **1**

(ii) Using (i), or otherwise, determine the number of stationary points on $y=f(x)$. **2**

(iii) Using (i), or otherwise, sketch the curve $y=f(x)$ indicating any asymptotes and x-intercepts. **3**

(iv) Using your graph, or otherwise, solve the inequality

$$\frac{x^2-x}{x-2}>0.$$ **1**

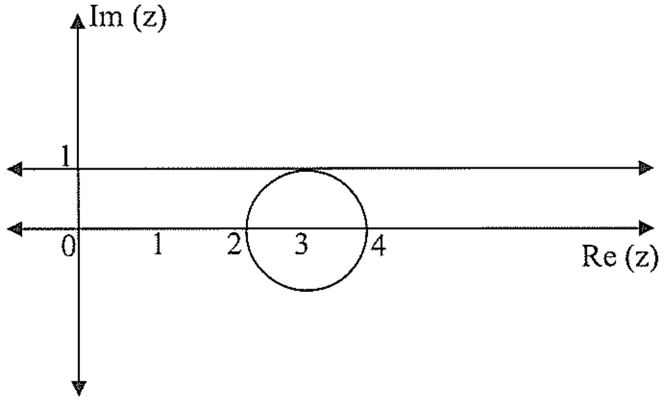
(d) Sketch $y=|x|+|x+2|$ for $-4\le x\le 2$.
Hence solve the inequality $|x|+|x+2|<4$. **2**

Year12	Mathematics Extension 2	Half Yearly Examination 2005
Question No1	Solutions and Marking Guidelines	

Outcomes Addressed in this Question

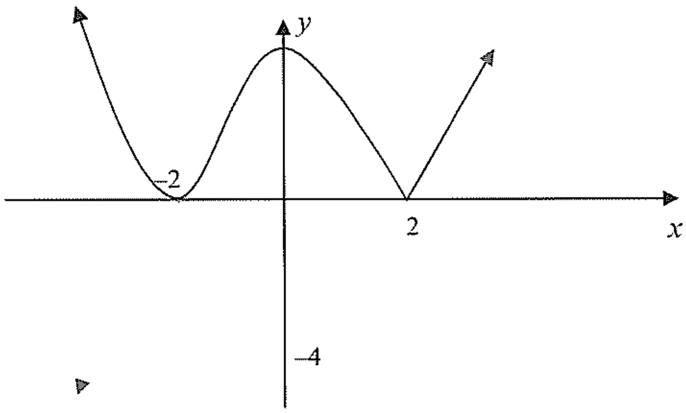
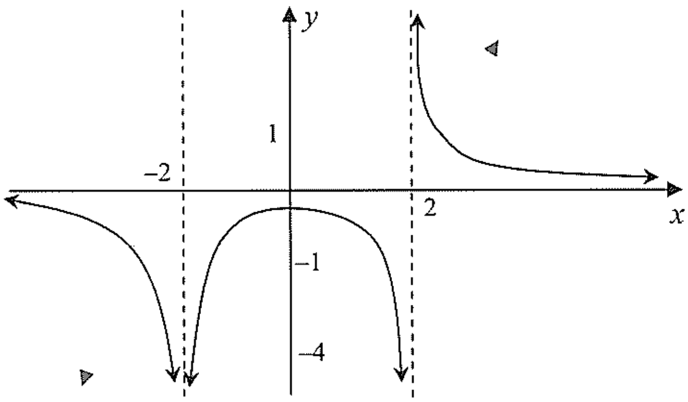
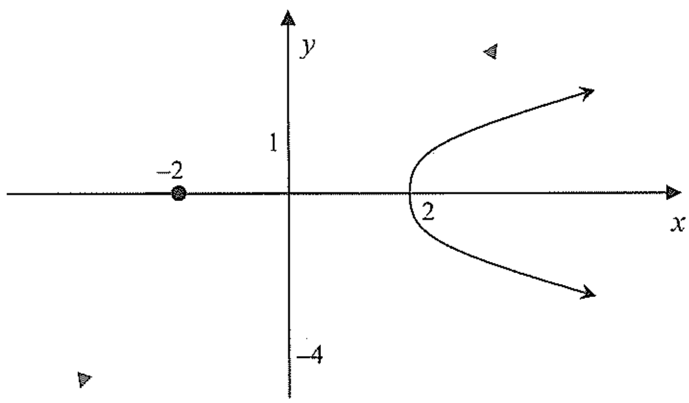
E3 Uses the relationship between algebraic and geometric representations of complex numbers

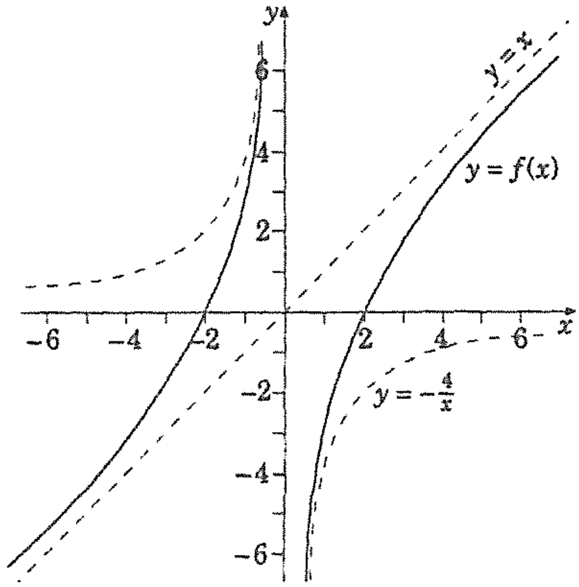
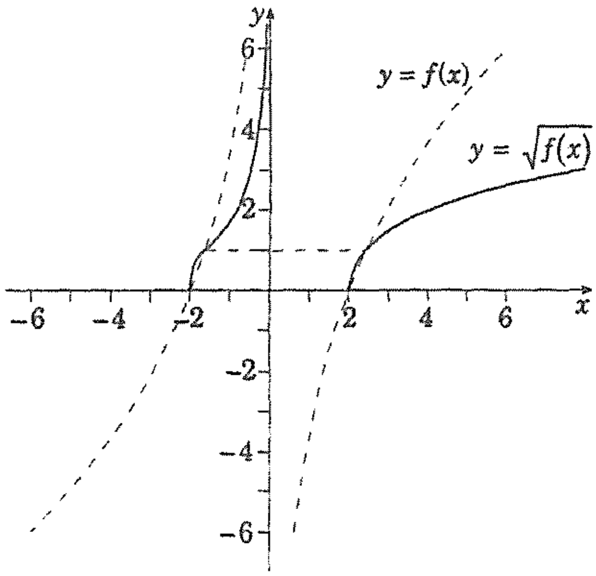
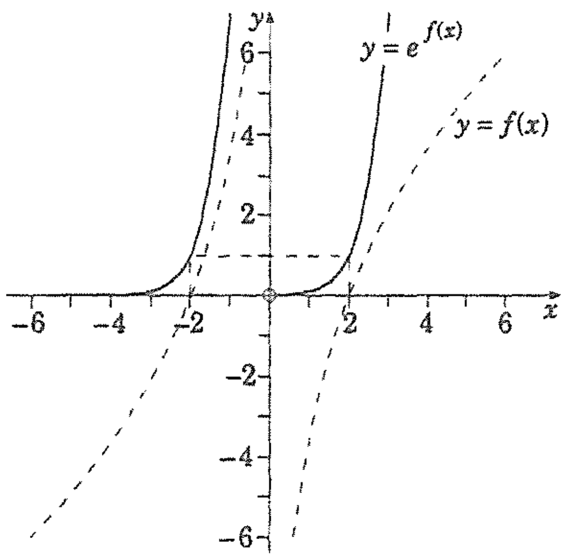
Outcome	Solutions	Marking Guidelines
E3 (a)(i)	$\overline{u}v = (4-2i)(3-i)$ $= 12 - 10i + 2i^2$ $= 10 - 10i$	Award 1 for correct answer
E3 (a)(ii)	$v^2 = (3-i)^2$ $= 9 - 6i + i^2$	Award 1 for correct answer
E3 (b)(i)	$\dfrac{\lvert z-1\rvert}{\lvert z\rvert} = \dfrac{1}{2}$ so $\left\lvert\dfrac{z-1}{z}\right\rvert = \dfrac{1}{2}$ Also $\arg(z-1) - \arg(z) = \dfrac{\pi}{3}$ $\therefore \arg\left(\dfrac{z-1}{z}\right) = \dfrac{\pi}{3}$ $\therefore \dfrac{z-1}{z} = \dfrac{1}{2} cis \dfrac{\pi}{3}$	Award 2 for correct solution Award 1 for reasonable attempt (failing to correctly show the result)
E3 (b)(ii)	$z - 1 = z\left(\dfrac{1}{2} cis \dfrac{\pi}{3}\right)$ $z - z\left(\dfrac{1}{2} cis \dfrac{\pi}{3}\right) = 1$ $z\left(1 - \dfrac{1}{2} cis \dfrac{\pi}{3}\right) = 1$ $z\left(1 - \dfrac{1}{2}\cos\dfrac{\pi}{3} - \dfrac{1}{2} i \sin\dfrac{\pi}{3}\right) = 1$ $z\left(\dfrac{3}{4} - i\dfrac{\sqrt{3}}{4}\right) = 1$ $\therefore z = \dfrac{1}{\frac{3}{4} - i\frac{\sqrt{3}}{4}} = \dfrac{1}{\frac{3}{4} - i\frac{\sqrt{3}}{4}} \times \dfrac{\frac{3}{4} + i\frac{\sqrt{3}}{4}}{\frac{3}{4} + i\frac{\sqrt{3}}{4}} = \dfrac{3 + i\sqrt{3}}{3}$	Award 2 for correct solution Award 1 for a substantially correct attempt
E3 (c)(i)	$\lvert z-3\rvert = 1$ represents a circle centre (3, 0) and radius 1 $\lvert z\rvert = \lvert z-2i\rvert$ represents the perpendicular bisector of the interval joining (0, 0) and (0, 2)	Award 2 for correct diagram with correct point of intersection Award 1 for correct diagram but no attempt to find point of intersection, **or** incorrect diagram (with one mistake) and an attempt to find the point of intersection



E3 (c)(ii)	The point of intersection represents the complex number $z = 3 + i$	Award 1 for correct answer
E3 (d)(i)	Let $z = cis\theta \; (\lvert z\rvert = 1)$ $(cis\theta)^3 = 1$ $\therefore cis3\theta = 1$ (by de Moivre's) Hence, $3\theta = 0, 2\pi, 4\pi, ... \Rightarrow \theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}, ...$ $\therefore$ The roots are $1, cis\frac{2\pi}{3}, cis\frac{4\pi}{3}$.	Award 1 for correct solution
E3 (d)(ii)	Roots are $cis0 = 1, cis\frac{2\pi}{3} = \omega, cis\frac{4\pi}{3} = \omega^2$ From equation $z^3 - 1 = 0$ Sum of roots $= 1 + \omega + \omega^2 = 0$	Award 1 for correct solution
E3 (d)(iii)	$(z+2)(z^2+z+1) = z^3 + 3z^2 + 3z + 2$ $\therefore z^3 + 3z^2 + 3z + 2 = 0$ has the same roots as $(z+2)(z^2+z+1) = 0$ $\therefore$ Roots are $z = -2, \omega, \omega^2$	Award 2 for correct solution Award 1 for a substantially correct attempt
E3 (e)	The vectors BA and OQ are equal. Hence, Q represents the complex number $z - u$. OP represents a clockwise rotation of OQ through $\frac{2\pi}{3}$. Hence $OP = (u - v) \times cis\left(-\frac{2\pi}{3}\right)$ $= (u-v)\left(\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right)$ $= (u-v)\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$	Award 2 for correct solution Award 1 for either $z - w$ **or** $-\frac{1}{2} - i\frac{\sqrt{3}}{2}$

<table>
<tr><td>Year 12</td><td>Mathematics Extension 2</td><td>Half Yearly Examination 2005</td></tr>
<tr><td>Question No. 2</td><td>Solutions and Marking Guidelines</td><td></td></tr>
<tr><td colspan="3">Outcomes Addressed in this Question</td></tr>
<tr><td colspan="3">E6 combines the ideas of algebra and calculus to determine the important features of the graphs of a wide variety of functions</td></tr>
</table>

<table>
<tr><th>Outcome</th><th>Solutions</th><th>Marking Guidelines</th></tr>
<tr><td>E6 (a)(i)</td><td>
</td><td><u>1 mark</u> : correct graph</td></tr>
<tr><td>E6 (a)(ii)</td><td>
</td><td><u>2 marks</u> : correct graph with asymptotes
OR
<u>1 mark</u> : substantially correct attempt</td></tr>
<tr><td>E6 (a)(iii)</td><td>
</td><td><u>2 marks</u> : correct graph with (−2, 0) included
OR
<u>1 mark</u> : substantially correct attempt</td></tr>
</table>

Year 12	Mathematics Extension 2	Half Yearly Examination 2005
Question No. 2	Solutions and Marking Guidelines	
Outcome	**Solutions**	**Marking Guidelines**
E6 (b)(i)		<u>2 marks</u> : correct graph *with asymptotes* *OR* <u>1 mark</u> : substantially correct attempt
E6 (b)(ii)		<u>2 marks</u> : correct graph *with concavity changes* *OR* <u>1 mark</u> : substantially correct attempt
E6 (b)(iii)		<u>2 marks</u> : correct graph *OR* <u>1 mark</u> : substantially correct attempt

Year 12	Mathematics Extension 2	Half Yearly Examination 2005
Question No. 2	Solutions and Marking Guidelines	

Outcome	Solutions	Marking Guidelines
E6 (c)	$xy(x+y)+16=0$ using chain rule with $u = xy$, and $v = (x + y)$ $xy\left(1+\frac{dy}{dx}\right)+(x+y)\left(x\frac{dy}{dx}+y\right)+0=0$	<u>1 mark</u> : differentiating correctly
	now, when $\frac{dy}{dx}=-1$ $xy(1-1)+(x+y)(-x+y)+0=0$ which gives $x=-y$ or $x=y$	<u>1 mark</u> : substituting $y'=-1$ to find relationship between x and y
	$xy(x+y)+16=0$ $x=-y \Rightarrow \quad -y^2(-y+y)=-16$ $0=16$ ie no solution $y^2(y+y)+16=0$ $x=y \Rightarrow \quad y=-2$ $x=-2$	<u>1 mark</u> : finding the point where $y'=-1$ to be $(-2, -2)$
	Therefore $m=-1$ at the point $(-2, -2)$, so $y-y_1=m(x-x_1)$ $y+2=-1(x+2)$ $y+2=-x-2$ $\therefore x+y+4=0$	<u>1 mark</u> : finding the actual equation of the gradient at this point

Year 12	Mathematics Extension 2	HY Examination 2005
Question No. 3	Solutions and Marking Guidelines	

Outcomes Addressed in this Question

E4 uses efficient techniques for the algebraic manipulation required in dealing with questions such as those involving conic sections and polynomials

Outcome	**Solutions**	**Marking Guidelines**
E4 (a)(i)	$P(x) = x^4 - 5x^2 + 6$ $= (x^2 - 3)(x^2 - 2)$ $\therefore$ no rational roots	<u>1 mark</u> : factorising $P(x)$ over rationals *AND* <u>1 mark</u> : stating no rational roots
E4 (a)(ii)	$Q(x) = (x+1)(x-3)^3$ $= x^4 - 8x^3 + 18x^2 - 27$	<u>2 marks</u> : $Q(x)$ in factorised form *OR* <u>1 mark</u> : having one factor correct with one factor degree 1 and the other factor degree 2.
E4 (b)	$P(x) = x^3 + 2x^2 + 1$ $P(-i) = (-i)^3 + 2(-i)^2 + 1$ $= i - 2 + 1$ $= i - 1$	<u>1 mark</u> : using the remainder theorem, with $x = -i$ *AND* <u>1 mark</u> : finding remainder = $i - 1$
E4 (c)(i)	$x^3 + x^2 - 2x + 1 = 0$ For roots $2\alpha, 2\beta, 2\gamma$, replace x with $\frac{x}{2}$ $\left(\frac{x}{2}\right)^3 + \left(\frac{x}{2}\right)^2 - 2\left(\frac{x}{2}\right) - 3 = 0$ $\frac{x^3}{8} + \frac{x^2}{4} - x - 3 = 0$ $x^3 + 2x^2 - 8x - 24 = 0$	<u>1 mark</u> : using correct substitution $\left(\frac{x}{2}\right)$ *AND* <u>1 mark</u> : obtaining correct equation (1 simple algebraic error allowed)
E4 (c)(ii)	For roots $\frac{\alpha}{2}, \frac{\beta}{2}, \frac{\gamma}{2}$, replace x with $2x$ $(2x)^3 + (2x)^2 - 2(2x) - 3 = 0$ $8x^3 + 4x^2 - 4x - 3 = 0$	<u>1 mark</u> : using correct substitution $2x$ *AND* <u>1 mark</u> : obtaining correct equation (1 simple algebraic error allowed)
E4 (c)(iii)	For roots $\alpha - 2, \beta - 2, \gamma - 2$, replace x with $x + 2$ $(x+2)^3 + (x+2)^2 - 2(x+2) - 3 = 0$ $x^3 + 7x^2 + 14x + 5 = 0$	<u>1 mark</u> : using correct substitution $(x+2)$ *AND* <u>1 mark</u> : obtaining correct equation (1 simple algebraic error allowed)

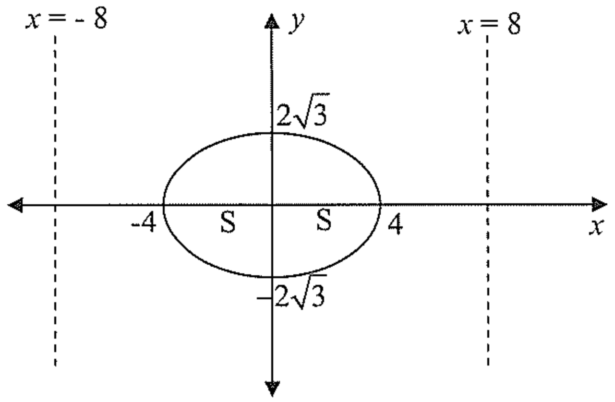
Outcome	Solutions	Marking Guidelines
E4 (d)(i)	$P(x) = x^4 + 3x^3 + 6x^2 + 12x + 8$ $P(2i) = (2i)^4 + 3(2i)^3 + 6(2i)^2 + 12(2i) + 8$ $= 16 - 24i - 24 + 24i + 8$ $= 0$ $\therefore$ no remainder	1 mark : showing $P(2i)=0$
E4 (d)(ii)	coefficients of P(x) are real $\therefore$ if $x - 2i$ is a factor, $x + 2i$ is also a factor *ie* $(x-2i)(x+2i) = x^2 + 4$ is a factor so $P(x) = x^4 + 3x^3 + 6x^2 + 12x + 8$ $= (x^2+4)(ax^2+bx+c)$ $= (x^2+4)(x^2+3x+2)$ $= (x-2i)(x+2i)(x+2)(x+1)$ $\therefore$ roots in the complex field are $-1, -2, 2i, -2i$	1 mark : factorising $P(x)$ at least over the real field *AND* 1 mark : stating the actual complex roots

Year 12	Mathematics Extension 2	Half Yearly Examination 2005
Question No. 4	Solutions and Marking Guidelines	

Outcomes Addressed in this Question

E3 uses the relationship between algebraic and geometric representations of conic sections

E4 uses efficient techniques for the algebraic manipulation required in dealing with questions involving conic sections

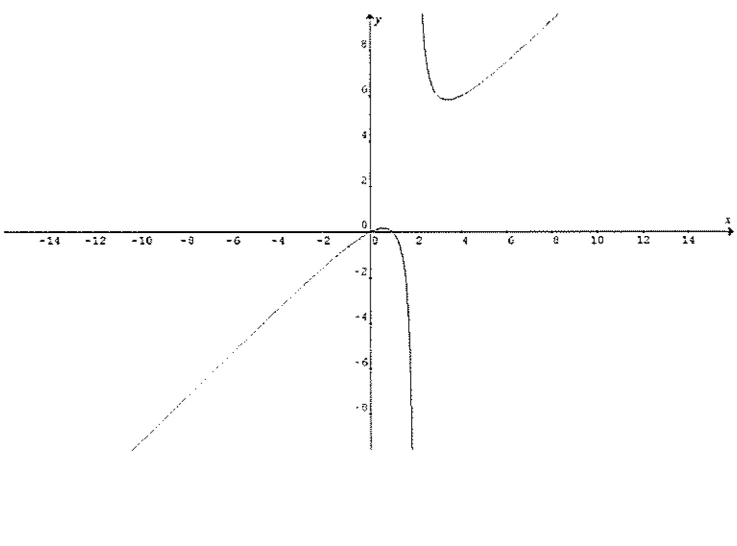
Outcome	Solutions	Marking Guidelines
E3 (i)	$e^2 = 1 - \frac{b^2}{a^2} = 1 - \frac{12}{16} = \frac{1}{4}$ $e = \frac{1}{2}$	Award 1 for correct answer
E3 (ii)	Foci = $(\pm ae, 0) = (\pm 2, 0)$ Directrices: $x = \pm\frac{a}{e} = \pm 8$	Award 1 for correct foci Award 1 for directrices
E3 (iii)		Award 2 for correct sketch showing all features and intercepts Award 1 for sketch showing 2 of the 3 requirements
E4 (iv)	$\frac{x^2}{16} + \frac{y^2}{12} = 1$ $\frac{2x}{16} + \frac{2y\frac{dy}{dx}}{12} = 0$ $\frac{dy}{dx} = -\frac{3x}{4y}$ At P(2, 3) $\frac{dy}{dx} = -\frac{1}{2}$ Equation of tangent is $y - 3 = -\frac{1}{2}(x-2)$ $\Rightarrow x + 2y - 8 = 0$ Equation of normal is $y - 3 = 2(x-2)$ $\Rightarrow 2x - y - 1 = 0$	Award 1 for determining $\frac{dy}{dx}$ Award 1 for equation of tangent Award 1 for equation of normal
E4 (v)	Tangent cuts y axis at $A(0, 4)$ Normal cuts y axis at $B(0, -1)$	Award 1 for correct coordinates of both A and B
E4 (vi)	$m_{AS} = \frac{4-0}{0-2} = -2$ and $m_{BS} = \frac{-1-0}{0-2} = \frac{1}{2}$ $\therefore m_{AS} \times m_{BS} = -1$ $\therefore \angle ASB = 90^0$	Award 2 for correct solution Award 1 for a solution that is substantially correct

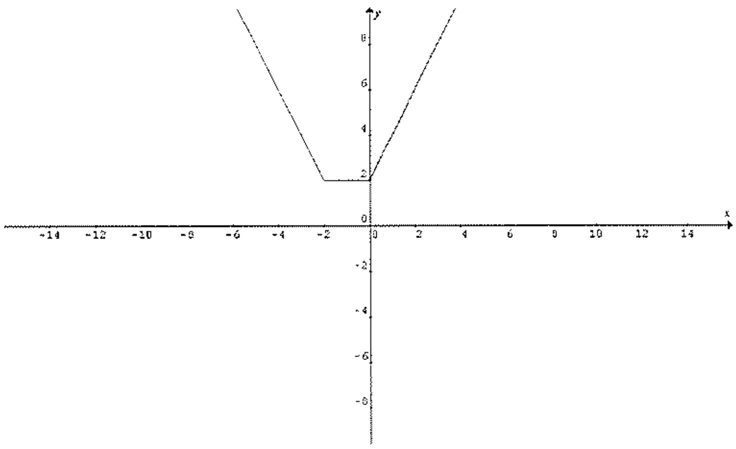
E3 (vii)	$\angle APB = 90°$ (tangent $\perp$ normal) $\therefore AB$ subtends equal angles of $90°$ at P and S $\therefore A$, P, S, and B are concyclic.	Award 1 for correct solution
E4 (viii)	Diameter of circle is AB Distance between A and $B = 5$ units Centre of circle is $\left(0, \frac{3}{2}\right)$ and radius is $\frac{5}{2}$ units	Award 3 for correct solution Award 2 for being able to identify AB as diameter and finding either centre or radius Award 1 for only finding centre, radius or diameter AB

Year12	Mathematics Extension 2	Half Yearly Examination 2005
Question No. 5	Solutions and Marking Guidelines	

Outcomes Addressed in this Question

E3 Uses the relationship between algebraic and geometric representations of complex numbers
E4 Uses efficient techniques for the algebraic manipulation required in dealing with questions involving polynomials
E6 Combines the ideas of algebra and calculus to determine the important features of the graphs of a wide variety of functions

Outcome	Solutions	Marking Guidelines
E4 (a)	$\dfrac{7x-4}{2x^2-3x-2}=\dfrac{A}{2x+1}+\dfrac{B}{x-2}$ $\therefore 7x+4=A(x-2)+B(2x+1)$ Let $x=2$, $10=5B\Rightarrow B=2$ Let $x=-\dfrac{1}{2}$, $-7\dfrac{1}{2}=-\dfrac{5}{2}A\Rightarrow A=3$	Award 1 mark for correct value of A Award 1 mark for correct value of B
E3 (b)	$\left(1+\cos 2\theta+i\sin 2\theta\right)^n=\left(1+2\cos^2\theta-1+i.2\sin\theta\cos\theta\right)^n$ $=\left(2\cos\theta\left(\cos\theta+i\sin\theta\right)\right)^n$ $=\left(2\cos\theta\right)^n\left(\cos\theta+i\sin\theta\right)^n$ $=2^n\cos^n\theta\left(\cos n\theta+i\sin n\theta\right)$	Award 1 mark for use of trig. Identities Award 1 mark for factorisation Award 1 mark for use of de Moivre's Theorem
E6 (c)(i)	$f(x)=\dfrac{x^2-x}{x-2}=\dfrac{x^2-x-2+2}{x-2}=\dfrac{(x-2)(x+1)+2}{x-2}$ $=x+1+\dfrac{2}{x-2}$	Award 1 mark for correct solution
E6 (c)(ii)	$f'(x)=1-\dfrac{2}{(x-2)^2}$ S.P's are where $f'(x)=0$ $\therefore (x-2)^2=2$ $\therefore x=2\pm\sqrt{2}$ $\therefore$ There are two stationary points	Award 1 mark for correct derivative Award 1 mark for correctly identifying the number of SP's
E6 (c)(iii)	 	Award 3 marks for correct sketch showing both SP's, asymptotes and x intercepts Award 2 marks for sketch showing only 2 of the required features Award 1 mark for sketch showing only 1 feature

	The x intercepts are (0, 0) and (1, 0) The asymptotes are For $x \to \pm\infty \quad y = x+1$ For $x \to 2^+ \quad y = \dfrac{2}{x-2}$ For $x \to 2^- \quad y = \dfrac{2}{x-2}$ $x = 2$ is a vertical asymptote	
E6 (c)(iv)	From the graph $\dfrac{x^2 - x}{x-2} > 0$ when $0 < x < 1$ and $x > 2$	Award 1 mark for correct solution
E6 (d)	 $\lvert x\rvert + \lvert x+2\rvert > 4$ when $-3 < x < 1$	Award 2 marks for correct sketch Award 1 mark for a sketch that is substantially correct Award 1 mark for correct solution