# HURLSTONE AGRICULTURAL HIGH SCHOOL

**MATHEMATICS – EXTENSION TWO**

# 2008 HSC

ASSESSMENT TASK 1

**Examiners ~ G Rawson, J Dillon**

**GENERAL INSTRUCTIONS**

- Reading Time – 3 minutes.
- Working Time – 40 MINUTES.
- Attempt **all** questions.
- **All** necessary working should be shown in every question.
- This paper contains two (2) questions.
- Marks may not be awarded for careless or badly arranged work.
- Board approved calculators may be used.
- **Each question is to be started on a new piece of paper.**
- This examination paper must **NOT** be removed from the examination room.

STUDENT NAME: ______________________________

TEACHER: ______________________________

**QUESTION ONE** **18 marks** *Start a SEPARATE sheet*

(a) Evaluate $i^{2007}$ **1**

(b) Simplify $\dfrac{14+3i}{i(4-5i)}$ **2**

(c) Let $z$ be a non-zero complex number.
If $z^2 = (\overline{z})^2$, show that $z$ is either real or purely imaginary. **2**

(d) Find the complex square roots of $15 + 8i$,
giving your answer in the form $a + ib$, where $a$, $b$ are real. **3**

(e) Let $z = \dfrac{\sqrt{2}}{2} - \dfrac{\sqrt{2}}{2}i$. Find

(i) $|z|$ **1**

(ii) Arg $(z)$ **1**

(f) Simplify $\dfrac{\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)}{\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)}$ **3**

(g) Let $z = 1 + \cos\theta + i\sin\theta$ for $0 < \theta < \pi$.

(i) Show that $z = 2\cos\dfrac{\theta}{2}\left(\cos\dfrac{\theta}{2} + i\sin\dfrac{\theta}{2}\right)$ **2**

(ii) Find $|z|$ and arg $(z)$ in terms of $\theta$. **2**

(h) It is given that $2+i$ is a root of $P(z) = z^3 + rz^2 + sz + k$.

Under what conditions would $2 - i$ also be a root of $P(z)$? **1**

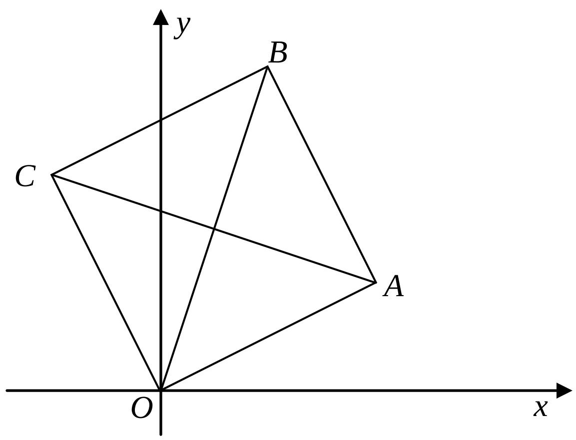
**QUESTION TWO** **18 marks** *Start a SEPARATE sheet*

(a) (i) Find the equation of the locus of the point representing the complex number $z$ which satisfies the equation $|z-2+i|=2$. **2**

(ii) Describe this locus geometrically. **1**

(b) Find the locus of the point representing the complex number $z$ which satisfies the equation $|z-2+i|=|z+1+3i|$. **2**

(c) The points $A$, $B$ and $C$ represent the complex numbers $z_1$, $z_2$ and $z_3$ respectively




Suppose $OABC$ is a square, and $z_1 = a + ib$,

(i) Which vectors would represent $z_1 + z_3$ and $z_3 - z_1$? **2**

(ii) Express $z_3$ and $z_2$ in terms of $a$ and $b$. **2**

(ii) Find $\dfrac{z_3 - z_1}{z_2}$ **2**

(d) Sketch the region in the complex plane where the inequalities

$$|z-1-i| < 2 \text{ and } 0 < \arg(z-1-i) < \frac{\pi}{4}$$

hold simultaneously. **3**

(e) Given $z = x + iy$, sketch the locus of $z$ if $\arg\left(\dfrac{z-1}{z+1}\right) = \dfrac{\pi}{4}$.

Show all important features . **4**

| Year 12 | Mathematics Extension 2 | Ass Task 1 2008 HSC |
|---|---|---|
| Question No. 1 | Solutions and Marking Guidelines | |

**Outcomes Addressed in this Question**

E2 chooses appropriate strategies to construct arguments an proofs in both concrete and abstract settings

E3 uses the relationship between algebraic and geometric representations of complex numbers

| Outcome | Solutions | Marking Guidelines |
|---|---|---|
| **(a)** | $i^{2007} = \left(i^3\right)\left(i^{2004}\right) = (-i)(1) = -i$ | <u>1 mark</u> : correct answer |
| **(b)** | $\frac{14+3i}{i(4-5i)} = \frac{14+3i}{5+4i} \times \frac{5-4i}{5-4i}$<br>$= \frac{70-56i+15i+12}{25+16}$<br>$= \frac{82-41i}{41}$<br>$= 2-i$ | <u>2 marks</u> :<br><u>1 mark</u> : |
| **(c)** | Let $z = x + iy$<br>Then $z^2 = \bar{z}^2$<br>$(x+iy)^2 = (x-iy)^2$<br>$x^2 - y^2 + 2xyi = x^2 - y^2 - 2xyi$<br>$xy = -xy$<br>$xy = 0$<br>$\Rightarrow x = 0$ or $y = 0$<br>$\therefore z$ is real or imaginary | <u>2 marks</u> :<br><u>1 mark</u> : |
| **(d)** | Let $x + iy = \sqrt{15+8i}$<br>$(x+iy)^2 = 15+8i$<br>$x^2 - y^2 + 2xyi = 15+8i$<br>$x^2 - y^2 = 15$ ...(i)<br>$2xy = 8$ ...(ii)<br>from (ii) $y = \frac{4}{x}$ ...(iii)<br>sub (iii) $\rightarrow$ (i), we have<br>$x^2 - \left(\frac{4}{x}\right)^2 = 15$<br>$x^4 - 15x^2 - 16 = 0$<br>$\left(x^2+1\right)\left(x^2-16\right) = 0$<br>$x = \pm 4$ or $\pm i$ but $x, y \in \mathbb{R}$<br>$\therefore x = 4, y = 1$<br>*or* $x = -4, y = -1$<br>ie $\sqrt{15+8i} = 4+i, -4-i$<br>$= \pm(4+i)$ | <u>3 marks</u> :<br><u>2 marks</u> :<br><u>1 mark</u> : |

| | | |
|---|---|---|
| **(e)(i)** | $\lvert z\rvert=\sqrt{\left(\frac{\sqrt{2}}{2}\right)^2+\left(-\frac{\sqrt{2}}{2}\right)^2}$ <br> $=\sqrt{\frac{1}{2}+\frac{1}{2}}$ <br> $=1$ | <u>1 mark</u> : correct solution |
| **(e)(ii)** | $Arg(z)=\tan^{-1}\left(\frac{-\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}\right)$ <br> $=\tan^{-1}(-1)$ <br> $=-\frac{\pi}{4}$ | <u>1 mark</u> : correct solution |
| **(f)** | $\frac{\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)}{\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right)}$ <br> $=\cos\left(\frac{\pi}{3}+\frac{\pi}{4}-\frac{\pi}{12}\right)+i\sin\left(\frac{\pi}{3}+\frac{\pi}{4}-\frac{\pi}{12}\right)$ <br> $=\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}$ <br> $=i$ | <u>3 marks</u> : <br> <u>2 marks</u> : <br> <u>1 mark</u> : |
| **(g)(i)** | $z=1+\cos\theta+i\sin\theta$ <br> $=1+\left(2\cos^2\frac{\theta}{2}-1\right)+i\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)$ <br> $=2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)$ | <u>2 marks</u> : <br> <u>1 mark</u> : |
| **(g)(ii)** | $\lvert z\rvert=2\cos\frac{\theta}{2}\quad and\quad \arg z=\frac{\theta}{2}$ | <u>2 marks</u> : <br> <u>1 mark</u> : |
| **(h)** | Complex roots occur in conjugate pairs when coefficients of $P(z)$ are real. <br> So if 2 + *i* is a root, 2 – *i* will also be a root <u>when *r*, *s* and *k* are all real.</u> | <u>1 mark</u> : |

| Year 12 | Mathematics Extension 2 | Ass Task 1 2008 HSC |
|---|---|---|
| Question No. 2 | Solutions and Marking Guidelines | |

**Outcomes Addressed in this Question**

E2 chooses appropriate strategies to construct arguments an proofs in both concrete and abstract settings

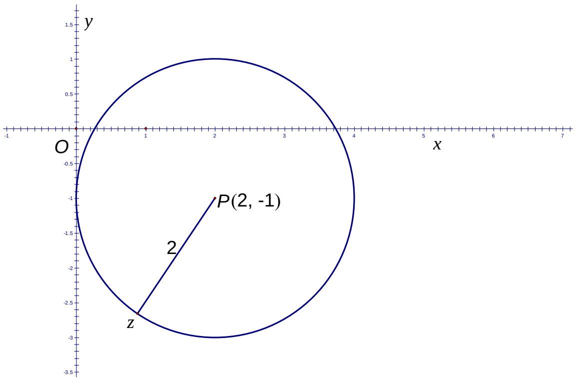
E3 uses the relationship between algebraic and geometric representations of complex numbers

<table>
<tr><th>Outcome</th><th>Solutions</th><th>Marking Guidelines</th></tr>
<tr><td>(a)(i)</td><td>

Let $z = x + iy$

Then
$$|z-2+i|=2$$
$$|(x+iy)-2+i|=2$$
$$|(x-2)+i(y+1)|=2$$
$$\sqrt{(x-2)^2+(y+1)^2}=2$$
$$(x-2)^2+(y+1)^2=4$$

Can be done by inspection (graphically)

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<u>2 marks</u> :

<u>1 mark</u> :

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<tr><td>(a)(ii)</td><td>

The locus is a circle [centre (2, -1), radius 2]

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<u>1 mark</u> :

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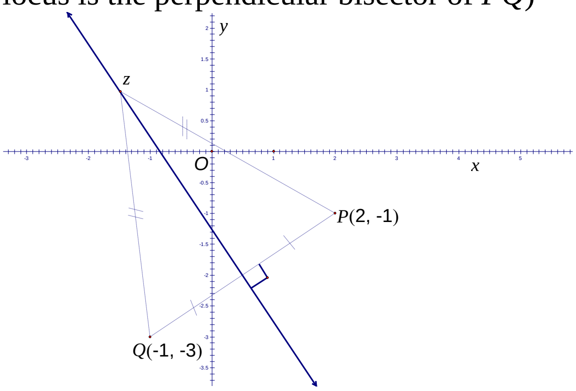
<tr><td>(b)</td><td>

Let $z = x + iy$

Then
$$|z-2+i|=|z+1+3i|$$
$$|x+iy-2+i|=|x+iy+1+3i|$$
$$|(x-2)+i(y+1)|=|(x+1)+i(y+3)|$$
$$(x-2)^2+(y+1)^2=(x+1)^2+(y+3)^2$$
$$x^2-4x+4+y^2+2y+1=x^2+2x+1+y^2+6y+9$$
$$6x+4y+5=0$$

(Again, this is can be done graphically by noting the locus is the perpendicular bisector of $PQ$)

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<u>2 marks</u> :

<u>1 mark</u> :

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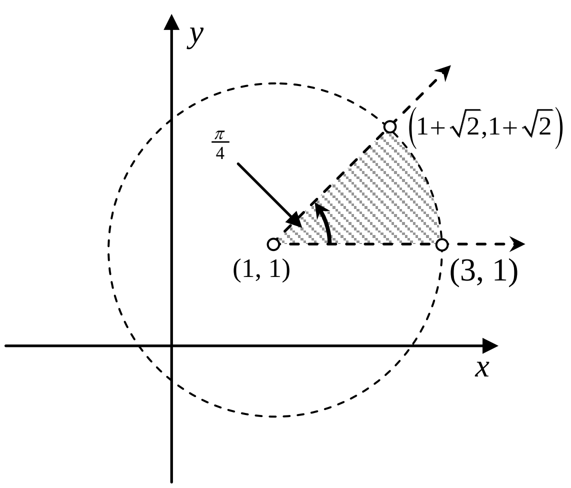
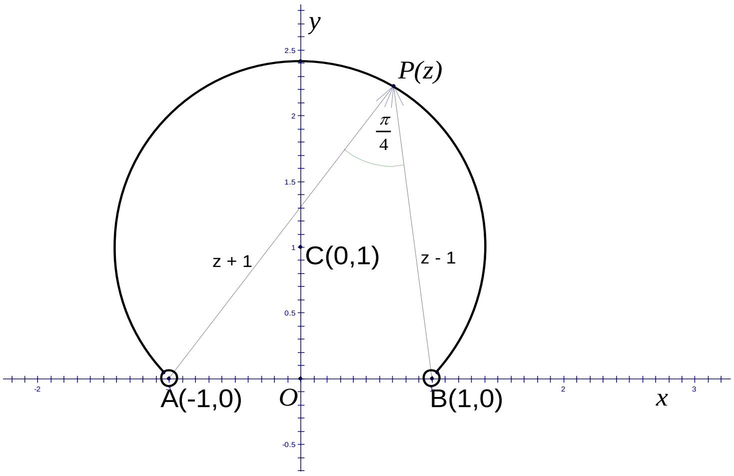
|---|---|---|
| **(c)(i)** | $z_1 + z_3$ is represented by $\overrightarrow{OB}$<br>$z_3 - z_1$ is represented by $\overrightarrow{AC}$ | <u>2 marks</u> :<br><u>1 mark</u> : |
| **(c)(ii)** | $\overrightarrow{OC}$ is $\overrightarrow{OA}$ rotated anticlockwise throught $90°$<br>so $z_3 = i.z_1$<br>$= i(a+ib)$<br>$= -b+ia$<br><br>Also, $z_2 = z_3 + z_1$<br>$= (-b+ai)+(a+bi)$<br>$= (a-b)+(a+b)i$ | |
| **(c)(iii)** | $\frac{z_3 - z_1}{z_2} = \frac{(-b+ai)-(a+bi)}{(a-b)+(a+b)i}$<br>$= \frac{(-a-b)+(a-b)i}{(a-b)+(a+b)i}$<br>$= \frac{[-(a+b)+(a-b)i]}{[(a-b)+(a+b)i]} \times \frac{[(a-b)-(a+b)i]}{[(a-b)-(a+b)i]}$<br>$= \frac{[-(a^2-b^2)+(a^2-b^2)]+[(a-b)^2+(a+b)^2]i}{(a-b)^2+(a+b)^2}$<br>$= i$<br><br>Alternatively…(better method)<br>$OB = AC$ and $OB \perp AC$ (diagonals of square)<br>so, $\overrightarrow{AC}$ is obtained by rotating $\overrightarrow{OB}$ anticlockwise through $90°$, and then translating (no change in length).<br>ie, $\overrightarrow{AC} = i.\overrightarrow{OB}$<br>$z_3 - z_1 = i.z_2$<br>$\therefore \frac{z_3 - z_1}{z_2} = i$ | <u>2 marks</u> :<br><u>1 mark</u> : |
| **(d)** |  | <u>3 marks</u> :<br><u>2 marks</u> :<br><u>1 mark</u> : |

| | | |
|---|---|---|
| **(e)** | $$\arg\left(\frac{z-1}{z+1}\right)=\frac{\pi}{4}$$ $$\arg(z-1)-\arg(z+1)=\frac{\pi}{4}$$ $$\arg(z+1)=\arg(z-1)-\frac{\pi}{4}$$   Algebraically: $$\frac{z-1}{z+1}=\frac{x+iy-1}{x+iy+1}$$ $$=\frac{[(x-1)+iy]}{[(x+1)+iy]}\cdot\frac{[(x+1)-iy]}{[(x+1)-iy]}$$ $$=\frac{[(x^2-1^2)+y^2]+[y(x+1)-y(x-1)]i}{(x+1)^2+y^2}$$ $$=\frac{[x^2+y^2-1]}{(x+1)^2+y^2}-\frac{2y}{(x+1)^2+y^2}i$$ $$\tan\left[\arg\left(\frac{z-i}{z+i}\right)\right]=\frac{-\frac{2y}{(x+1)^2+y^2}}{\frac{x^2+y^2-1}{(x+1)^2+y^2}}$$ $$=-\frac{2y}{x^2+y^2-1}$$ as $\tan\frac{\pi}{4}=1$ $$-\frac{2y}{x^2+y^2-1}=1$$ so the relation is $x^2+y^2+2y-1=0$<br><br>Note: this method alone will not show that the locus is above the $x$-axis only. | <u>4 marks</u> :<br><br><u>3 marks</u> :<br><br><u>2 marks</u> :<br><br><u>1 mark</u> : |