HURLSTONE AGRICULTURAL HIGH SCHOOL

PRO PATRIA

YEAR 12

MATHEMATICS EXTENSION 2

2008

HSC COURSE

HALF YEARLY EXAMINATION (ASSESSMENT TASK 2)

EXAMINERS ~ G. RAWSON AND J. DILLON

GENERAL INSTRUCTIONS

- Reading Time – 5 minutes.
- Working Time – Two Hours.
- Attempt all questions.
- Questions are of equal value.
- All necessary working should be shown in every question.
- **This paper contains Five (5) questions.**
- **Total Marks – 75 marks**
- Marks may not be awarded for careless or badly arranged work.
- Board approved Calculators and Templates may be used.
- Each question is to be started in a new Examination Booklet.
- This assessment task must **NOT** be removed from the Examination Room.

STUDENT NAME / NUMBER: ______________________________

TEACHER: ______________________________

Question 1: (Use a SEPARATE answer booklet)

Marks

(a) Two complex numbers are given by

$$z = 3 - 4i \text{ and } w = 2 - 2i$$

(i) Find, in the form $x + iy$, the product $\bar{z}\,w$. **1**

(ii) Find, in the form $x + iy$, the two square roots of z. **2**

(iii) (α) Express w in modulus – argument form. **1**

(β) Find, in the form $x + iy$, w^5. **2**

(b) The locus of a point P which moves in the complex plane is represented by the equation

$$\left|z - (3 + 4i)\right| = 5$$

(i) Sketch the locus of the point P **1**

(ii) (α) Find the maximum value of the modulus of z. **1**

(β) Write down the value of $\arg z$ when P is in the position of maximum modulus. **1**

(γ) Find the value of the modulus of z when $\arg z = \tan^{-1}\left(\frac{1}{2}\right)$ **1**

(c) The complex number z is given by $z = 1 - i$. **2**

Find real numbers a and b such that $az = 1 - \frac{b}{z}$.

(d) In an Argand diagram the point P represents the complex number z and the point Q represents the complex number w and the point R represents the complex number $z - w$ with O being the origin.

It is given $z - w = iz$.

(i) Describe the geometric properties of triangle POR, providing full reasoning for your answer. **2**

(ii) Find the size of angle QPR. **1**

QUESTION 2: **(USE A SEPARATE ANSWER BOOKLET)**

MARKS

Consider the function $g(x) = 4\sqrt{x} - 2x$.

(a) Write down the domain of $g(x)$. **1**

(b) Find the x intercepts of the graph of $y = g(x)$. **1**

(c) Show that the curve $y = g(x)$ is concave downwards for all $x > 0$. **1**

(d) Find the coordinates of the stationary point and determine its nature. **1**

(e) Sketch the graph of $y = g(x)$, clearly showing all essential features. **1**

(f) Hence, by consideration of your graph of $y = g(x)$, sketch each of the following on separate diagrams, showing all essential features.

(i) $y = |g(x)|$ **2**

(ii) $y = g(x-2)$ **2**

(iii) $y = g(|x|)$ **2**

(iv) $|y| = g(x)$ **2**

(v) $y = \dfrac{1}{g(x)}$ **2**

<u>QUESTION 3:</u> **(USE A SEPARATE ANSWER BOOKLET)**

MARKS

(a) A polynomial has a remainder of 5 when divided by $(x-3)$ and a remainder of 12 when divided by $(x-4)$. **2**
What is the remainder when the polynomial is divided by $(x-3)(x-4)$?

(b) The polynomial $P(x) = x^3 + 3x - 5$ has roots α, β and γ. **2**
Find the value of $\alpha^3 + \beta^3 + \gamma^3$.

(c) The equation $x^3 + 3x^2 - 2 = 0$ has roots α, β and γ. **3**
Find the cubic equation with roots α^2, β^2 and γ^2.

(d) Given that $x = 2 + i$ is a root of the polynomial $P(x) = x^3 - 2x^2 - 3x + 10$, **3**
factorise $P(x)$ over the complex field, $\mathbb{C}$.

(e) If $P(x) = x^4 + 2x^3 - 12x^2 + 14x - 5$ has a triple zero, find all the zeros of $P(x)$. **2**

(f) Consider the function $P(x) = x^3 + ax + b$. **3**
Show that the function will have three distinct roots if $4a^3 + 27b^2 < 0$.

QUESTION 4: (USE A SEPARATE ANSWER BOOKLET)

MARKS

(a) The ellipse $\mathscr{E}$ has the equation

$$4x^2 + 9y^2 = 36$$

(i) Write down:

(α) its eccentricity **1**

(β) the coordinates of its foci S and S' **1**

(γ) the equation of each directrix **1**

(δ) the length of the major axis. **1**

(ii) Sketch the ellipse $\mathscr{E}$. Show the x and y intercepts as well as the features found in parts (β) and (γ) of part (i) above. **1**

(iii) Without using the formula for the area of an ellipse, show by integration that the area of the ellipse $\mathscr{E}$ is 6π square units. **4**

(b) (i) Show that if $y = px + q$ is a tangent to the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ then $p^2a^2 - b^2 = q^2$ **3**

(ii) Hence find the equations of the tangents from the point (1, 3) to the hyperbola $\dfrac{x^2}{4} - \dfrac{y^2}{15} = 1$. **3**

QUESTION 5: (USE A SEPARATE ANSWER BOOKLET)

MARKS

(a) (i) Use De Moivre's theorem, or otherwise, to show that **2**

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta .$$

(ii) Hence, solve $8x^3 - 6x - 1 = 0$. **2**

(iii) Deduce that $\sin\dfrac{\pi}{18} = \sin\dfrac{7\pi}{18} + \sin\dfrac{11\pi}{18}$. **1**

(b) The polynomial $P(x) = x^3 + qx^2 + rx + s$ has real coefficients. It has three distinct zeros $\alpha, -\alpha$ and β.

(i) Prove that $qr = s$. **3**

(ii) The polynomial does not have three real zeros. Show that two of the the zeros are purely imaginary. (A number is purely imaginary if it is of the form iy, with y real and $y \neq 0$.) **2**

(c) You are given

$$\begin{aligned}\cos 2A &= \cos^2 A - \sin^2 A\\ &= 2\cos^2 A - 1\\ &= 1 - 2\sin^2 A\end{aligned}$$

and

$$\sin 2A = 2\sin A\cos A$$

If n is any integer, prove that:

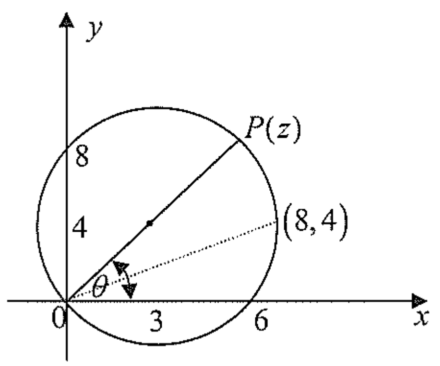
(i) $\left(1+\cos\dfrac{2\pi}{n} + i\sin\dfrac{2\pi}{n}\right)^n = -2^n\cos^n\dfrac{\pi}{n}$. **2**

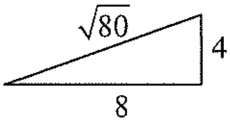
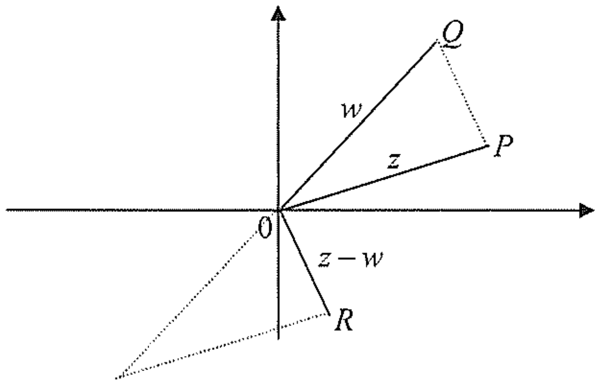
(ii) $\left(1+i\sqrt{3}\right)^{2n} + \left(1-i\sqrt{3}\right)^{2n}$ has either the value 2^{2n+1} or -2^{2n}, according as to whether n is or is not a multiple of 3. **3**

Question 1 Solutions and Marking Guidelines

Outcomes Addressed in this Question

E3 uses the relationship between algebraic and geometric representations of complex numbers and of conic sections

Part	Solutions	Marking Guidelines
(a) (i)	$\bar{z}w = (3+4i)(2-2i)$ $= 14+2i$	**Award 1** correct solution
(ii)	Let $\sqrt{3-4i} = a+ib$ $\therefore 3-4i = a^2-b^2+2abi$ $\therefore\ a^2-b^2=3$ .... (1) $2ab=-4 \Rightarrow ab=-2$ .... (2) Since $(a^2+b^2)^2 = (a^2-b^2)^2+4a^2b^2$ $=9+16$ $=25$ Then $a^2+b^2=5$ .... (3) $(1)+(3) \Rightarrow 2a^2=8$ $\therefore a=\pm 2,\ b=\mp 1$ The two square roots are $2-i,\ -2+i$.	**Award 2** correct solution **Award 1** establish initial equations only
(iii)(α)	$2-2i = 2\sqrt{2}\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i\right) = 2\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right)$	**Award 1** correct solution
(β)	$w^5 = (2\sqrt{2})^5\left(\cos\left(-\frac{5\pi}{4}\right)+i\sin\left(-\frac{5\pi}{4}\right)\right)$ $=128\sqrt{2}\left(-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)$ $=-128+128i$	**Award 2** correct solution **Award 1** attempts to apply De Moivres Theorem
(b) (i)		**Award 1** correct sketch
(ii) (α)	Maximum value of $\lvert z\rvert$ is 10 (i.e. a diameter)	**Award 1** correct solution
(β)	When P is in maximum modulus $\arg z = \theta = \tan^{-1}\frac{8}{6} = \tan^{-1}\frac{4}{3}$	**Award 1** correct solution

(γ)	 When $\arg z = \tan^{-1}\frac{1}{2}$, $\lvert z\rvert = \sqrt{80} = 4\sqrt{5}$	**Award 1** correct solution
(c)	$az = 1 - \frac{b}{z}$ $\therefore\ a(1-i) = 1 - \frac{b}{1-i}$ $= 1 - \frac{b(1+i)}{2}$ $= \frac{2-b-bi}{2}$ $\therefore\ a - ai = 1 - \frac{b}{2} - \frac{b}{2}i$ $\therefore\ a = 1 - \frac{b}{2}$ and $a = \frac{b}{2}$. Solving simultaneously gives $a = \frac{1}{2}$, $b = 1$.	**Award 2** correct solution **Award 1** substitution and attempt to realise the denominator
(d) (i)	 If $z - w = iz$, then we need to rotate P about O through $\frac{\pi}{2}$. $\therefore \angle POR = \frac{\pi}{2}$ $\lvert z-w\rvert = \lvert iz\rvert = \lvert z\rvert$ $\therefore OR = OP$ $\therefore \Delta OPR$ is right-angled isosceles.	**Award 2** correct solution ~ with full reasoning **Award 1** correct approach ~ with incorrect of insufficient reasoning
(ii)	$\angle OPR = \frac{\pi}{4}$ and $\angle OPQ = \frac{\pi}{2}$ $\therefore \angle QPR = \frac{3\pi}{4}$	**Award 1** correct solution

Question 3 Solutions and Marking Guidelines

Outcomes Addressed in this Question

E4 uses efficient techniques for the algebraic manipulation required in dealing with questions such as those involving conic sections and polynomials

Part	Solutions	Marking Guidelines
(a)	Let $P(x)=(x-3)(x-4)Q(x)+ax+b$ $P(3)=5=3a+b \dots (1)$ $P(4)=12=4a+b \dots (2)$ $(2)-(1) \quad a=7$ and $b=-16$. $\therefore$ Remainder $=7x-16$.	**Award 2** correct solution **Award 1** uses remainder theorem
(b)	Let $x=\alpha,\ \alpha^3+3\alpha-5=0$ Let $x=\beta,\ \beta^3+3\beta-5=0$ Let $x=\gamma,\ \gamma^3+3\gamma-5=0$ $\therefore \alpha^3+\beta^3+\gamma^3+3(\alpha+\beta+\gamma)-15=0$ $\therefore \alpha^3+\beta^3+\gamma^3+3.0-15=0$ $\therefore \alpha^3+\beta^3+\gamma^3=15.$	**Award 2** correct solution **Award 1** substantial progress towards solution
(c)	Let $y=x^2 \Rightarrow x=\sqrt{y}$ The desired equation is given by $\left(\sqrt{y}\right)^3+3\left(\sqrt{y}\right)^2-2=0$ $y\sqrt{y}+3y-2=0$ $y\sqrt{y}=2-3y$ $y^3=4-12y+9y^2$ $\therefore y^3-9y^2+12y-4=0$	**Award 3** correct solution **Award 2** substantial progress towards solution **Award 1** establishes required transformation
(d)	If $x=2+i$ is a root, then $x=2-i$ is also a root. $\therefore \left(x-(2+i)\right)\left(x-(2-i)\right)$ $=\left((x-2)-i\right)\left((x-2)+i\right)$ $=x^2-4x+5$ is a factor. $\therefore x^3-2x^2-3x+10=\left(x^2-4x+5\right)(x+2)$ by inspection. $\therefore x^3-2x^2-3x+10=(x-2+i)(x-2-i)(x+2)$ over □ .	**Award 3** correct solution **Award 2** determines quadratic factor **Award 1** uses conjugate root theorem
(e)	$P(x)=x^4+2x^3-12x^2+14x-5$ $P'(x)=4x^3+6x^2-24x+14$ $P''(x)=12x^2+12x-24=12\left(x^2+x-2\right)=12(x+2)(x-1)$	**Award 2** correct solution **Award 1** uses multiple root theorem ~ does not determine other root

	By observation $P''(1)=P'(1)=P(1)=0$ $\therefore\ x=1$ is the triple root. $\therefore\ 1+1+1+\alpha=-2$ $\therefore\ \alpha=-5.$ $\therefore$ Roots are 1, 1, 1, -5.	
(f)	$P(x)=x^3+ax+b$ $P'(x)=3x^2+a$ Stationary points occur @ $P'(x)=0$ $\therefore\ x=\pm\sqrt{-\frac{a}{3}}$ $P\left(\sqrt{-\frac{a}{3}}\right)=b+\frac{2}{3}a\sqrt{-\frac{a}{3}}$ .... (y_1) $P\left(-\sqrt{-\frac{a}{3}}\right)=b-\frac{2}{3}a\sqrt{-\frac{a}{3}}$ .... (y_2) 3 distinct roots $\Rightarrow y_1y_2<0$ $\left(b+\frac{2}{3}a\sqrt{-\frac{a}{3}}\right)\left(b-\frac{2}{3}a\sqrt{-\frac{a}{3}}\right)<0$ $b^2-\frac{4}{9}a^2.-\frac{a}{3}<0$ $b^2+\frac{4a^3}{27}<0$ $27b^2+4a^2<0$	**Award 3** correct solution **Award 2** substantial progress towards solution **Award 1** establishes the x coordinates of the SP only

Outcomes Addressed in this Question

E2 chooses appropriate strategies to construct arguments and proofs in both concrete and abstract settings

E3 uses the relationship between algebraic and geometric representations of complex numbers and of conic sections

E4 uses efficient techniques for the algebraic manipulation required in dealing with questions such as those involving conic sections and polynomials

Part	Solutions	Marking Guidelines
(a) (i) E2 E3	$(\cos\theta + i\sin\theta)^3$ $= \cos^3\theta + 3i\cos^2\theta\sin\theta + 3i^2\cos\theta\sin^2\theta + i^3\sin^3\theta$ $= \cos^3\theta - 3\cos\theta\sin^2\theta + i\left(3\cos^2\theta\sin\theta - \sin^3\theta\right)$ $= \cos 3\theta + i\sin 3\theta$ Equating imaginary parts, $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$ $= 3\left(1-\sin^2\theta\right)\sin\theta - \sin^3\theta$ $= 3\sin\theta - 4\sin^3\theta$	**Award 2** correct solution **Award 1** attempt to expand or to use De Moivres Theorem.
(ii) E2	Let $x = \sin\theta$ and $\sin 3\theta = -\frac{1}{2}$. $-\frac{1}{2} = 3x - 4x^3$ $\therefore\ 8x^3 - 6x - 1 = 0$ Solutions of this equation are the solutions of $\sin 3\theta = -\frac{1}{2}$ $\therefore\ 3\theta = \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}$ $\therefore\ \theta = \frac{7\pi}{18}, \frac{11\pi}{18}, \frac{19\pi}{18}$ $\therefore\ x = \sin\frac{7\pi}{18}, \sin\frac{11\pi}{18}, \sin\frac{19\pi}{18}$ $= \sin\frac{7\pi}{18}, \sin\frac{11\pi}{18}, \sin\left(-\frac{\pi}{18}\right)$ $= \sin\frac{7\pi}{18}, \sin\frac{11\pi}{18}, -\sin\frac{\pi}{18}$	**Award 2** correct solution **Award 1** establishes appropriate relationships
(iii) E4	$\therefore\ \sin\frac{7\pi}{18} + \sin\frac{11\pi}{18} + \left(-\sin\frac{\pi}{18}\right) = -\frac{b}{a} = 0$ $\therefore\ \sin\frac{7\pi}{18} + \sin\frac{11\pi}{18} = \sin\frac{\pi}{18}$	**Award 1** correct solution

(b) (i) E4	$\therefore\ \alpha+(-\alpha)+\beta=-q$ $\therefore\ \beta=-q$ $\therefore\ P(-q)=(-q)^3+q(-q)^2+r(-q)+s$ $=-q^3+q^3-qr+s$ $=0$ $\therefore\ s-qr=0$ $\therefore\ s=qr$	**Award 3** correct solution **Award 2** find an expression for two of q, r and s in terms of α and β. **Award 1** find an expression for one of q, r and s in terms of α and β.
(ii) E2 E4	Since $s=qr$ $P(x)=x^3+qx^2+rx+qr$ $=(x+q)(x^2+r)$ So the roots of $P(x)$ are $-q$ and the roots of $x^2+r=0$. If $r=-\gamma$ then the roots are $-q$ and $\pm\sqrt{\gamma}$ $\Rightarrow$ all real $\therefore$ contradicts initial assumption. If $r=0$ then the roots are $-q$ and 0 $\Rightarrow$ all real $\therefore$ contradicts initial assumption. Hence, $r>0$ and two of the roots of $P(x)$ are the purely imaginary numbers $\pm i\sqrt{r}$.	**Award 2** correct solution **Award 1** observe that two of the roots are complex conjugates
(c) (i) E2 E3	$\left(1+\cos\frac{2\pi}{n}+i\sin\frac{2\pi}{n}\right)^n$ $=\left(1+2\cos^2\frac{\pi}{n}-1+2i\sin\frac{\pi}{n}\cos\frac{\pi}{n}\right)^n$ $=\left(2\cos^2\frac{\pi}{n}+2i\sin\frac{\pi}{n}\cos\frac{\pi}{n}\right)^n$ $=\left(2\cos\frac{\pi}{n}\left(\cos\frac{\pi}{n}+i\sin\frac{\pi}{n}\right)\right)^n$ $=2^n\cos^n\frac{\pi}{n}(\cos\pi+i\sin\pi)$ $=-2^n\cos^n\frac{\pi}{n}$	**Award 2** correct solution **Award 1** attempts to use given trigonometric relationships
(ii) E2 E3	$\left(1+i\sqrt{3}\right)^{2n}+\left(1-i\sqrt{3}\right)^{2n}$ $=\left(2\text{cis}\frac{\pi}{3}\right)^{2n}+\left(2\text{cis}\left(-\frac{\pi}{3}\right)\right)^{2n}$ $=2^{2n}\text{cis}\frac{2n\pi}{3}+2^{2n}\text{cis}\left(-\frac{2n\pi}{3}\right)$ $=2^{2n}\left(\cos\frac{2n\pi}{3}+i\sin\frac{2n\pi}{3}+\cos\frac{2n\pi}{3}-i\sin\frac{2n\pi}{3}\right)$ $=2^{2n}\times2\cos\frac{2n\pi}{3}$	**Award 3** correct solution **Award 2** substantial progress towards solution **Award 1** limited progress towards solution

If n is a multiple of 3, $n = 3k$ where $k \in \mathbb{Z}$

$$\begin{aligned}
\therefore\ 2^{2n+1}\cos\frac{2n\pi}{3} &= 2^{2n+1}\cos\frac{6k\pi}{3} \\
&= 2^{2n+1}\cos 2k\pi \\
&= 2^{2n+1}.1 \\
&= 2^{2n+1}
\end{aligned}$$

If n is not a multiple of 3, $n = 3k-1$ or $n = 3k+1$ where $k \in \mathbb{Z}$

$$\begin{aligned}
\therefore\ 2^{2n+1}\cos\frac{2n\pi}{3} &= 2^{2n+1}\cos\frac{2(3k-1)\pi}{3} \\
&= 2^{2n+1}\cos\left(2k\pi - \frac{2\pi}{3}\right) \\
&= 2^{2n+1}.-\frac{1}{2} \\
&= -2^{2n}
\end{aligned}$$

or

$$\begin{aligned}
\therefore\ 2^{2n+1}\cos\frac{2n\pi}{3} &= 2^{2n+1}\cos\frac{2(3k+1)\pi}{3} \\
&= 2^{2n+1}\cos\left(2k\pi + \frac{2\pi}{3}\right) \\
&= 2^{2n+1}.-\frac{1}{2} \\
&= -2^{2n}
\end{aligned}$$