HURLSTONE AGRICULTURAL HIGH SCHOOL

HSC COURSE

ASSESSMENT TASK 2

Mathematics Extension 2

Half Yearly Examination

2009

Examiners ~G. Rawson, G. Huxley

GENERAL INSTRUCTIONS

- Reading time – 5 minutes.
- Working time – 90 minutes.
- This test has 4 questions. Attempt all questions.
- Each question is worth 20 marks. Total: 80 marks.
- All necessary working should be shown in each question. Marks may not be awarded for careless or badly arranged work.
- **Start each question on a new answer booklet.** Write your name on every sheet.
- This test must **NOT** be removed from the examination venue.

QUESTION 1 (20 marks) Start a new Answer Booklet.

Marks

(a) If $z_1 = 3 + 2i$ and $z_2 = 4 - i$, find

(i) $z_1 \overline{z_1}$ **1**

(ii) $\dfrac{z_1}{z_2}$ **1**

(b) (i) Express $1 + \sqrt{3}\,i$ in mod-arg form. **1**

(ii) Hence or otherwise, find a and b such that $\left(1 + \sqrt{3}\,i\right)^5 = a + bi$. **2**

(c) Find $\sqrt{-15 - 8i}$ **3**

(d) (i) On the same Argand diagram sketch the following:

z_1 , such that $\arg z_1 = \dfrac{\pi}{4}$

$z_2 = iz_1$

$z_3 = -z_1$ **1**

(ii) Hence or otherwise, find $\arg\left(z_2 + z_3\right)$ **1**

(e) (i) Show all the roots of $z^5 + 1 = 0$ on a unit circle in an Argand diagram. **2**

(ii) Factorising the equation above into irreducible factors with real coefficients gives the solution $z^5 + 1 = (z+1)\left(z^2 - 2z\cos\dfrac{\pi}{5} + 1\right)\left(z^2 - 2z\cos\dfrac{3\pi}{5} + 1\right)$.

Deduce that $\cos\dfrac{\pi}{5} + \cos\dfrac{3\pi}{5} = \dfrac{1}{2}$ and $\cos\dfrac{\pi}{5}\cos\dfrac{3\pi}{5} = -\dfrac{1}{4}$ **2**

(iii) Write a quadratic equation with integer coefficients which has roots $\cos\dfrac{\pi}{5}$ and $\cos\dfrac{3\pi}{5}$.

Hence find $\cos\dfrac{\pi}{5}$ and $\cos\dfrac{3\pi}{5}$ as surds. **3**

(f) Sketch the following region on an Argand diagram, where the following conditions hold simultaneously for z:

$|z| \le 2$

$0 \le \arg z \le \dfrac{\pi}{2}$

$\mathrm{Re}(z) \le 1$ **3**

QUESTION 2 (20 marks) Start a new Answer Booklet.

Marks

(a) (i) Find all the zeros of $P(x) = x^4 - 5x^2 + 6$ over the rational numbers. **2**

(ii) Find $P(x)$, given that $P(x)$ is monic, of degree 4, with -1 as a single zero and 3 as a zero of multiplicity 3. **1**

(b) The equation $x^3 + kx + 2 = 0$ has roots α, β and γ.

(i) Write down the values, in terms of k, of $\Sigma\alpha$, $\Sigma\alpha\beta$ and $\Sigma\alpha\beta\gamma$. **1**

(ii) Find the value of $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}$. **2**

(iii) Find the monic equation with roots $\alpha^2, \beta^2, \gamma^2$. **2**

(iv) Show that the value of $\alpha^3+\beta^3+\gamma^3$ is independent of k. **2**

(c) Let $x = \alpha$ be a root of the quartic polynomial $P(x) = x^4 + Ax^3 + Bx^2 + Ax + 1$ where $(2+B)^2 \neq 4A^2$.

(i) Show that α cannot be 0, 1, -1. **2**

(ii) Show that $x = \frac{1}{\alpha}$ is a root. **2**

(iii) Deduce that if α is a multiple root, then its multiplicity is 2 and $4B = 8 + A^2$. **3**

(d) Find real numbers A, B and C such that $\frac{2}{(t+1)(t^2+1)} = \frac{A}{t+1} + \frac{Bt+C}{t^2+1}$ **3**

QUESTION 3 (20 marks) Start a new Answer Booklet.

Marks

(a) Consider the function $f(x)=\dfrac{x-1}{x+3}$

(i) Sketch the graph of $y = f(x)$ showing clearly the coordinates of any points of intersection with the x axis or the y axis, and the equations of any asymptotes. **3**

(ii) Show that the line $y = x$ is a tangent to the curve $y = f(x)$ and find the coordinates of its point of contact. Draw the tangent line on the graph and show its point of contact. **3**

(iii) On separate axes, sketch the graphs of $y=\dfrac{1}{f(x)}$ and $y=f^{-1}(x)$.

In each case, show clearly the coordinates of any points of intersection with the x axis or the y axis, the equations of any asymptotes, and the line $y = x$. **6**

(b) (i) The x coordinates of the stationary points on the graph $y=\{f(x)\}^2$ are the same as the x coordinates of the stationary points or intercepts on the x axis of the graph $y = f(x)$.

Show that this is true by solving $\dfrac{dy}{dx}=0$ when $y=\{f(x)\}^2$. **2**

(ii) Hence or otherwise, on the same axes, draw the graphs of $y = f(x)$ and $y=\{f(x)\}^2$

if $f(x)=\sin^2 x-\dfrac{1}{2}, \qquad 0\le x\le\pi$ **3**

(c) Find the gradient of the tangent to the curve $3xy-y^3=1-2x$ at the point (0, -1) on the curve. **3**

QUESTION 4 (20 marks) Start a new Answer Booklet.

Marks

(a) (i) If $x = \alpha$ is a double root of the polynomial equation $P(x) = 0$, show that $x = \alpha$ is a root of the equation $P'(x) = 0$. **2**

(ii) If the equation $x^3 - px + q = 0$ has a double root, show that $4p^3 - 27q^2 = 0$. **3**

(iii) Find the equation of the line through the origin which is a tangent to the curve $y = x^3 + 1$. **3**

(b) Suppose that $a > 0$, $b > 0$, $c > 0$.

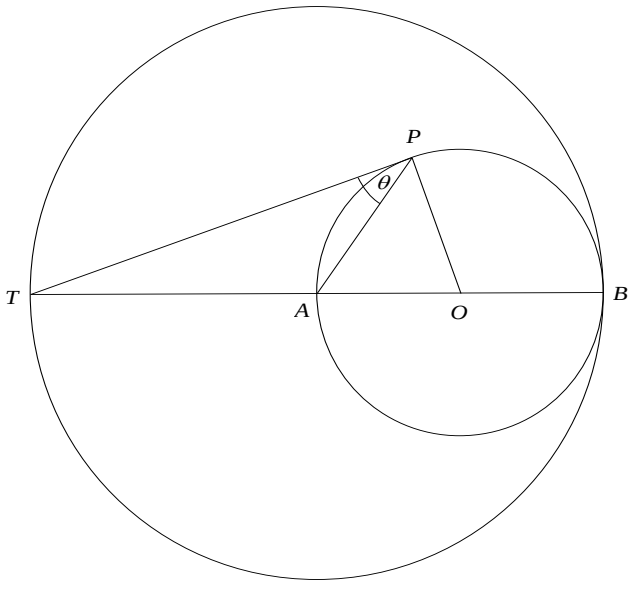
(i) Prove that $a^2 + b^2 \geq 2ab$. **1**

(ii) Hence prove that $a^2 + b^2 + c^2 \geq ab + bc + ca$. **2**

(iii) Given $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$, prove that $a^3 + b^3 + c^3 \geq 3abc$. **2**

(iv) Assume $x > 0, y > 0, z > 0$. Show that $x + y + z \geq 3(xyz)^{\frac{1}{3}}$. **2**

(c) Two circles touch at *B*.
A is the centre of the larger circle, and *O* is the centre of the smaller circle.
TAOB is a diameter of the larger circle, and *AOB* is a diameter of the smaller circle.
PT is a tangent to the smaller circle from *T*.



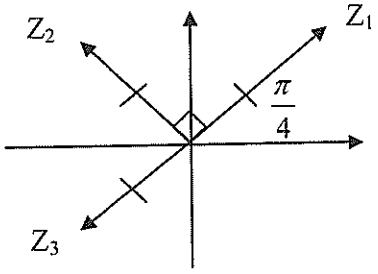
(i) If $\angle TPA = \theta$, show that $\cos\theta = \frac{\sqrt{6}}{3}$. **3**

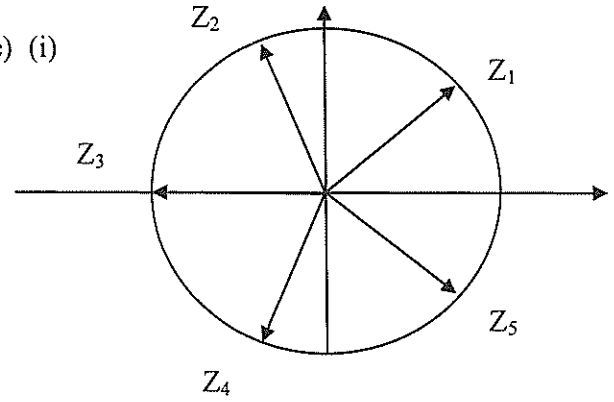
(ii) Show that $PB = \frac{\sqrt{6}AB}{3}$. **2**

Year 12	Mathematics Extension 2	Half Yearly Examination 2009
Question No. 1	Solutions and Marking Guidelines	

Outcomes Addressed in this Question

E3 uses the relationship between algebraic and geometric representations of complex numbers and of conic sections.

Outcome	Solutions	Marking Guidelines
E3	(a) (i) $z_1\overline{z_1} = (3+2i)(3-2i)$ $= 13$	**2 marks:** correct solution **1 mark:** partially correct
	(ii) $\frac{z_1}{z_2} = \frac{3+2i}{4-i} \times \frac{4+i}{4+i}$ $= \frac{10+11i}{17}$	**1 mark:** correct solution
E3	(b) (i) $1+\sqrt{3}\,i = 2cis\frac{\pi}{3}$	**1 mark:** correct solution (completely)
	(ii) $\left(1+\sqrt{3}\,i\right)^5 = 2^5\left(\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}\right)$ $= 32\left(\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right)$ $\therefore a = 16, b = 16\sqrt{3}$	**2 marks:** correct solution **1 mark:** partially correct
	(c) $x+iy = \sqrt{-15-8i}$ $\therefore x^2 - y^2 = -15$ $2xy = -8$ $x+iy = \pm(1-4i)$	**2 marks:** correct solution **1 mark:** partially correct
	(d) (i) 	**2 marks:** correct solution **1 mark:** partially correct
	(ii) $\arg(z_1 + z_2) = \pi$	

E3	(e) (i)  	**2 marks:** correct solution **1 mark:** substantial progress towards correct solution **2 marks:** correct solution **1 mark:** substantial progress towards correct solution **3 marks:** correct solution **2 marks:** substantial progress towards correct solution **1 mark:** partial progress towards correct solution

E3+		**3 marks:** correct solution **2 marks:** substantial progress towards correct solution **1 mark:** partial progress towards correct solution

Year 12	Mathematics Extension 2	Half Yearly Examination 2009
Question No. 2	Solutions and Marking Guidelines	

Outcomes Addressed in this Question

E4 uses efficient techniques for the algebraic manipulation required in dealing with questions such as those involving polynomials

Outcome	Solutions	Marking Guidelines
E4	(a) (i) $x^4-5x^2+6=\left(x^2-2\right)\left(x^2-3\right)$ $x=\pm\sqrt{2},\pm\sqrt{3}$ $\therefore$ there are no rational zeros	**2 marks:** correct solution **1 mark:** partially correct
	(ii) $P(x)=(x-3)^3(x+1)$ $=x^4-8x^3+18x^2-27$	**1 mark:** correct solution
E4	(b) (i) $\Sigma\alpha=-\frac{b}{a}=0$ $\Sigma\alpha\beta=\frac{c}{a}=k$ $\Sigma\alpha\beta\gamma=-\frac{d}{a}=-2$	**1 mark:** correct solution (completely)
	(ii) $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{\Sigma\alpha\beta}{\Sigma\alpha\beta\gamma}$ $=-\frac{k}{2}$	**2 marks:** correct solution **1 mark:** partially correct
	(iii) let $x=\alpha^2\Rightarrow\alpha=\sqrt{x}$ so the equation becomes $\left(\sqrt{x}\right)^3+k\left(\sqrt{x}\right)+2=0$ $x\sqrt{x}+k\sqrt{x}=-2$ $(x+k)^2x=4$ $x^3+2kx^2+k^2x-4=0$	**2 marks:** correct solution **1 mark:** partially correct
	(iv) $\alpha^3+k\alpha+2=0$ $\beta^3+k\beta+2=0$ $\gamma^3+k\gamma+2=0$ so $\alpha^3+\beta^3+\gamma^3+k(\alpha+\beta+\gamma)+6=0$ $\alpha^3+\beta^3+\gamma^3+k(0)+6=0$ $\alpha^3+\beta^3+\gamma^3=-6$	**2 marks:** correct solution **1 mark:** partially correct

E4

(c)(i)

$P(x) = x^4 + Ax^3 + Bx^2 + Ax + 1$

$P(0) = 1 \qquad \therefore 0$ is not a root

$P(1) = 1 + A + B + A + 1$

$0 = 2A + B + 2$ if 1 is a root

$2 + B = -2A$

$(2+B)^2 = 4A^2$, which can't happen $\therefore 1$ is not a root

$P(-1) = 1 - A + B - A + 1$

$0 = -2A + B + 2$ if -1 is a root

$2 + B = 2A$

$(2+B)^2 = 4A^2$, which can't happen $\therefore -1$ is not a root

2 marks: correct solution

1 mark: substantial progress towards correct solution

(ii) $P\left(\frac{1}{\alpha}\right) = \frac{1}{\alpha^4} + \frac{A}{\alpha^3} + \frac{B}{\alpha^2} + \frac{A}{\alpha} + 1$

$= \frac{1}{\alpha^4}\left(1 + A\alpha + B\alpha^2 + A\alpha^3 + \alpha^4\right)$

$= \frac{1}{\alpha^4}(0)$ as α is a root of $P(x)$

$= 0 \qquad \therefore \frac{1}{\alpha}$ is a root

2 marks: correct solution

1 mark: substantial progress towards correct solution

(iii) $\frac{1}{\alpha}$ is a root, α is at least a double root

let the 4th root be k.

so, the sum of the roots of $P(x)$ is

$$\alpha \times \alpha \times \frac{1}{\alpha} \times k = \frac{e}{a}$$

$$\frac{\alpha^2 k}{\alpha} = 1$$

$$k = \frac{1}{\alpha}$$

$\therefore$ the roots are $\alpha, \alpha, \frac{1}{\alpha}, \frac{1}{\alpha}$

ie, α is a double root

3 marks: correct solution

2 marks: substantial progress towards correct solution

1 mark: partial progress towards correct solution

For $P(x)$,

$$\Sigma\alpha = -\frac{b}{a} \qquad \Sigma\alpha\beta = \frac{c}{a}$$

$$\alpha + \alpha + \frac{1}{\alpha} + \frac{1}{\alpha} = -A \qquad \alpha^2 + 2 + 2 + \frac{1}{\alpha^2} = B$$

$$\alpha + \frac{1}{\alpha} = -\frac{A}{2} \qquad \alpha^2 + \frac{1}{\alpha^2} = B - 4$$

$$\left(\alpha + \frac{1}{\alpha}\right)^2 = \frac{A^2}{4}$$

$$\alpha^2 + \frac{1}{\alpha^2} = \frac{A^2}{4} - 2$$

so, $B - 4 = \frac{A^2}{4} - 2$

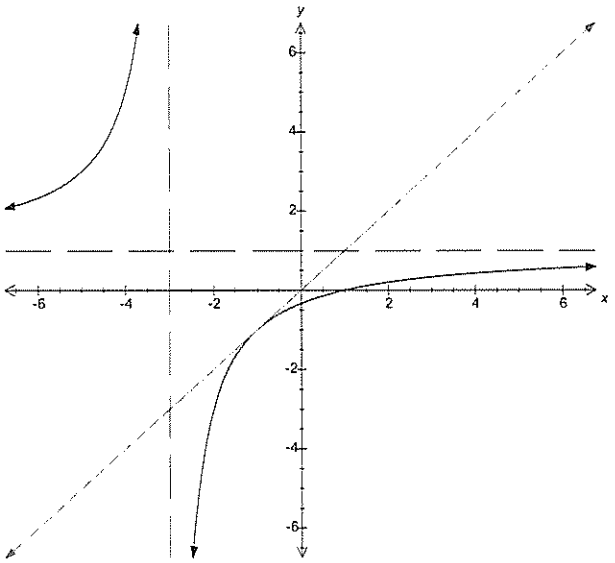
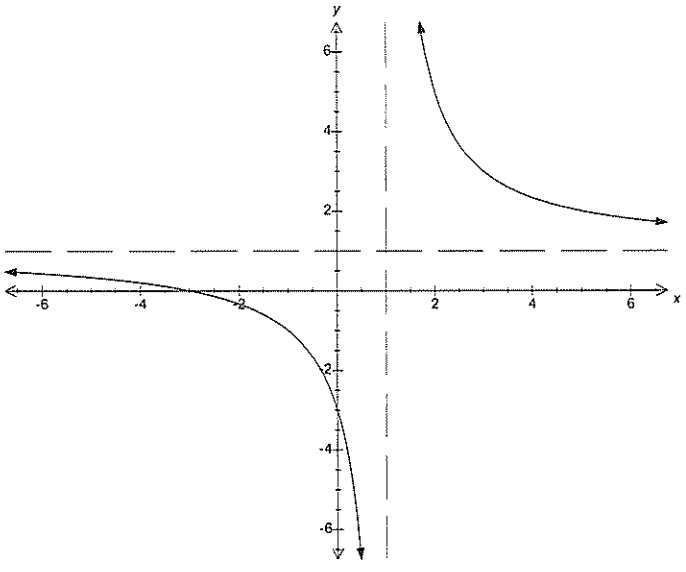
ie $\quad 4B = A^2 + 8$

E4	(d) $\dfrac{2}{(t+1)(t^2+1)} = \dfrac{A}{t+1} + \dfrac{Bt+C}{t^2+1}$ $2 = A(t^2+1) + (Bt+C)(t+1)$ let $t=-1$ $2 = A((-1)^2+1) + (B(-1)+C)(-1+1)$ $2 = 2A$ $A = 1$ let $t=0$ $2 = 1((0)^2+1) + (B(0)+C)(0+1)$ $2 = 1+C$ $C = 1$ let $t=1$ $2 = 1((1)^2+1) + (B(1)+1)(1+1)$ $2 = 2+2B+2$ $B = -1$ $\therefore A=1,\ B=-1,\ C=1$	**3 marks:** correct solution **2 marks:** substantial progress towards correct solution **1 mark:** partial progress towards correct solution

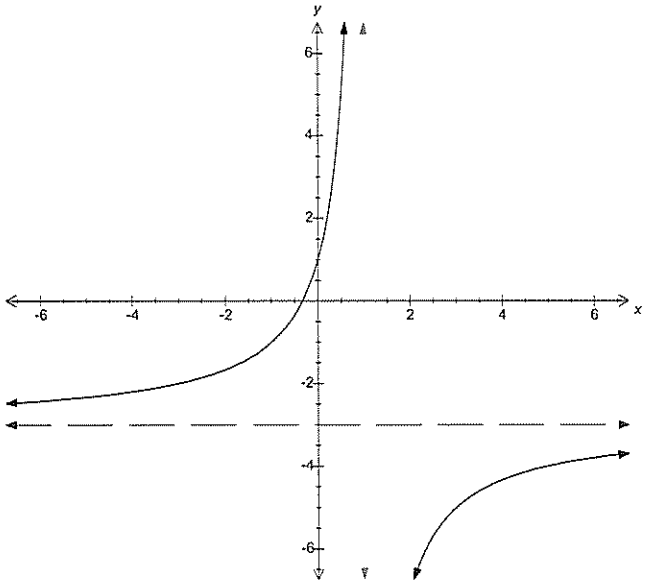
Year 12	Mathematics Extension 2	Half Yearly Examination 2009
Question No. 3	Solutions and Marking Guidelines	

Outcomes Addressed in this Question

E6 combines the ideas of algebra and calculus to determine the important features of the graph of a wide variety of functions

Outcome	Solutions	Marking Guidelines
E6	(a) (i)  	**3 marks:** correct solution, including drawing and labelling of asymptotes, x and y intercepts, and the shape of the graph. **2 marks:** substantial progress towards correct solution **1 mark:** partial progress towards correct solution
	(ii) Solving $y = x$ and $y = \dfrac{x-1}{x+3}$ simultaneously Gives: $(x+1)^2 = 0$ Since this polynomial has a double root at $x = 1$, then the line is a tangent to the curve. Point of contact is (1, 1)	**3 marks:** correct solution, including drawing and labelling tangent on graph, finding point of contact and algebraically justifying that the line is a tangent. **2 marks:** substantial progress towards correct solution **1 mark:** partial progress towards correct solution
	(iii) $y = \dfrac{1}{f(x)}$  	**3 marks:** correct solution, including drawing and labelling of asymptotes, x and y intercepts, and the shape of the graph. **2 marks:** substantial progress towards correct solution **1 mark:** partial progress towards correct solution

$y = f^{-1}(x)$



3 marks: correct solution, including drawing and labelling of asymptotes, x and y intercepts, and the shape of the graph.

2 marks: substantial progress towards correct solution

1 mark: partial progress towards correct solution

(b) (i)

$$y = f(x)^2$$

$$\frac{dy}{dx} = 2 \times f(x) \times f'(x)$$

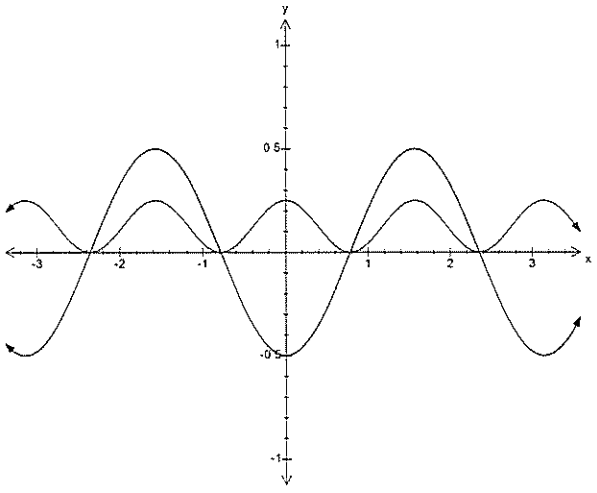
$$\therefore \frac{dy}{dx} = 0 \text{ when } f(x) = 0, \text{ or when } f'(x) = 0$$

i.e at x intercepts or at stationary points of $f(x)$.

2 marks: correct solution

1 mark: partially correct

(ii)



3 marks: correct solution.

2 marks: substantial progress towards correct solution

1 mark: partial progress towards correct solution

(c)

$3xy - y^3 = 1 - 2x$ Use implicit differentiation

$$3x\frac{dy}{dx} + 3y - 3y^2\frac{dy}{dx} = -2$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2-3y}{3(x-y^2)}$$

When $x = 0$, and $y = 1$,

$$\frac{dy}{dx} = -\frac{1}{3}$$

3 marks: correct solution

2 marks: substantial progress towards correct solution

1 mark: partial progress towards correct solution

Year 12	Mathematics Extension 2	Half Yearly Examination 2009
Question No. 4	Solutions and Marking Guidelines	

Outcomes Addressed in this Question

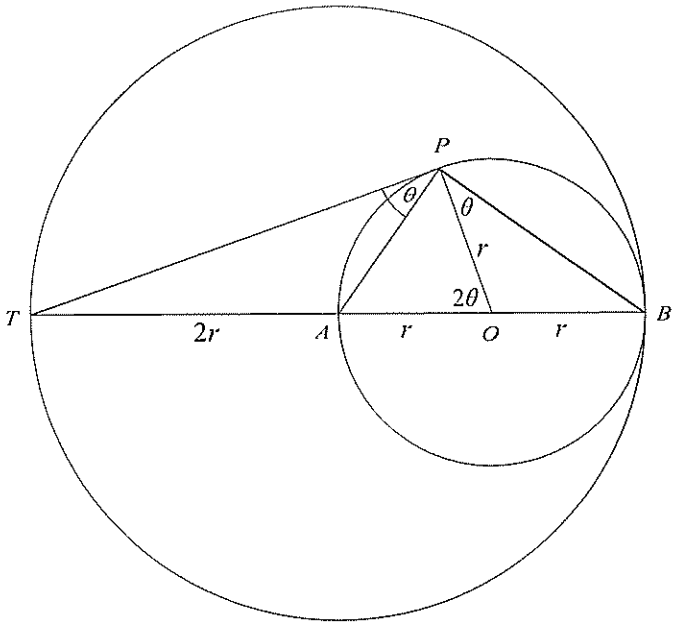
E2 chooses appropriate strategies to construct arguments and proofs in both concrete and abstract settings

E4 uses efficient techniques for the algebraic manipulation required in dealing with questions such as those involving polynomials

Outcome	Solutions	Marking Guidelines
E4	(a) (i) let α be a double root of $P(x)$ then $(x-\alpha)^2$ is a factor of $P(x)$ ie $P(x)=(x-\alpha)^2 Q(x)$ $P'(x)=2(x-\alpha)Q(x)+(x-\alpha)^2 Q'(x)$ $=(x-\alpha)\left[2Q(x)+(x-\alpha)Q'(x)\right]$ $\therefore (x-\alpha)$ is a factor of $P'(x)$ ie α is a root of $P'(x)$	**2 marks:** correct solution **1 mark:** partially correct
E4	(ii) let $f(x)=x^3-px+q$ $f'(x)=3x^2-p=0$ $x^2=\frac{p}{3}$ $x=\frac{\pm\sqrt{p}}{\sqrt{3}}$ must also satisfy $f(x)=0$ $f\left(\frac{\pm\sqrt{p}}{\sqrt{3}}\right)=\left(\frac{\pm\sqrt{p}}{\sqrt{3}}\right)^3-p\left(\frac{\pm\sqrt{p}}{\sqrt{3}}\right)+q$ $-q=\frac{\pm\sqrt{p}}{\sqrt{3}}\left(\frac{p}{3}-p\right)=\frac{\pm\sqrt{p}}{\sqrt{3}}\left(\frac{-2p}{3}\right)$ $-q=\frac{\mp 2p\sqrt{p}}{3\sqrt{3}}$ $q^2=\frac{4p^3}{27}$ $4p^3-27q^2=0$	**3 marks:** correct solution **2 marks:** substantial progress towards correct solution **1 mark:** partial progress towards correct solution
E4	(iii) straight line through origin has equation $y=mx$. Solve simultaneously with $y=x^3+1$ and find double root for tangent. ie $x^3+1=mx$ $x^3-mx+1=0$ $\left(\text{from (ii), let } p=m, q=1\right)$ ie $4m^3-27=0$ for double root $m=\frac{3}{\sqrt[3]{4}}$ $\therefore$ tangent has equation $y=\frac{3}{\sqrt[3]{4}}x$ $\sqrt[3]{4}y=3x$	**3 marks:** correct solution **2 marks:** substantial progress towards correct solution **1 mark:** partial progress towards correct solution

E2	(b) (i) $(a-b)^2 \geq 0$ $a^2 - 2ab + b^2 \geq 0$ $a^2 + b^2 \geq 2ab$	**1 mark:** correct solution
E2	(ii) from (i) $a^2 + b^2 \geq 2ab$ $b^2 + c^2 \geq 2bc$ $c^2 + a^2 \geq 2ca$ adding $2(a^2 + b^2 + c^2) \geq 2(ab + bc + ca)$ $a^2 + b^2 + c^2 \geq ab + bc + ca$	**2 marks:** correct solution **1 mark:** progress towards correct solution
E2	(iii) from (ii) $a^2 + b^2 + c^2 \geq ab + bc + ca$ $a^2 + b^2 + c^2 - ab - bc - ca \geq 0$ $(a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca) \geq 0$ $a^3 + b^3 + c^3 - 3abc \geq 0$ $a^3 + b^3 + c^3 \geq 3abc$	**2 marks:** correct solution **1 mark:** progress towards correct solution
E2	(iv) from (iii) let $a^3 = x, b^3 = y, c^3 = z$ $a^3 + b^3 + c^3 \geq 3abc$ $x + y + z \geq 3\sqrt[3]{x}.\sqrt[3]{y}.\sqrt[3]{z}$ $x + y + z \geq 3(xyz)^{\frac{1}{3}}$	**2 marks:** correct solution **1 mark:** progress towards correct solution

(c)



E2

(i) let $OA = OP = OB = r$ (radius small circle)
$AT = AB = 2r$ (radius large circle)

$\angle TPO = 90°$ (tangent $\perp$ radius)
$\angle APO = 90° - \theta$ (adj comp $\angle$'s)
$\angle PAO = 90° - \theta$ (equal $\angle$'s in isos ΔAPO)
$\angle AOP = 180° - 2(90° - \theta)$ ($\angle$ sum ΔAPO)
$= 2\theta$

In ΔOPT,

$$\cos 2\theta = \frac{OP}{OT} = \frac{r}{3r}$$

$$2\cos^2\theta - 1 = \frac{1}{3}$$

$$\cos^2\theta = \frac{2}{3}$$

$$\cos\theta = \frac{\sqrt{2}}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{6}}{3}$$

3 marks: correct solution

2 marks: substantial progress towards correct solution

1 mark: partial progress towards correct solution

E2

(ii) $\angle APB = 90°$ (angle in a semicircle)
$\angle OPB = 90° - (90° - \theta)$ (adj comp $\angle$'s)
$= \theta$
$\angle PBA = \angle OPB = \theta$ (ΔOPB isosceles)

Now, in ΔOPB

$$\cos B = \cos\theta = \frac{PB}{AB}$$

$$PB = AB\cos\theta$$

$$= \frac{AB\sqrt{6}}{3}$$

2 marks: correct solution

1 mark: progress towards correct solution