HURLSTONE AGRICULTURAL HIGH SCHOOL

MATHEMATICS – EXTENSION TWO

2012 HSC

ASSESSMENT TASK 1

Examiners ~ S Gee, J Dillon

GENERAL INSTRUCTIONS

- Reading Time – 3 minutes.
- Working Time – 40 MINUTES.
- Attempt **all** questions.
- **All** necessary working should be shown in every question.
- This paper contains two (2) questions.
- Marks may not be awarded for careless or badly arranged work.
- Board approved calculators and Mathematical templates may be used.
- **Each question is to be started on a new piece of paper.**
- This examination paper must **NOT** be removed from the examination room.

STUDENT NAME: ______________________________

TEACHER: ______________________________

QUESTION 1 (COMMENCE A NEW ANSWER BOOKLET) **MARKS**

Answer the following multiple choice question by writing the letter of the correct response on your answer sheet.

a) Given the coordinates of S are $(0,6)$ and S' is $(0,-6)$ and the conic obeys the condition $|PS|+|PS'|=20$.

The length of the semi minor axis is

A. 20 B. 16 C. 10 D. 8

b) The locus condition for any conic is defined by $PS = ePM$ where e is the eccentricity.

The value of e that will define a parabola is

A. $e=0$ B. $0<e<1$ C. $e=1$ D. $e>1$

c) Given the coordinates of S are $(3,0)$ and S' is $(-3,0)$, the quadrants on the number plane for which the conic $|PS|-|PS'|=10$ is defined are

A. 1 and 4 B. 2 and 3 C. only 1 D. 1, 2, 3 and 4

d) For the ellipse: $E = x^2 + 16y^2 = 25$.

Find

i) The eccentricity **1**

ii) The coordinates of the foci **1**

iii) The equations of the directrices **1**

iv) Draw a sketch. Mark in the foci, directrices, and the x and y intercepts **1**

v) Determine the length of the latus rectum. **1**

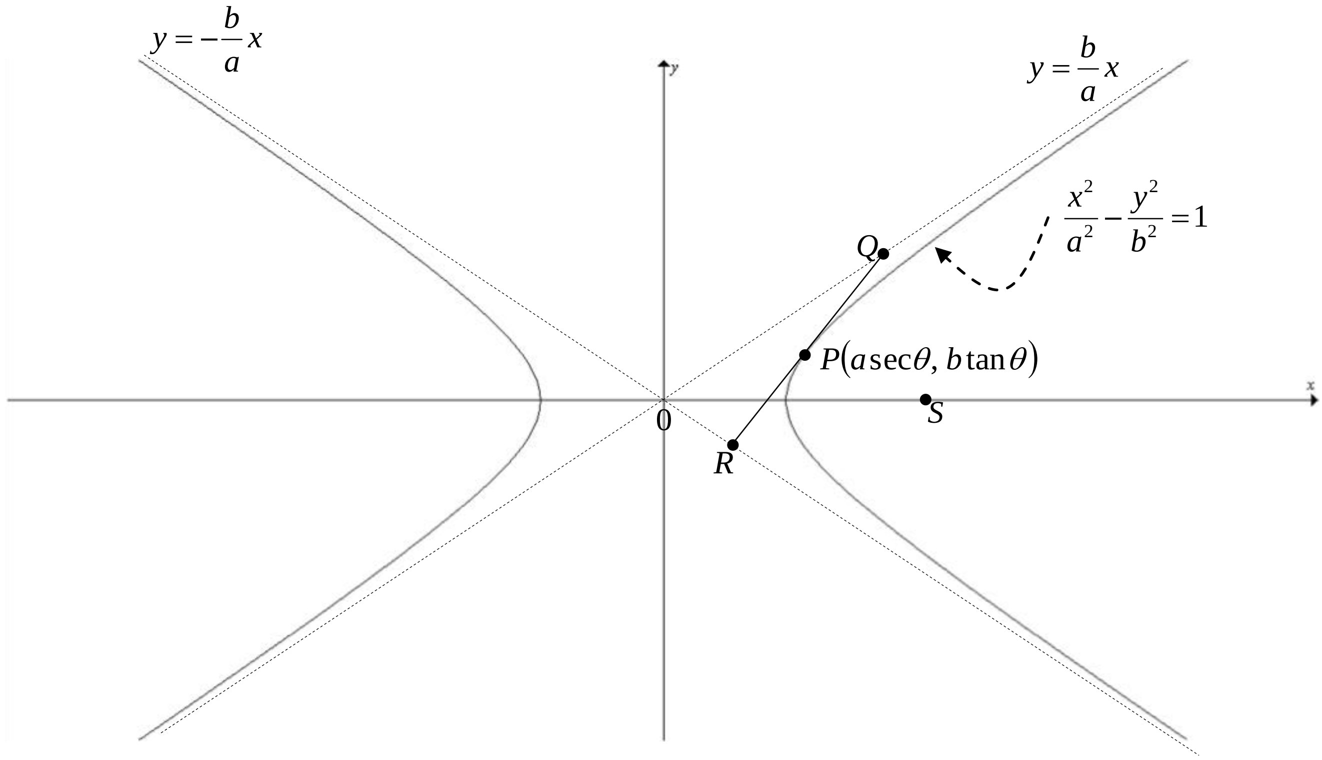
vi) State the equation of the auxiliary circle. **1**

vii) Find the equation of the tangent at the point P (3,1) **2**

viii) Find the equation of the normal at the point P (3,1) **1**

ix) If the normal meets the major axis in G and OQ is the perpendicular from the centre O of the ellipse E to the tangent at P, show that the value of PG. OQ is equal to the square of the semi-minor axis. **3**

QUESTION 2 (COMMENCE A NEW ANSWER BOOKLET) **MARK**



(a)

In the diagram $P(a\sec\theta, b\tan\theta)$ is a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

S is the focus of the hyperbola. The tangent to the hyperbola at P meets the asymptotes $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$ at the points Q and R respectively.

(i) Show that the tangent to the hyperbola at P has equation $bx\sec\theta - ay\tan\theta = ab$ **2**

(ii) Show that Q and R have coordinates $\left(a(\sec\theta + \tan\theta), b(\sec\theta + \tan\theta)\right)$ and $\left(a(\sec\theta - \tan\theta), -b(\sec\theta - \tan\theta)\right)$ respectively. **2**

(iii) Show that P is the midpoint of QR. **1**

(iv) Show that $OQ \times OR = OS^2$ where O is the origin. **3**

(b) $P\left(p, \frac{1}{p}\right)$ and $Q\left(q, \frac{1}{q}\right)$ are two points on the rectangular hyperbola $xy = 1$.

M is the midpoint of the chord PQ.

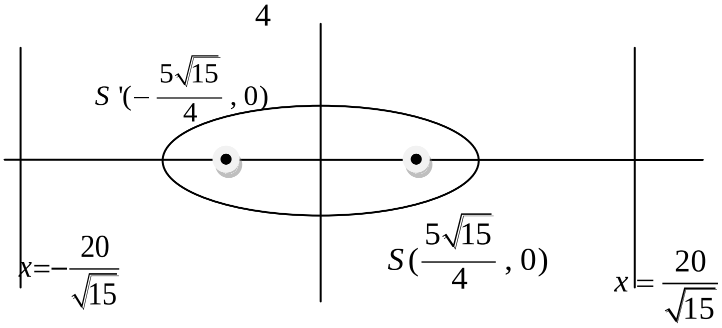
(i) Show that the chord PQ has equation $x + pqy - (p+q) = 0$. **2**

(ii) If P and Q move on the rectangular hyperbola such that the perpendicular distance of the chord PQ from the origin $O(0,0)$ is always $\sqrt{2}$, show that $(p+q)^2 = 2\left(1 + p^2q^2\right)$. **1**

(iii) Hence find the equation of the locus of M, stating any restrictions on its domain and range. **4**

Outcomes Addressed in this Question

E4 uses efficient techniques for the algebraic manipulation required in dealing with questions such as those involving (conic sections and) polynomials.

Outcome	Solutions	Marking Guidelines
	a. D	1 mark correct answer
	b. C	1 mark correct answer
	c. No correct answer accept all foci should have been on the x axis	1 mark correct answer
	d. $E: \quad \frac{x^2}{25}+\frac{y^2}{\frac{25}{16}}=1$	
	$\therefore a=5 \; b=\frac{5}{4}$	
	$i) \quad b^2=a^2\left(1-e^2\right)$	1 mark correct answer
	$\frac{\frac{25}{16}}{25}=\left(1-e^2\right)$	
	$e^2=\frac{15}{16}$	
	$e=\frac{\sqrt{15}}{4}$	1 mark correct answer
	$ii) \quad S(ae,0), S'(-ae,0)$	
	$S(5\times\frac{\sqrt{15}}{4},0), S'(-5\times\frac{\sqrt{15}}{4},0)$	
	$S(\frac{5\sqrt{15}}{4},0), S'(\frac{-5\sqrt{15}}{4},0)$	1 mark correct answer
	$iii) \quad x=\frac{\pm a}{e}=\frac{\pm 5}{\frac{\sqrt{15}}{4}}=\pm\frac{20}{\sqrt{15}}$	
		1 mark correct graph
		1 mark correct answer
	$iv) \quad latus\,rectum=2\times y$ value at the focus	
	$2\times\frac{5}{16}=\frac{5}{8}$	1 mark correct answer
	$v) \quad x^2+y^2=25$	

	$vii)$ $x^2+16y^2=25$ differentiating implicitly to find gradient of the tangent $2x+32y\frac{dy}{dx}=0$ $\frac{dy}{dx}=\frac{-x}{16y}$ at $(3,1)$ gradient of the tangent $m=\frac{-3}{16}$ eqn of tangent $y-1=\frac{-3}{16}(x-3)$ $3x+16y-25=0$	2 marks correct method leading to correct answer. 1 mark substantial progress towards correct answer.
	$viii)$ eqn of normal $y-1=\frac{16}{3}(x-3)$ $16x-3y-45=0$	1mark correct answer.
	$ix)$ The semi minor axis has length $\frac{5}{4}\therefore$ its square is $\frac{25}{16}$ The Normal meets the x axis when $y=0$ $\therefore 16x=45\, and\, G\left(\frac{45}{16},0\right)$ $PG=\sqrt{\left(\frac{45}{16}-3\right)^2+(0-1)^2}=\frac{\sqrt{234}}{16}$ $OQ=\left\lvert\frac{3\times0+16\times0-25}{\sqrt{3^2+16^2}}\right\rvert=\frac{25}{\sqrt{234}}$ $PG\times OQ=\frac{\sqrt{234}}{16}\times\frac{25}{\sqrt{234}}=\frac{25}{16}$ the semi minor axis.	3 marks correct method leading to correct answer with reasons. 2 marks substantial progress towards correct answer with reasons. 1 mark some progress towards correct solution with reasons

Outcomes Addressed in this Question

E3 uses the relationship between algebraic and geometric representations of complex numbers and of conic sections

E4 uses efficient techniques for the algebraic manipulation required in dealing with questions such as those involving conic sections and polynomials

Outcomes	Solutions	Marking Guidelines
(a) (i)	$x = a\sec\theta \qquad y = b\tan\theta$ $\dfrac{dx}{d\theta} = a\sec\theta\tan\theta \qquad \dfrac{dy}{d\theta} = b\sec^2\theta$ $\therefore \dfrac{dy}{dx} = \dfrac{b\sec^2\theta}{a\sec\theta\tan\theta} = \dfrac{b\sec\theta}{a\tan\theta} = m_{\text{tangent}}$ Equation is given by $y - b\tan\theta = \dfrac{b\sec\theta}{a\tan\theta}(x - a\sec\theta)$ $ay\tan\theta - ab\tan^2\theta = bx\sec\theta - ab\sec^2\theta$ $bx\sec\theta - ay\tan\theta = ab(\sec^2\theta - \tan^2\theta) = ab$	**Award 2** Correct solution (with sufficient detail provided) **Award 1** Substantial progress towards solution
(ii)	At Q on the tangent, $ay = bx$ $\therefore bx(\sec\theta - \tan\theta) = ab$ $bx(\sec^2\theta - \tan^2\theta) = ab(\sec\theta + \tan\theta)$ $\therefore x = a(\sec\theta + \tan\theta)$ $\therefore y = b(\sec\theta + \tan\theta)$ At R on the tangent, $ay = -bx$ $\therefore bx(\sec\theta + \tan\theta) = ab$ $bx(\sec^2\theta - \tan^2\theta) = ab(\sec\theta - \tan\theta)$ $\therefore x = a(\sec\theta - \tan\theta)$ $\therefore y = -b(\sec\theta - \tan\theta)$	**Award 2** Correct coordinates for both points **Award 1** Substantial progress towards determination of points
(iii)	At the midpoint of QR, $x = \dfrac{1}{2}\{a(\sec\theta + \tan\theta) + a(\sec\theta - \tan\theta)\} = a\sec\theta$ $y = \dfrac{1}{2}\{b(\sec\theta + \tan\theta) - b(\sec\theta - \tan\theta)\} = b\tan\theta$ Hence, P is the midpoint of QR.	**Award 1** Correct solution

(iv)	$b^2 = a^2\left(e^2-1\right) \Rightarrow a^2+b^2 = a^2e^2$ Hence, $OQ^2 = \left(a^2+b^2\right)\left(\sec\theta+\tan\theta\right)^2$ $OQ = ae(\sec\theta+\tan\theta)$ $OR^2 = \left(a^2+b^2\right)\left(\sec\theta-\tan\theta\right)^2$ $OQ = ae(\sec\theta-\tan\theta)$ $\therefore OQ\times OR = \left(ae\right)^2\left(\sec^2\theta-\tan^2\theta\right) = \left(ae\right)^2 = OS^2$	**Award 3** Correct solution **Award 2** Substantial progress towards solution **Award 1** Limited progress towards solution
(b) (i)	Chord PQ has gradient, $\dfrac{\frac{1}{p}-\frac{1}{q}}{p-q} = \dfrac{q-p}{pq\left(p-q\right)} = \dfrac{-1}{pq}$ and equation $y-\dfrac{1}{p} = \dfrac{-1}{pq}\left(x-p\right)$ $pqy-q = -x+p$ $x+pqy-\left(p+q\right) = 0$	**Award 2** Correct solution (with sufficient detail provided) **Award 1** Substantial progress towards solution
(ii)	$\left\lvert\dfrac{-\left(p+q\right)}{\sqrt{1^2+\left(pq\right)^2}}\right\rvert = \sqrt{2}$ $\therefore \left(p+q\right)^2 = 2\left(1+p^2q^2\right)$	**Award 1** Correct solution
(iii)	At M, $x = \dfrac{1}{2}\left(p+q\right)$ and $y = \dfrac{1}{2}\left(\dfrac{1}{p}+\dfrac{1}{q}\right) = \dfrac{\frac{1}{2}\left(p+q\right)}{pq}$ $\dfrac{x^2}{y^2} = p^2q^2 = \dfrac{1}{2}\left(p+q\right)^2-1 = 2x^2-1$ $\therefore y^2 = \dfrac{x^2}{2x^2-1}$ This relation has domain $\lvert x\rvert > \dfrac{1}{\sqrt{2}}$ Rearrangement gives $x^2+y^2 = 2x^2y^2$ which is symmetric in x and y. Hence, relation has range $\lvert y\rvert > \dfrac{1}{\sqrt{2}}$	**Award 4** Correct algebraic solution with correct restrictions on <u>both</u> domain and range **Award 3** Correct algebraic solution with correct restrictions on <u>either</u> domain or range **Award 2** Correct algebraic solution **Award 1** Limited progress towards solution