HURLSTONE AGRICULTURAL HIGH SCHOOL

PRO PATRIA

MATHEMATICS – EXTENSION TWO

2011 HSC

ASSESSMENT TASK 3

Examiners ~ G Huxley, G Rawson

GENERAL INSTRUCTIONS

- Reading Time – 3 minutes.
- Working Time – 40 minutes.
- Attempt **all** questions.
- **All** necessary working should be shown in every question.
- This paper contains two (2) questions.
- Marks may not be awarded for careless or badly arranged work.
- Board approved calculators may be used.
- **Each question is to be started in a new booklet.**
- This examination paper must **NOT** be removed from the examination room.

STUDENT NAME:______________________________

TEACHER: ______________________________

QUESTION ONE **15 marks** *Start a SEPARATE booklet.* **Marks**

(a) (i) Show that $P(x) = x^5 - 3x^4 + 4x^3 - 4x^2 + 3x - 1$ has $x = 1$ as a zero of multiplicity 3. **2**

(ii) Verify that $x = i$ is also a zero of $P(x)$. **1**

(iii) Hence solve the equation $P(x) = 0$. **2**

(b) Let α, β, γ be the roots of the equation $x^3 - 2x^2 - 5x - 1 = 0$. Form an equation whose roots are $\dfrac{1}{\alpha}, \dfrac{1}{\beta}, \dfrac{1}{\gamma}$ **2**

(c) (i) Solve $z^5 + 1 = 0$ by De Moivre's Theorem, leaving your solutions in modulus argument form. **2**

(ii) Prove that the solutions of $z^4 - z^3 + z^2 - z + 1 = 0$ are the non-real Solutions of $z^5 + 1 = 0$. **1**

(iii) Given that $z^n + \dfrac{1}{z^n} = 2n\cos\theta$, show that if $z^4 - z^3 + z^2 - z + 1 = 0$ then $4\cos^2\theta - 2\cos\theta - 1 = 0$. **3**

(iv) Hence find the exact value, fully simplified, of $\sec\dfrac{3\pi}{5}$. **2**

QUESTION TWO **15 marks** *Start a SEPARATE booklet.* **Marks**

(a) Evaluate $\int_0^1 xe^{-x^2}\,dx$. **2**

(b) Using the substitution $u = e^x$, or otherwise,

find $\int \frac{e^x\,dx}{\sqrt{1-e^{2x}}}$. **2**

(c) (i) Express $\frac{x^2+2x}{(x^2+4)(x-2)}$ as the sum of partial fractions **2**

(ii) Hence, find $\int \frac{x^2+2x}{(x^2+4)(x-2)}\,dx$ **2**

(d) Find $\int \frac{x^2}{1+4x^2}\,dx$ **3**

(e) Use the substitution $u=\sqrt{x-1}$ to evaluate $\int_2^3 \frac{1+x}{\sqrt{x-1}}\,dx$ **4**

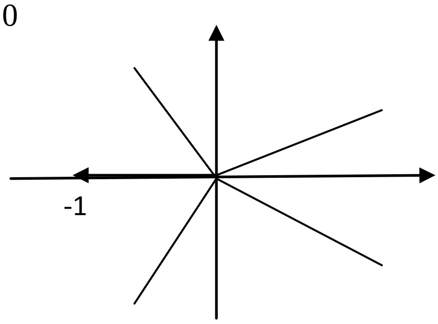
Standard Integrals Sheet.

Year 12 Mathematics Extension 2 HSC Task 3 Jun 2011

Question No. 1 Solutions and Marking Guidelines

Outcomes Addressed in this Question

E4-uses efficient techniques for the algebraic manipulation required in dealing with polynomials.

Outcome	**Solutions**	**Marking Guidelines**
E4 (a)	(i) $P(x)=x^5-3x^4+4x^3-4x^2+3x-1$ $P(1)=1-3+4-4+3-1=0$ $P'(x)=5x^4-12x^3+12x^2-8x+3$ $P'(1)=5-12+12-8+3=0$ $P''(x)=20x^3-36x^2+24x-8$ $P''(1)=20-36+24-8=0$ $\therefore P(1)=P'(1)=P''(1)=0$ So *x=1* is a zero of multiplicity 3.	**2 marks:** Showing information given. Note: To give the absolutely perfect answer, you should also show that $P'''(1)\neq 0$, so that *x=1* is not of multiplicity 4. **1 mark:** Not showing that all 3 equations =0, Or not showing the substitution for this particular polynomial.
	(ii) $P(i)=i^5-3i^4+4i^3-4i^2+3i-1$ $=i-3-4i+4+3i-1$ Therefore *i* is a zero $=0+0i$	**1 mark:** Correct answer
	(iii) Coefficients are real, therefore complex roots are complex conjugate pairs. $P(x)=0\rightarrow x=1,\pm i$	**2 marks:** Correct answer. **1 mark:** Partially correct.
(b)	Replace roots with their reciprocals, and simplify: $\left(\frac{1}{y}\right)^3-2\left(\frac{1}{y}\right)^2-5\left(\frac{1}{x}\right)-1=0$ $1-2y-5y^2-y^3=0 \quad\rightarrow\quad x^3+5x^2+2x-1=0$	**2 marks:** Correct answer. **1 mark:** Partially correct.
(c)	(i) $z^5+1=0$  $z=cis\pm\frac{\pi}{5},\ cis\pm\frac{3\pi}{5},cis\,\pi\,(or-1)$	**2 marks:** Correct answer. **1 mark:** Partially correct: possibly zeroes missing of having incorrect arguments that still differ by $\frac{2\pi}{5}$.

	(ii) $z^5+1=(z+1)(z^4-z^3+z^2-z+1)$ $z=-1 \textit{ is the solution of } z+1=0$ Since $z=-1$ is the only real solution, the unreal solutions of $z^5+1=0$ are the solutions of $z^4-z^3+z^2-z+1=0$	**<u>1 mark:</u>** Correct solution, including the factorisation.
	(iii) Divide the equation by z^2. $z^2\left(z^2-z+1-\frac{1}{z}+\frac{1}{z^2}\right)=0$, where $z\neq 0$ $\therefore\left(z^2+\frac{1}{z^2}\right)-\left(z+\frac{1}{z}\right)+1=0$ $2\cos 2\theta-2\cos\theta+1=0$ $2(2\cos^2\theta-1)-2\cos\theta+1=0$ $4\cos^2\theta-2\cos\theta-1=0$	**<u>3 marks:</u>** Correct solution. **<u>2 marks:</u>** 1 error in working. **<u>1 mark:</u>** Only one part of solution correct.
	(iv) $\cos\theta=\frac{1\pm\sqrt{5}}{4}$ $\sec\frac{3\pi}{5}<0, \qquad \therefore \sec\frac{3\pi}{5}=\frac{4}{1-\sqrt{5}}=-(1+\sqrt{5})$	**<u>2 marks:</u>** Correct solution. **<u>1 mark:</u>** Correct value of $\cos\theta$, or simplified calculation of $\sec\theta$ from incorrect $\cos\theta$.

Year 12	Mathematics Extension 2	HSC Task 3 Jun 2011

Question No. 2 Solutions and Marking Guidelines

Outcomes Addressed in this Question

E8- Applies further techniques of integration, including partial fractions, integration by parts and recurrence formulae.

Outcome	**Solutions**	**Marking Guidelines**
E8 (a)	$\int_0^1 xe^{-x^2}\,dx = -\frac{1}{2}\int_0^1 -2xe^{-x^2}\,dx$ $= -\frac{1}{2}\left[e^{-x^2}\right]_0^1$ $= \frac{e-1}{2e}$	**<u>2 marks:</u>** Correct answer, or equivalent **<u>1 mark:</u>** Correct primitive, or correct calculation from incorrect primitive.
(b)	$\int \frac{e^x dx}{\sqrt{1-e^{2x}}} = \int \frac{du}{\sqrt{1-u^2}}$ Let $u = e^x$, $du = e^x dx$ $= \sin^{-1} u + c$ $= \sin^{-1}(e^x) + c$	**<u>2 marks:</u>** Correct answer. **<u>1 mark:</u>** Correct substitution, or correct calculation from incorrect substitution.
(c)	(i) $\frac{x^2+2x}{(x^2+4)(x-2)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+4}$ $x^2+2x \equiv A(x^2+4) + (Bx+C)(x-2)$ $\to A=1; \quad B=0; \quad C=2$ $\therefore \frac{x^2+2x}{(x^2+4)(x-2)} = \frac{1}{x-2} + \frac{2}{x^2+4}$	**<u>2 marks:</u>** Correct solution. Must include *B=0* **<u>1 mark:</u>** Partially correct.
	(ii) $\int \frac{1}{x-2} + \frac{2}{x^2+4}\,dx = \ln\lvert x-2\rvert + \tan^{-1}\frac{x}{2} + c$	**<u>2marks:</u>** Correct answer, CFPA. **<u>1 mark:</u>** One term correctly integrated.
(d)	$\int \frac{x^2}{4x^2+1}\,dx = \int \frac{1}{4}\left(\frac{4x^2+1-1}{4x^2+1}\right)dx$ $= \frac{1}{4}\int 1 - \frac{1}{4x^2+1}\,dx$ $= \frac{1}{4}\left(x - \frac{1}{2}\tan^{-1}2x\right) + c$	**<u>3 marks:</u>** Correct answer. **<u>2 marks:</u>** Significant progress. **<u>1 mark:</u>** Correct manipulation of fraction; or correct integration of one term.

(e)	Let $u = \sqrt{x-1}$ $\therefore x = u^2 + 1$ $dx = 2u\,du$ $x = 3 \rightarrow u = \sqrt{2}$ $x = 2 \rightarrow u = 1$ $\int_2^3 \frac{1+x}{\sqrt{x-1}}\,dx = \int_1^{\sqrt{2}} \frac{2+u^2}{u} 2u\,du$ $= 2\int_1^{\sqrt{2}} 2 + u^2\,du$ $= 2\left[2u + \frac{u^3}{3}\right]_1^{\sqrt{2}}$ $= \frac{2}{3}\left(8\sqrt{2} - 7\right)$	**<u>4 marks:</u>** Correct solution, or equivalent. Includes: substitution; adjusting boundaries; finding primitive; calculating integral. **<u>3 marks:</u>** Significant progress. **<u>2 marks:</u>** 2 parts of solution correct. **<u>1 mark:</u>** Only one part of solution correct..