Student Name:______________________________

Teacher:___________________________________

HURLSTONE
AGRICULTURAL
HIGH
SCHOOL

2021

Year 12
TASK ONE

Mathematics Extension 2

Examiners Mr G Rawson, Mrs P Biczo, Mr G Huxley

Total marks: 33

Section 1 – 3 marks

- Attempt Questions 1-3
- Allow about 4 minutes for this section

Section 2 – 30 marks

- Attempt Questions 4-6
- Allow about 36 minutes for this section

General Instructions

- Reading time – 3 minutes
- Working time – 40 minutes
- Write using black pen.
- Calculators approved by NESA may be used.
- A reference sheet is provided.
- A multiple choice answer sheet is provided at the back of the paper.
- In Questions 4-6 show relevant mathematical reasoning and/or calculations.
- Write your name at the top of each booklet. Answer the questions in the space provided.
- Extra booklets are available if you require more space.

Written by 2021 HAHS Maths Faculty, rewritten by Ariq Abdullah

Section 1

3 marks

Attempt Questions 1-3

1. Given $z = 1 - i$, which expression is equal to z^3?

a. $\sqrt{2}e^{-\frac{3\pi i}{4}}$

b. $2\sqrt{2}e^{-\frac{3\pi i}{4}}$

c. $\sqrt{2}e^{\frac{3\pi i}{4}}$

d. $2\sqrt{2}e^{\frac{3\pi i}{4}}$

2. The complex number z satisfies $\arg\left(\frac{z-2}{z+2i}\right) = -\frac{\pi}{2}$.

What is the maximum value of $|z|$?

a. $\sqrt{2}$

b. $2\sqrt{2}$

c. $2 - \sqrt{2}$

d. $2 + \sqrt{2}$

3. Which of the following is equivalent to $\left(\frac{e^{-\frac{7i\pi}{6}}}{e^{i\pi}}\right)$?

a. $\frac{1}{2} + \frac{\sqrt{3}}{2}i$

b. $\frac{1}{2} + \frac{\sqrt{3}}{2}i$

c. $-\frac{\sqrt{3}}{2} + \frac{1}{2}i$

d. $\frac{\sqrt{3}}{2} - \frac{1}{2}i$

Section 2

30 marks

Attempt Questions 4-6

Allow about 36 minutes for this section

Question 4 (10 marks) **Marks**

(a) Let $z = 4 + i$ and $w = \bar{z}$. Find, in the form $x + iy$,

(i) w **1**

(ii) $w - z$ **1**

(b) Let $\alpha = 1 + i\sqrt{3}$ and $\beta = 1 + i$.

(i) Find $\frac{\alpha}{\beta}$ in the form $x + iy$ **1**

(ii) Express α in modulus-argument form. **2**

(iii) Given that β has the modulus-argument form **1**

$$\beta = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

Find the modulus-argument form of $\frac{\alpha}{\beta}$.

(iv) Hence, find the exact value of $\sin\frac{\pi}{12}$. **1**

(c) Let $z = 2(\cos\theta + i\sin\theta)$.

(i) Find $\overline{1 - z}$. **1**

(ii) Show that the real part of $\frac{1}{1-z}$ is $\frac{1-2\cos\theta}{5-4\cos\theta}$. **2**

Question 5 (10 marks) **Marks**

(a)

(i) Express $-\sqrt{3} - i$ in modulus-argument form. **2**

(ii) De Moivre's theorem states that $(rcis\theta)^n = r^n cisn\theta$ for any integer n. **2**

Show that $\left(-\sqrt{3} - i\right)^6$ is a rea number.

(b) Sketch the region in the complex plane where the inequalities $1 \leq |z| \leq 2$ and $0 \leq z + \bar{z} \leq 3$ hold simultaneously. **3**

(c) Let $z = 2(\cos\theta + i\sin\theta)$. **3**

Question 6 (10 marks) **Marks**

(a) Given that $z = 3 + i$ is a root of $z^2 + pz + q = 0$, where p and q are real, find the values of p and q. **3**

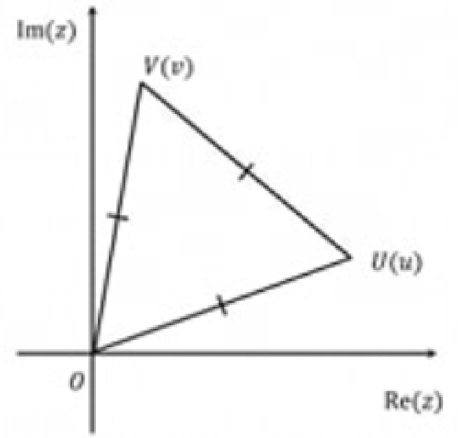
(b) Solve the equation $\left|e^{2i\theta} + e^{-2i\theta}\right| = 1$ where $-\pi \leq \theta \leq \pi$. **3**

Question 6 continued on next page

(c) Let the complex numbers $V(v)$, $U(u)$ and the origin form a triangle of area $1\,u^2$ on the Argand Diagram.

M is the midpoint of UV.

Let $\angle UMO = \theta$



(i) Show that $|u+v||u-v|\sin\theta = 4$. **2**

(ii) If $\theta = \frac{\pi}{2}$, prove that $(|u|-|v|)(|u|+|v|) = 0$. **2**

End of Exam

Year 12	Mathematics Extension 2	Ass Task 1 2021 HSC
MULTIPLE CHOICE	Solutions and Marking Guidelines	
Outcome	**Solutions**	**Marking Guidelines**

MEX12-4

1. $|1-i|=\sqrt{1^2+(-1)^2}=\sqrt{2}$

1 mark

$$\arg(1-i)=-\frac{\pi}{4}$$

$$\therefore 1-i=\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right)$$

$$(1-i)^3=\left(\sqrt{2}\right)^3\left(\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right)^3$$

$$=2\sqrt{2}\left(\cos\left(-\frac{3\pi}{4}\right)+i\sin\left(-\frac{3\pi}{4}\right)\right)$$

$$=2\sqrt{2}e^{-\frac{3\pi i}{4}} \qquad \therefore B$$

MEX12-4

2. $\arg\left(\dfrac{z-2}{z+2i}\right)=-\dfrac{\pi}{2}$

$\therefore \arg(z-2)-\arg(z-(-2i))=-\dfrac{\pi}{2}$, and so *z* lies on a semicircle (excluding the endpoints) whose diameter is the line joining the points 2 and $-2i$ on the argand diagram.
Need to test whether top or bottom half of the semicircle.
Testing if *z* is at (0, 0):

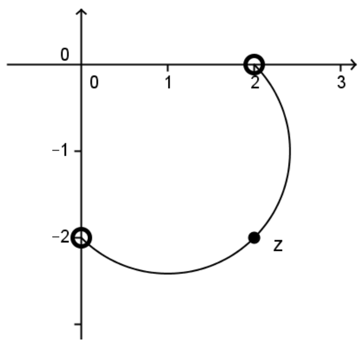
1 mark

$$\arg(z-2)-\arg(z+2i)=180^\circ-90^\circ\neq-\frac{\pi}{2}.$$

If *z* is $(2,-2)$, $\arg(z-2)-\arg(z+2i)=-90^\circ-0^\circ=-\dfrac{\pi}{2}$.

$\therefore z$ is on the semicircle containing the point $(2,-2)$.

From the diagram, the position of *z* that gives the maximum value of $|z|$ is when *z* is at $(2,-2)$.



$\therefore |z|$ is $d(0,0),(2,-2)=\sqrt{8}$ u $\qquad \therefore B$

MEX12-4	3. $e^{-\left(\frac{7i\pi}{6}\right)} \div e^{i\pi} = e^{-\left(\frac{7i\pi}{6}\right)-i\pi}$ $= e^{-i\left(\frac{7\pi+6\pi}{6}\right)}$ $= e^{-\frac{13i\pi}{6}}$ $= \cos\left(-\frac{13\pi}{6}\right) + i\sin\left(-\frac{13\pi}{6}\right)$ $= \cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)$ $= \cos\frac{\pi}{6} - i\sin\frac{\pi}{6}$ $= \frac{\sqrt{3}}{2} - \frac{1}{2}i \qquad \therefore D$	1 mark

Year 12	Mathematics Extension 2	Ass Task 1 2021 HSC
Question No. 4	Solutions and Marking Guidelines	

Outcomes Addressed in this Question

MEX12-4 uses the relationship between algebraic and geometric representations of complex numbers and complex number techniques to prove results, model and solve problems

Part / Outcome	Solutions	Marking Guidelines
(a)(i)	$\begin{aligned} w &= \bar{z} \\ &= \overline{4+i} \\ &= 4-i \end{aligned}$	1 mark : correct solution
(a)(ii)	$\begin{aligned} w-z &= (4-i)-(4+i) \\ &= -2i \end{aligned}$	1 mark : correct solution
(b)(i)	$\begin{aligned} \frac{\alpha}{\beta} &= \frac{1+i\sqrt{3}}{1+i} \times \frac{1-i}{1-i} \\ &= \frac{1-i+i\sqrt{3}+\sqrt{3}}{1+1} \\ &= \frac{(1+\sqrt{3})+(\sqrt{3}-1)i}{2} \\ &= \left(\frac{1+\sqrt{3}}{2}\right)+\left(\frac{\sqrt{3}-1}{2}\right)i \end{aligned}$	1 mark : correct solution *NB: the question asked for this to be in the form x + iy, which really means the real and imaginary parts should be fully separated*
(b)(ii)	$\begin{aligned} \alpha &= 1+i\sqrt{3} \\ \lvert\alpha\rvert &= \sqrt{1+3} = 2 \\ \arg(\alpha) &= \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3} \\ \therefore \alpha &= 2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right) \end{aligned}$	2 marks : correct solution 1 mark : substantially correct solution *NB: if using degrees, the degree symbol (°) MUST be used.* *If no symbol is used, you are automatically referring to radians. 60 radians is not the correct angle here!*
(b)(iii)	$\begin{aligned} \frac{\alpha}{\beta} &= \frac{2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)}{\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)} \\ &= \frac{2}{\sqrt{2}}\left(\cos\left(\frac{\pi}{3}-\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{3}-\frac{\pi}{4}\right)\right) \\ &= \sqrt{2}\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right) \end{aligned}$	1 mark : correct solution

Question 4 continued…

(b)(iv)

Comparing (i) and (iii), and equating imaginary parts:

$$\sqrt{2}\sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2}$$

$$\sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

1 mark : correct solution

(c)(i)
MEX12-4

$$1-z = 1-2\cos\theta - 2i\sin\theta$$

$$\overline{1-z} = (1-2\cos\theta) + 2i\sin\theta$$

1 mark : correct solution

(c)(ii)

$$\begin{aligned}\frac{1}{1-z} &= \frac{1}{1-z}\times\frac{\overline{1-z}}{\overline{1-z}}\\ &= \frac{\overline{1-z}}{|1-z|^2}\\ &= \frac{(1-2\cos\theta)+2i\sin\theta}{(1-2\cos\theta)^2+4\sin^2\theta}\\ &= \frac{(1-2\cos\theta)+2i\sin\theta}{1-4\cos\theta+4\cos^2\theta+4\sin^2\theta}\\ &= \frac{(1-2\cos\theta)+2i\sin\theta}{1-4\cos\theta+4}\\ &= \frac{(1-2\cos\theta)+2i\sin\theta}{5-4\cos\theta}\end{aligned}$$

$$\text{Re}\left(\frac{1}{1-z}\right) = \frac{1-2\cos\theta}{5-4\cos\theta}$$

2 marks : correct solution

1 mark : substantially correct solution

NB: as this is a 'show' question, it should have been clearly stated that $\sin^2 x + \cos^2 x = 1$ was being used to help simplify the denominator

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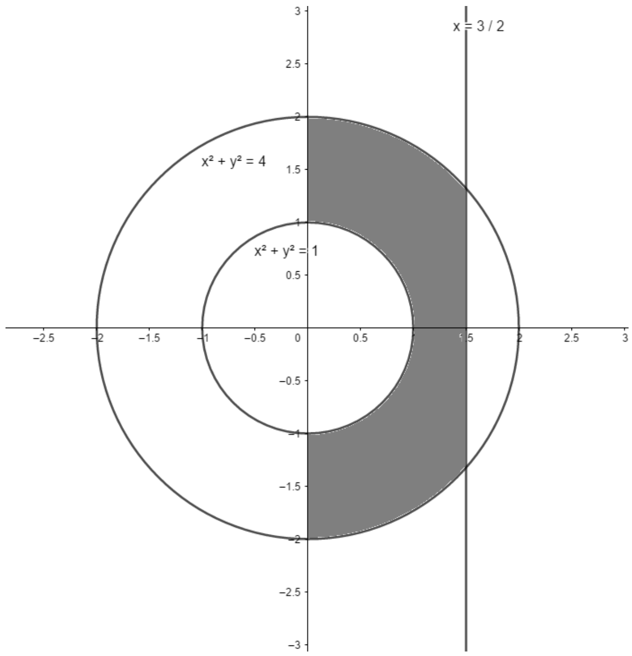
Question No. 5 | Solutions and Marking Guidelines

Outcomes Addressed in this Question

MEX12-4 uses the relationship between algebraic and geometric representations of complex numbers and complex number techniques to prove results, model and solve problems

Outcome	Solutions	Marking Guidelines
(a) **MEX12-4**	(i) $-\sqrt{3}-i=2\text{cis}\left(-\frac{5\pi}{6}\right)$ (ii) $\left(-\sqrt{3}-i\right)^6=2^6\text{ cis}\left(-\frac{5\pi}{6}\times 6\right)$ $=2^6\text{ cis}\left(-5\pi\right)$ $=2^6\left(-1\right)=-2^6$, which is real.	**(a) (i) 2 marks**: Both modulus and principal argument correct. **1 mark**: 1 correct component. **(ii) 2 marks**: Correct use of DeMoivre's Theorem and correct real number **1 mark**: 1 component correct.
(b) **MEX12-4**	$0\le z+\bar{z}\le 3 \quad\rightarrow\quad 0\le 2\text{Re}\left(z\right)\le 3 \quad\rightarrow\quad 0\le x\le \frac{3}{2}$ $1\le\lvert z\rvert\le 2$ represents the area between concentric circles. 	**(b)** **(i) 3 marks**: Complete solution. **2 marks**: One error in solving method. **1 mark**: Some relevant progress.
(c) **MEX12-4**	$\lvert z\rvert=\sqrt{\left(1-\cos 2\theta\right)^2+\left(\sin 2\theta\right)^2}$ $=\sqrt{1+2\cos 2\theta+\cos^2 2\theta+\sin^2 2\theta}$ $=\sqrt{2-2\cos 2\theta}$ $=\sqrt{2\left(1-\left(1-2\sin^2\theta\right)\right)}$ $=\sqrt{4\sin^2\theta}$ $=2\sin\theta$	**(c)** **(i)** 3 marks: Correct proof, showing all relevant steps **2 marks**: One error in solving method. **1 mark**: Some relevant progress.

Year 12	Mathematics Extension 2	Task 1 2021
Question No. 6	Solutions and Marking Guidelines	

Outcomes Addressed in this Question

MEX12-4 uses the relationship between algebraic and geometric representations of complex numbers and complex number techniques to prove results, model and solve problems

Outcome	**Solutions**	**Marking Guidelines**
MEX12-4	(a) As real coefficients, complex roots must occur in conjugate pairs. $\therefore$ as $3+i$ is a root, $3-i$ is a root. Sum of roots of $z^2+pz+q=0$ is $\frac{-p}{1}$ and product of the roots is $\frac{q}{1}$. $\therefore 3+i+3-i=-p$ and $(3+i)(3-i)=q$ $\therefore p=-6$ and $q=10$.	3 marks: correct solution 2 marks: partially correct solution 1 mark: finds either p or q or the other root
MEX12-4	(b) $\left\|e^{2i\theta}+e^{-2i\theta}\right\|=1$ $\therefore \left\|\cos 2\theta+i\sin 2\theta+\cos(-2\theta)+i\sin(-2\theta)\right\|=1$ $\therefore \left\|\cos 2\theta+i\sin 2\theta+\cos 2\theta-i\sin 2\theta\right\|=1$ $\therefore \left\|2\cos 2\theta\right\|=1$ $\therefore \left\|\cos 2\theta\right\|=\frac{1}{2}$ $\therefore \cos 2\theta=\pm\frac{1}{2}$ All quadrants; basic angle $60°$; required to solve for $-\pi\le\theta\le\pi$, hence $-2\pi\le 2\theta\le 2\pi$. $\therefore 2\theta=\frac{\pi}{3},\ \pi-\frac{\pi}{3},\ \pi+\frac{\pi}{3},2\pi-\frac{\pi}{3},\frac{\pi}{3}-2\pi,-\pi-\frac{\pi}{3},-\pi+\frac{\pi}{3},-\frac{\pi}{3}$ $\therefore 2\theta=\pm\frac{\pi}{3},\ \pm\frac{2\pi}{3},\ \pm\frac{4\pi}{3},\pm\frac{5\pi}{3}$ $\therefore \theta=\pm\frac{\pi}{6},\ \pm\frac{\pi}{3},\ \pm\frac{2\pi}{3},\pm\frac{5\pi}{6}$	3 marks: correct solution 2 marks: partially correct solution 1 mark: uses Euler's formula and simplifies

MEX12-4	*Question 6 continued…* (c) (i) Completing the parallelogram $OWPZ$, $\overrightarrow{OW} = w$, $\overrightarrow{OZ} = z, \overrightarrow{OP} = w + z, \overrightarrow{WZ} = z - w.$ As the diagonals of a parallelogram bisect one another, $\overrightarrow{OM} = \frac{1}{2}(z+w)$ and $\overrightarrow{MZ} = \frac{1}{2}(z-w)$ Also, as triangles OMW and OMZ have the same base length and same perpendicular height, their areas are equal. $\therefore$ area $\Delta OMZ = \frac{1}{2}$, given area $\Delta OWZ = 1$. $\therefore \frac{1}{2} \times OM \times MZ \times \sin\theta = \frac{1}{2}$ (using $\frac{1}{2}ab\sin C$ in ΔOMZ) $\therefore \frac{1}{2} \times \left\|\frac{1}{2}(z+w)\right\| \times \left\|\frac{1}{2}(z-w)\right\| \times \sin\theta = \frac{1}{2}$ $\therefore \frac{1}{8} \times \|z+w\| \times \|z-w\| \times \sin\theta = \frac{1}{2}$ $\therefore \|z+w\|\|z-w\| \times \sin\theta = 4.$ (ii) If $\theta = \frac{\pi}{2}$, then the parallelogram is a rhombus, hence all sides are equal. $\therefore \|z\| = \|w\|$ $\therefore \|z\| - \|w\| = 0$ $\therefore (\|z\| - \|w\|)(\|z\| + \|w\|) = 0 \times (\|z\| + \|w\|)$ $= 0.$	2 marks: correct solution 1 mark: partially correct solution 2 marks: correct solution 1 mark: partially correct solution