Student Name: ________________

Teacher: ________________

2021

Year 12

Mathematics Extension 2
Assessment Task 2

Examiners: Mr G. Huxley, Mrs P. Biczo, Mr G. Rawson

General Instructions

- Reading time - 3 minutes.
- Working time – 60 minutes.
- Write using black or blue pen.
- Diagrams may be drawn in pencil.
- Board-approved calculators may be used.
- Write your full name on the question booklet.
- Write your name on each answer booklet.
- A hand-written one page double-sided reference sheet may be used.

Total marks – 34

Attempt all questions.

The test contains four multiple choice questions.

- Questions 1 - 4 worth 1 mark each.

  Answer on the multiple choice sheet attached.

- Questions 5 and 6 are worth 15 marks each.

- Questions 5 and 6: start each question in a separate answer booklet.

Section A: Multiple Choice Questions: 1 mark each

Answer on the multiple choice sheet attached.

1. Consider the conditional statement for $n > 2$:

"If n is a prime number, then n is odd."

What is the contrapositive of the conditional statement?

A If n is prime, then n is not odd.

B If n is not prime, then n is not odd.

C If n is odd, then n is not prime.

D If n is not odd, then n is not prime.

2. Consider the statement:

"$\exists$ a real number x such that $x^2 = 4$ and $x < 1$."

What is the negation of this statement?

A $\exists$ a real number x such that $x^2 \neq 4$ and $x \geq 1$.

B $\forall$ real numbers x, $x^2 \neq 4$ or $x \geq 1$.

C $\exists$ a real number x such that $x^2 = 4$ or $x > 1$.

D $\forall$ real numbers x, $x^2 \neq 4$ and $x > 1$.

3. Which complex number is a 6^{th} root of i?

A $-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i$

B $-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i$

C $-\sqrt{2}+\sqrt{2}i$

D $-\sqrt{2}-\sqrt{2}i$

4. Let $\alpha = 1 - i$.

Which of the following is true about the value of α^{10}?

A It is purely real

B It is purely imaginary

C 0

D $32e^{\frac{5\pi i}{2}}$

Section B: (Questions 5 & 6)

Question 5 (15 marks) Use a SEPARATE writing booklet **Marks**

Show all necessary working.

(a) Prove that $\log_3 5$ is irrational. **2**

(b) Prove that n is divisible by 6 if and only if n is divisible by 2 and 3. **2**

(c) (i) Given $a, b > 0$, show that $\frac{a}{b} + \frac{b}{a} \geq 2$. **1**

(ii) Hence, prove that if $x^k, y^k > 0$ for all $k = 1, 2, 3, \ldots, n$ then

$$\sum_{k=1}^{n} \left(\frac{x_k}{y_k} + \frac{y_k}{x_k}\right) \geq 2n.$$ **2**

(d) Prove by Induction, for $n \geq 4$, that $n! > n^2$. **4**

(e) Consider the statement: **2**

"If $y'' = 0$ at $x = a$, then there is a point of inflexion at $x = a$."

Prove or disprove this statement.

(f) Three positive real numbers a, b and c are such that
$a + b + c = 1$ and $a \leq b \leq c$.

By considering the expansion of $(a + b + c)^2$ or otherwise, **2**
show that $5a^2 + 3b^2 + c^2 \leq 1$

Question 6 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Given that $z = \cos\theta + i\sin\theta$,

(i) Show that $z^n + \dfrac{1}{z^n} = 2\cos n\theta$ **1**

(ii) Hence, by using (i) and binomial expansion, write $\cos^4\theta$ in terms of $\cos n\theta$ (where n = 2, 4) **3**

(b) Let n be an integer greater than 2.
Suppose ω is an n-th root of unity and $\omega \neq 1$.

(i) By expanding the left-hand side, show that

$$\left(1 + 2\omega + 3\omega^2 + 4\omega^3 + \cdots + n\omega^{n-1}\right)\left(\omega - 1\right) = n$$ **2**

(ii) Using the identity $\dfrac{1}{z^2 - 1} = \dfrac{z^{-1}}{z - z^{-1}}$, or otherwise, prove that

$$\frac{1}{\cos 2\theta + i\sin 2\theta - 1} = \frac{\cos\theta - i\sin\theta}{2i\sin\theta}$$ **1**

(iii) Hence, if $\omega = \cos\dfrac{2\pi}{n} + i\sin\dfrac{2\pi}{n}$, find the real part of $\dfrac{1}{\omega - 1}$ **1**

(iv) Deduce that $1 + 2\cos\dfrac{2\pi}{5} + 3\cos\dfrac{4\pi}{5} + 4\cos\dfrac{6\pi}{5} + 5\cos\dfrac{8\pi}{5} = -\dfrac{5}{2}$ **1**

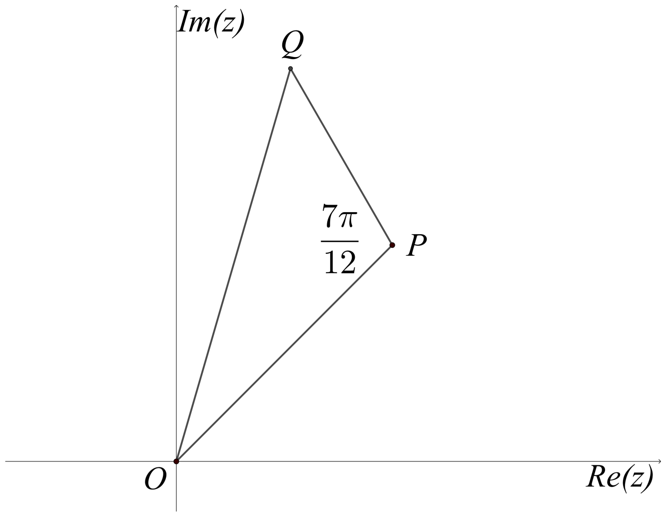
(v) By expressing the left hand side of the equation in part (iv) in terms of

$\cos\dfrac{\pi}{5}$ and $\cos\dfrac{2\pi}{5}$, find the exact value, in surd form, of $\cos\dfrac{\pi}{5}$. **3**

Question 6 continued next page…

Question 6 continued…

(c) The diagram below shows triangle OPQ in an Argand plane with the point P being represented by the complex number $\frac{3}{\sqrt{2}} + i\frac{3}{\sqrt{2}}$. **3**



If $\angle OPQ = \frac{7\pi}{12}$ and $|PQ| = 2$, find, in Cartesian form, the complex number representing the point Q.

End of Examination

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NSW Education Standards Authority

2020 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a+b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\dfrac{d}{a}$

Relations

$$(x-h)^2 + (y-k)^2 = r^2$$

Financial Mathematics

$$A = P(1+r)^n$$

Sequences and series

$$T_n = a + (n-1)d$$

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right] = \frac{n}{2}(a+l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1}, \ r \neq 1$$

$$S = \frac{a}{1-r}, \ |r| < 1$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

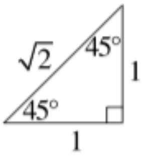
$$a^x = e^{x \ln a}$$

Trigonometric Functions

$$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$$

$$A = \frac{1}{2}ab\sin C$$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

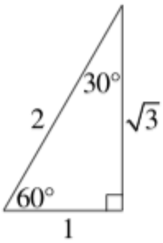


$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$l = r\theta$$

$$A = \frac{1}{2}r^2\theta$$



Trigonometric identities

$$\sec A = \frac{1}{\cos A}, \ \cos A \neq 0$$

$$\operatorname{cosec} A = \frac{1}{\sin A}, \ \sin A \neq 0$$

$$\cot A = \frac{\cos A}{\sin A}, \ \sin A \neq 0$$

$$\cos^2 x + \sin^2 x = 1$$

Compound angles

$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

If $t = \tan\frac{A}{2}$ then $\sin A = \dfrac{2t}{1 + t^2}$

$$\cos A = \frac{1 - t^2}{1 + t^2}$$

$$\tan A = \frac{2t}{1 - t^2}$$

$$\cos A\cos B = \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$$

$$\sin A\sin B = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$$

$$\sin A\cos B = \frac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$$

$$\cos A\sin B = \frac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$$

$$\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$$

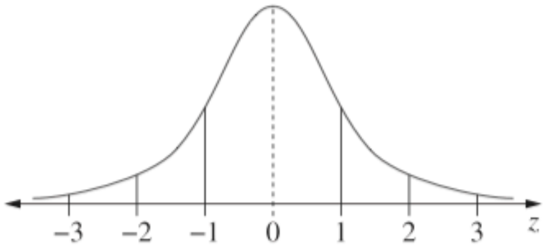
$$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$$

Statistical Analysis

$$z = \frac{x - \mu}{\sigma}$$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between −1 and 1
- approximately 95% of scores have z-scores between −2 and 2
- approximately 99.7% of scores have z-scores between −3 and 3

$$E(X) = \mu$$

$$\operatorname{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$$

Probability

$$P(A \cap B) = P(A)P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$$

Continuous random variables

$$P(X \leq x) = \int_a^x f(x)\,dx$$

$$P(a < X < b) = \int_a^b f(x)\,dx$$

Binomial distribution

$$P(X = r) = {}^nC_r p^r(1 - p)^{n - r}$$

$$X \sim \operatorname{Bin}(n, p)$$

$$\Rightarrow \ P(X = x)$$

$$= \binom{n}{x}p^x(1 - p)^{n - x}, \ x = 0, 1, \ldots, n$$

$$E(X) = np$$

$$\operatorname{Var}(X) = np(1 - p)$$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\dfrac{dy}{dx} = nf'(x)\left[f(x)\right]^{n-1}$
$y = uv$	$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$
$y = \dfrac{u}{v}$	$\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\dfrac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\dfrac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\dfrac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\dfrac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{\sqrt{1-\left[f(x)\right]^2}}$
$y = \cos^{-1} f(x)$	$\dfrac{dy}{dx} = -\dfrac{f'(x)}{\sqrt{1-\left[f(x)\right]^2}}$
$y = \tan^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{1+\left[f(x)\right]^2}$

Integral Calculus

$$\int f'(x)\left[f(x)\right]^n dx = \frac{1}{n+1}\left[f(x)\right]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}\,dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln\left|f(x)\right| + c$$

$$\int f'(x)a^{f(x)}\,dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2-\left[f(x)\right]^2}}\,dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2+\left[f(x)\right]^2}\,dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

$$\int_a^b f(x)\,dx$$

$$\approx \frac{b-a}{2n}\left\{f(a)+f(b)+2\left[f\left(x_1\right)+\cdots+f\left(x_{n-1}\right)\right]\right\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$$^nP_r = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^nC_r = \frac{n!}{r!\,(n-r)!}$$

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^r + \cdots + a^n$$

Vectors

$$|\underset{\sim}{u}| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^2+y^2}$$

$$\underset{\sim}{u}\cdot\underset{\sim}{v} = |\underset{\sim}{u}||\underset{\sim}{v}|\cos\theta = x_1x_2 + y_1y_2,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$\begin{aligned} z = a + ib &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

$$\begin{aligned} \left[r(\cos\theta + i\sin\theta)\right]^n &= r^n(\cos n\theta + i\sin n\theta) \\ &= r^n e^{in\theta} \end{aligned}$$

Mechanics

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$x = a\cos(nt+\alpha) + c$$

$$x = a\sin(nt+\alpha) + c$$

$$\ddot{x} = -n^2(x-c)$$

HURLSTONE AGRICULTURAL HIGH SCHOOL

Mathematics Extension 2

MULTIPLE CHOICE ANSWER SHEET

2021 Task 2

Student Name ______________________

Teacher ______________________

Select the letter A, B, C or D that best answers the question. Fill in the response oval completely.

Sample $3 + 5 =$ (A) 7 (B) 8 (C) 9 (D) 10

A ○ **B** ● **C** ○ **D** ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ○ **B** ⊗ **C** ○ **D** ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word '*correct*' and drawing an arrow as follows:

A ⊗ **B** ⊗ **C** ○ **D** ○

Correct

Question

1. **A** ○ **B** ○ **C** ○ **D** ○

2. **A** ○ **B** ○ **C** ○ **D** ○

3. **A** ○ **B** ○ **C** ○ **D** ○

4. **A** ○ **B** ○ **C** ○ **D** ○

Question	Marks
MC	/4
5	/15
6	/15
Total	/34

Year 12	Mathematics Extension 2	Task 2 2021
Question No. 5	Solutions and Marking Guidelines	

Outcomes Addressed in this Question

MEX12-1 understands and uses different representations of numbers and functions to model, prove results and find solutions to problems in a variety of contexts
MEX12-2 chooses appropriate strategies to construct arguments and proofs in both practical and abstract settings
MEX12-8 communicates and justifies abstract ideas and relationships using appropriate language, notation and logical argument

Outcome	Solutions	Marking Guidelines
MEX12-2	(i) Assume $\log_3 5 = \dfrac{p}{q}$ where p & q are integers, reduced to simplest form $\therefore 3^{\frac{p}{q}} = 5$ Raising both sides to the power of q, $3^p = 5^q$ But the LHS and RHS represent unique prime factorisations, so they can't be equal. $\therefore 3^p = 5^q$ has no solution and so there are no integers p & q such that $\log_3 5 = \dfrac{p}{q}$. $\therefore$ $\log_3 5$ is not rational, by contradiction.	2 marks: correct solution 1 marks: partially correct solution
	(ii) RTP that n is divisible by 6 if and only if n is divisible by 2 and 3. If n is divisible by 6, then $n = 6p$, where p is an integer $\therefore n = 2(3p) = 2t$ where t is an integer and $n = 3(2p) = 3m$ where m is an integer $\therefore$ If n is divisible by 6, n is divisible by 2 and 3. If n is divisible by 2 and divisible by 3, then $n = 2\times3\times x$ where x is an integer. $\therefore n = 6x$ $\therefore$ If n is divisible by 2 and 3, then n is divisible by 6. $\therefore$ n is divisible by 6 if and only if n is divisible by 2 and 3.	2 marks: correct solution 1 mark: partially correct solution
MEX12-1	(iii) a) $\left(\sqrt{\frac{a}{b}} - \sqrt{\frac{b}{a}}\right)^2 \geq 0$ $\therefore \frac{a}{b} + \frac{b}{a} - 2 \geq 0$ $\therefore \frac{a}{b} + \frac{b}{a} \geq 2$	1 mark: correct solution
	b) From a) any number plus its reciprocal ≥ 2 $\therefore \frac{x_1}{y_1} + \frac{y_1}{x_1} \geq 2$ $\frac{x_2}{y_2} + \frac{y_2}{x_2} \geq 2$	2 marks: correct solution 1 mark: partially correct solution

<table>
<tr>
<td>MEX12-4</td>
<td>
$\frac{x_3}{y_3} + \frac{y_3}{x_3} \geq 2$
….
….
….
$\frac{x_n}{y_n} + \frac{y_n}{x_n} \geq 2$
Adding $\sum_{k=1}^{n} \left(\frac{x_k}{y_k} + \frac{y_k}{x_k}\right) \geq 2n$
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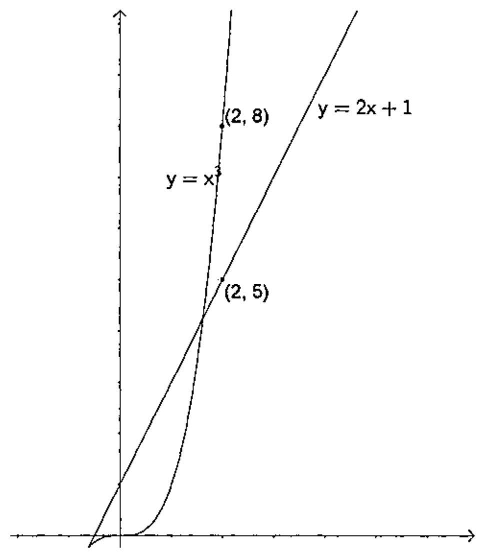
<tr>
<td>MEX12-2</td>
<td>
(iv) RTP for integers $n \geq 4$, that $n! > n^2$
Let $P(n)$ represent the proposition
$P(4)$: LHS = 4! = 24
RHS = 4^2= 16
LHS > RHS $\therefore$ $P(4)$ true
Assume $P(k)$ is true for $k \geq 4$ i.e. assume $k! > k^2$
RTP $P(k+1)$ true ie. that $(k+1)! > (k+1)^2$
Consider $(k+1)! - (k+1)^2$
$= (k+1)\,k! - k^2 - 2k - 1$
$> (k+1)\,k^2 - k^2 - 2k - 1$
$= k^3 + k^2 - k^2 - 2k - 1$
$= k^3 - (2k+1)$
> 0 as $k^3 > 2k + 1$ for $k \geq 4$
$\therefore P(k) \Rightarrow P(k+1)$
$\therefore P(n)$ is true for integers $n \geq 4$


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4 marks: correct solution with justification.
3 marks: correct induction steps without full justification.
2 marks: partially correct solution
1 mark: significant progress.
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(v) For a point of inflexion, need $y'' = 0$ and a change in concavity either side of $x = a$.
Counter example: $y = x^4$ has $y'' = 0$ at $x = 0$. This is not a point of inflexion, it is a minimum turning point.
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2 marks: correct solution
1 mark: partially correct solution
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MEX12-8 MEX12-2	(vi) $a+b+c=1$ $\therefore (a+b+c)^2 = 1^2 = 1$ $(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ac)$ $\geq a^2+b^2+c^2+2(a^2+b^2+a^2)$ (since $b \geq a$, $c \geq b$, $c \geq a$ given $a \leq b \leq c$) $\therefore (a+b+c)^2 \geq a^2+b^2+c^2+2a^2+2b^2+2a^2$ $\therefore 1 \geq 5a^2+3b^2+c^2$ Hence $5a^2+3b^2+c^2 \leq 1$	2 marks: correct solution 1 mark: significant progress
MEX12-8	Question 1: Contrapositive is: If n is not odd, then n is not prime. $\therefore$ D	1 mark: correct answer
MEX12-8	Question 2: Negation of statement is: $\forall$ real numbers x such that $x^2 \neq 4$ or $x \geq 1$. $\therefore$ B	1 mark: correct answer
MEX12-4	Question 3: $6\theta = \frac{\pi}{2} \therefore \theta = \frac{\pi}{12}$ Other roots spaced $\frac{2\pi}{6} = \frac{\pi}{3}$ apart $z_1 = cis\frac{\pi}{12},\ z_2 = cis\frac{5\pi}{12},\ z_3 = cis\frac{3\pi}{4},\ z_4 = cis\frac{13\pi}{12},$ $z_5 = cis\frac{17\pi}{12},\ z_6 = cis\frac{7\pi}{4}$ $z_3 = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ $z_6 = \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$ $\therefore$ A	1 mark: correct answer
MEX12-4	Question 4: $(1-i)^{10} = \left(\sqrt{2}cis\left(-\frac{\pi}{4}\right)\right)^{10}$ $= \left(\sqrt{2}e^{-\frac{\pi}{4}i}\right)^{10}$ $= 32e^{-\frac{5\pi}{2}i}$ $= 32\left(\cos\left(-\frac{5\pi}{2}\right) + i\sin\left(-\frac{5\pi}{2}\right)\right)$ $= 32(0-i)$ $= -32i$ which is purely imaginary $\therefore$ B	1 mark: correct answer

Year 12	Mathematics Extension 2	Ass Task 2 2021 HSC
Question No. 6	Solutions and Marking Guidelines	

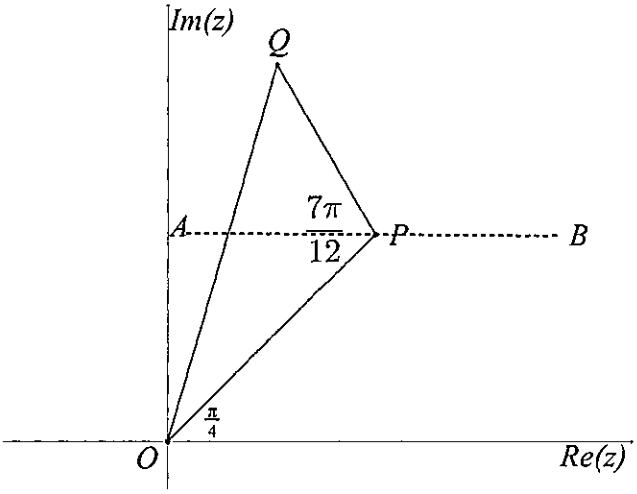
Outcomes Addressed in this Question

MEX12-4 uses the relationship between algebraic and geometric representations of complex numbers and complex number techniques to prove results, model and solve problems

Part / Outcome	Solutions	Marking Guidelines
(a)(i) **MEX12-4**	$z^n + \frac{1}{z^n} = z^n + z^{-n}$ $= (\cos\theta + i\sin\theta)^n + (\cos\theta + i\sin\theta)^{-n}$ $= \cos n\theta + i\sin n\theta + \cos(-n\theta) + i\sin(-n\theta)$ $= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$ $= 2\cos n\theta$	1 mark : correct solution
(a)(ii) **MEX12-4**	From (i), let $n = 1$ $\left(z+\frac{1}{z}\right)^4 = (2\cos\theta)^4$ $= 16\cos^4\theta$ and $(z+z^{-1})^4 = z^4 + 4z^3z^{-1} + 6z^2z^{-2} + 4zz^{-3} + z^{-4}$ $= z^4 + 4z^2 + 6 + 4z^{-2} + z^{-4}$ $= (z^4 + z^{-4}) + 4(z^2 + z^{-2}) + 6$ $= 2\cos 4\theta + 4.2\cos 2\theta + 6$ so, $16\cos^4\theta = 2\cos 4\theta + 8\cos 2\theta + 6$ $\cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$	3 marks : correct solution 2 marks : substantially correct solution 1 mark : partially correct solution

(b)(i) **MEX12-4**	Since ω is an n^{th} root of unity, $\omega^n = 1$ and $1+\omega+\omega^2+\omega^3+\ldots+\omega^{n-1}=0$ Hence, $\left(1+2\omega+3\omega^2+4\omega^3+\ldots+n\omega^{n-1}\right)\left(\omega-1\right)$ $= \omega+2\omega^2+3\omega^3+4\omega^4+\ldots+\left(n-1\right)\omega^{n-1}+n\omega^n$ $-1-2\omega-3\omega^2-4\omega^3-5\omega^4-\ldots-n\omega^{n-1}$ $=-1-\omega-\omega^2-\omega^3-\omega^4-\ldots-\omega^{n-1}+n\omega^n$ $=-\left(1+\omega+\omega^2+\omega^3+\ldots+\omega^{n-1}\right)+n$ using $\omega^n=1$ $=n$ using $1+\omega+\omega^2+\omega^3+\ldots+\omega^{n-1}=0$	<u>2 marks</u> : correct solution <u>1 mark</u> : substantially correct solution
(b)(ii) **MEX12-4**	Let $z=\cos\theta+i\sin\theta$ $\overline{z}=\cos\theta-i\sin\theta$ $z-\overline{z}=2i\sin\theta$ $z^2-1=\cos 2\theta+i\sin 2\theta-1$ Hence, $\dfrac{1}{\cos 2\theta+i\sin 2\theta-1}=\dfrac{1}{z^2-1}$ $=\dfrac{z^{-1}}{z-z^{-1}}$ $=\dfrac{\overline{z}}{z-\overline{z}}$ $\left(\text{since } \lvert z\rvert=1\right)$ $=\dfrac{\cos\theta-i\sin\theta}{2i\sin\theta}$ Note that if $\sin\theta=0$, then $\cos 2\theta+i\sin 2\theta-1=1-2\sin^2\theta+2i\sin\theta\cos\theta-1$ $=\sin\theta\left(-2\sin\theta+2i\cos\theta\right)$ $=0$ Thus when $\sin\theta=0$, both expressions are undefined	<u>1 mark</u> : correct solution
(b)(iii)	$\dfrac{1}{\omega-1}=\dfrac{1}{\cos\dfrac{2\pi}{n}+i\sin\dfrac{2\pi}{n}-1}$ $=\dfrac{\cos\dfrac{\pi}{n}-i\sin\dfrac{\pi}{n}}{2i\sin\dfrac{\pi}{n}}$, using (ii) with $\theta=\dfrac{\pi}{n}$ $=-\dfrac{1}{2}-\dfrac{i}{n}\cot\dfrac{\pi}{n}$ $\therefore\ \text{Re}\left(\dfrac{1}{\omega-1}\right)=-\dfrac{1}{2}$	<u>1 mark</u> : correct solution

(b)(iv)	From (i), we have $$1+2\omega+3\omega^2+4\omega^3+\ldots+n\omega^{n-1}=\frac{n}{\omega-1}$$ With $n=5$, this becomes $$1+2\omega+3\omega^2+4\omega^3+5\omega^4=\frac{5}{\omega-1} \qquad \ldots[1]$$ where $\omega=\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}$ $$\begin{aligned}\operatorname{Re}\left(\omega^r\right)&=\operatorname{Re}\left[\left(\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}\right)^r\right]\\&=\operatorname{Re}\left(\cos\frac{2\pi r}{5}+i\sin\frac{2\pi r}{5}\right), \quad \text{de Moivre's Theorem}\\&=\cos\frac{2\pi r}{5} \qquad \ldots[2]\end{aligned}$$ Hence, from [1]: $$\operatorname{Re}\left(1+2\omega+3\omega^2+4\omega^3+5\omega^4\right)=\operatorname{Re}\left(\frac{5}{\omega-1}\right)$$ $$1+2\operatorname{Re}(\omega)+3\operatorname{Re}\left(\omega^2\right)+4\operatorname{Re}\left(\omega^3\right)+5\operatorname{Re}\left(\omega^4\right)=5\operatorname{Re}\left(\frac{1}{\omega-1}\right)$$ $$1+2\cos\frac{2\pi}{5}+3\cos\frac{4\pi}{5}+4\cos\frac{6\pi}{5}+5\cos\frac{8\pi}{5}=-\frac{5}{2}$$ where we have used: $[2]$ on the LHS, with $r=1,2,3,4$ and (iii) on the RHS	1 mark : correct solution
(b)(v)	Using $\cos\frac{8\pi}{5}=\cos\frac{2\pi}{5}$ and $\cos\frac{6\pi}{5}=\cos\frac{4\pi}{5}=-\cos\frac{4\pi}{5}$, Part (iv) becomes $$\begin{aligned}1+2\cos\frac{2\pi}{5}-3\cos\frac{\pi}{5}-4\cos\frac{\pi}{5}+5\cos\frac{2\pi}{5}&=-\frac{5}{2}\\7\cos\frac{2\pi}{5}-7\cos\frac{\pi}{5}&=-\frac{7}{2}\\\cos\frac{2\pi}{5}-\cos\frac{\pi}{5}&=-\frac{1}{2}\\2\cos^2\frac{\pi}{5}-1-\cos\frac{\pi}{5}&=-\frac{1}{2}\\2\cos^2\frac{\pi}{5}-\cos\frac{\pi}{5}-\frac{1}{2}&=0\\4\cos^2\frac{\pi}{5}-2\cos\frac{\pi}{5}-1&=0\\\cos\frac{\pi}{5}&=\frac{2+\sqrt{20}}{8}, \quad \left(\cos\frac{\pi}{5}>0\right)\\&=\frac{1+\sqrt{5}}{4}\end{aligned}$$	3 marks : correct solution 2 marks : substantially correct solution 1 mark : partial progress towards correct solution

(c) **MEX12-4**	*Question 6 continued...*  $\arg\overrightarrow{OP} = \tan^{-1}\left(\frac{\frac{3}{\sqrt{2}}}{\frac{3}{\sqrt{2}}}\right) = \frac{\pi}{4}$ $\angle APO = \frac{\pi}{4}\quad(\text{alt}\,\angle)$ $\angle APQ = \frac{7\pi}{12} - \frac{\pi}{4} = \frac{\pi}{3}$ $\therefore\ \angle QPB = \frac{2\pi}{3}$ so, $\overrightarrow{PQ} = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$ $= -1 + i\sqrt{3}$ $\overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PQ}$ $= \frac{3}{\sqrt{2}} + \frac{3}{\sqrt{2}}i + \left(-1 + i\sqrt{3}\right)$ $= \frac{3}{\sqrt{2}} - 1 + i\left(\frac{3}{\sqrt{2}} + \sqrt{3}\right)$ $= \frac{3\sqrt{2}-2}{2} + \frac{3\sqrt{2}+2\sqrt{3}}{2}i$	<u>3 marks</u> : correct solution <u>2 marks</u> : substantially correct solution <u>1 mark</u> : partial progress towards correct solution

HSC examiners notes re 2005 Question 5b (Q 6b in this task)

(b) (i) Some (brief) comment was needed as to why the sum $(1+\omega+\omega^2+\ldots+\omega^{n-1})$ was zero, which many candidates overlooked.

(ii) This part was generally done well, with candidates showing a good understanding of inverses and de Moivre's theorem.

(iii) Many candidates found the real part, although not always from correct working. Many did not see the relationship between this and the previous part and this it made it more difficult to get the answer.

(iv) While this part was attempted by a large number of candidates, setting out, which was crucial to success, was often rather poor. Many tried to simply put $\omega = \cos\frac{2\pi}{5}$. Another common error was to claim that the real part of $\frac{1}{\omega-1} = -\frac{1}{2}$ implied that the real part of $\omega - 1$ was -2.

(v) There were many attempts at this part. It was accessible even to those who had managed to do little else in part (b). A reasonable number of those who attempted the question managed to write the terms in the sum in terms of $\cos\frac{2\pi}{5}$ and $\cos\frac{\pi}{5}$ or $\cos\frac{4\pi}{5}$ and $\cos\frac{2\pi}{5}$. Many were then able to form a quadratic in either $\cos\frac{2\pi}{5}$ or $\cos\frac{\pi}{5}$. There were a number of candidates who correctly reached this stage but then made careless errors in solving the quadratic equation.