Student Name: ________________

Teacher: ________________

2023	Hurlstone Agricultural High School **Year 12 Assessment Task 2**

Mathematics Extension 2

Examiners
- **Mr D Potaczala**
- **Mr G Huxley**
- **Mr G Rawson**

General Instructions
- Reading time – 3 minutes
- Working time – 40 minutes
- Write using black or blue pen
- NESA-approved calculators may be used
- A Reference sheet is provided for your use
- An A4 handwritten summary sheet may be used
- In Questions 7 to 9, show relevant mathematical reasoning and/or calculations

Total marks: 30

Section I – 6 marks (page 2–3)
- Attempt Questions 1–6
- Allow about 8 minutes for this section

Section II – 24 marks (pages 4–6)
- Attempt Questions 7–9
- Allow about 32 minutes for this section
- For each question use a separate answer booklet

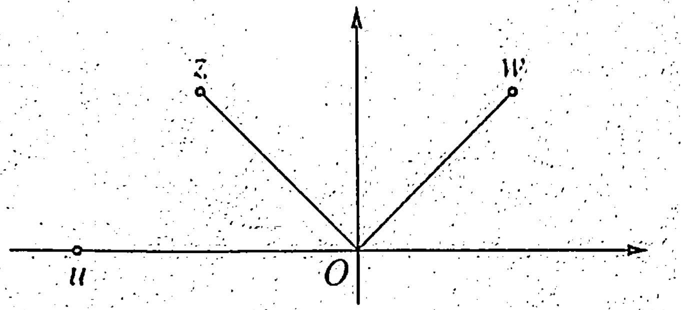
Section I

6 marks
Attempt Questions 1 to 6
Allow about 8 minutes for this section

Use the multiple-choice answer sheet for Questions 1 to 6

---

1. The argand diagram shows the complex numbers w, z, and u, where w lies in the first quadrant, z lies in the second quadrant and u lies on the negative real x.



Which statement could be true?

A. $u = zw$ and $u = z + w$

B. $u = zw$ and $u = z - w$

C. $z = uw$ and $u = z + w$

D. $z = uw$ and $u = z - w$

2. It is known that a particular complex number z is NOT a real number.

Which of the following could be true for this number z?

A. $\bar{z} = iz$

B. $\bar{z} = |z^2|$

C. $\text{Re}(iz) = \text{Im}(z)$

D. $\text{Arg}(z^3) = \text{Arg}(z)$

3. Which complex number lies in the region $2 < |z - 1| < 3$?

A. $1 + \sqrt{3}i$

B. $1 + 3i$

C. $2 + i$

D. $3 - i$

Scanned with CamScanner

4. Given that $z = 1 - i$, which expression is equal to z^3 ?

A. $\sqrt{2}\left(\cos\left(\frac{-3\pi}{4}\right) + i\sin\left(\frac{-3\pi}{4}\right)\right)$ $2\sqrt{2}$

B. $2\sqrt{2}\left(\cos\left(\frac{-3\pi}{4}\right) + i\sin\left(\frac{-3\pi}{4}\right)\right)$

C. $\sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)$

D. $2\sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)$

5. Which statement is true?

A. $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ such that $xy = 6$.

B. $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ such that $x + y = 6$.

C. $\exists x \in \mathbb{R}$ such that $\forall y \in \mathbb{R}, x + y = 6$.

D. $\exists x \in \mathbb{R}$ such that $\forall y \in \mathbb{R}, xy = 6$.

6. Consider the statement P.

P: For all integers $n \geq 1$, if n is a prime number then $\frac{n(n+1)}{2}$ is a prime number.

Which of the following is true about this statement and its converse?

A. The statement P and its converse are both true.

B. The statement P and its converse are both false.

C. The statement P is true and its converse is false.

D. The statement P is false and its converse is true.

~ END OF SECTION I ~

Section II

18 marks
Attempt Questions 7 to 9
Allow about 32 minutes for this section

Answer each question in a new answer booklet.

All necessary working should be shown in every question.

Question 7 (8 marks) **Marks**

(a) Let $z = 5 - i$.

(i) Find $z + 2\bar{z}$. **1**

(ii) Find $\frac{i}{z}$ in the form $x + iy$. **1**

(b) Prove that $2i$ is a root of $x^3 - 3x^2 + 4x - 12 = 0$. Hence, find all other roots. **3**

(c) Given the complex number $z = e^{i\theta}$, show that $w = \frac{z^2-1}{z^2+1}$ is purely imaginary **3**

~ End of question 7 ~

Question 8 (8 marks) **Marks**

(a) (i) Sketch the region on the Argand diagram where the following inequalities hold simultaneously: **3**

$$|z-1| < \sqrt{2} \text{ and } 0 \le \arg(z+i) \le \frac{\pi}{4}$$

(ii) Let w be the complex number of minimum modulus satisfying the inequalities in part (i). Express w in the form $x + iy$, where x and y are real number. **1**

(b) Let $z = \cos\theta + i\sin\theta$.

(i) Show that $z^n - \frac{1}{z^n} = 2i\sin n\theta$. **2**

(ii) Using the expansion of $\left(z - \frac{1}{z}\right)^5$, show that: **2**

$$\sin^5\theta = \frac{\sin 5\theta - 5\sin 3\theta + 10\sin\theta}{16}$$

~ End of question 8 ~

Question 9 (8 marks) **Marks**

(a) Consider the following statement, which is of the form $P \rightarrow Q$.

"If the pet is a stallion, then it must be a horse."

(i) Write the converse of the statement. **1**

(ii) Write the contrapositive of the statement. **1**

(iii) Write the negation of the statement **1**

(b) Write a proof by contradiction to show that: **2**

There are no integers a and b such that $21a + 3b = 1$.

(c) (i) Write an inequality that describes the statement: **1**

The area of a rectangle cannot be greater than the square of the average length of its two sides.

(ii) Prove the statement in part (i).

~ END OF SECTION II ~

~ END OF PAPER ~

2023 Yr12 HSC Assessment Task 2

Multiple Choice Solutions and Marking Guidelines

Solutions	Marking Guidelines

1)

Since z and w have positive imaginary components $u = z - w$ can be true as the imaginary parts can cancel to give only a real.

Also, can be true $u = zw$ as the addition of an acute and obtuse argument can result in a straight angle.

Option B

2)

Since z is not real, we have either $z = x + yi$ or $x = yi$

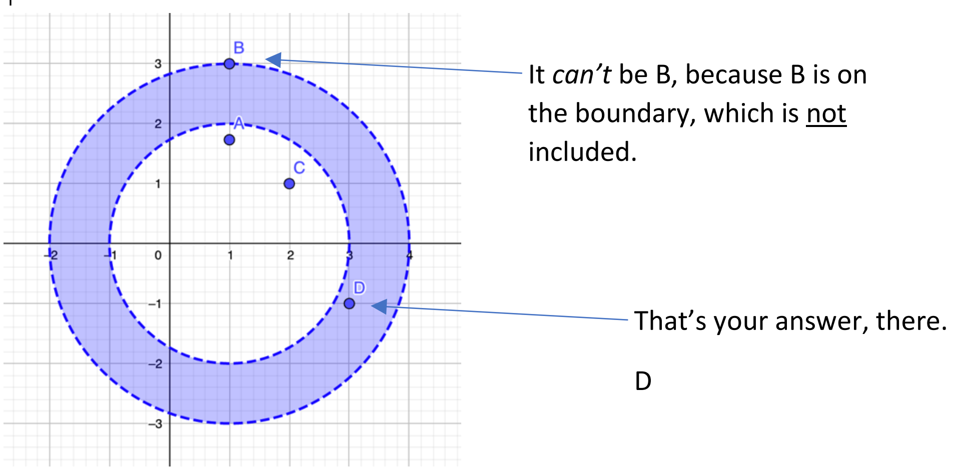
B is not an option as the modulus is a real number
C is not an option as $Re(z) = 0$ is a possibility.
D $\arg\left(r^3 e^{3i\theta}\right) = \arg\left(re^{i\theta}\right) \to 3i\theta = i\theta$ which is not true

Hence option A must be true ($\bar{z} = iz$ is true if $z = y + yi$).

Option A

3)

$2 < |z - 1| < 3$



D

Option D

4)

$$z^3 = (1 - i)^3$$
$$= \left(\sqrt{2}e^{-i\frac{\pi}{4}}\right)^3$$
$$= 2\sqrt{2}e^{-i\frac{3\pi}{4}}$$
$$= 2\sqrt{2}\left(\cos\left(\frac{-3\pi}{4}\right) + i\sin\left(\frac{-3\pi}{4}\right)\right)$$

Option B

5)
Interpretations in words:

A. For all real x, there is a y-value such that xy=6. This is false, with x=0 a counter-example.

B. For all real x, there is a y-value such that x + y = 6. This is true.

C. There exists a real x-value that has all y-values satisfying x + y=6. This is false because every x-value has a single unique y-value that makes the equation true.

D. There exists a real x-value that has all y-values satisfying xy=6. This is false, because every x-value in the domain has a single unique y-value that makes the equation true.

Option B

6)
The 4 response choices consider the combinations of the statement and its converse. Consider each statement:

Statement P is false. When testing prime numbers, test the number 2 (the only even prime) and an odd prime. n =3 is a counter-example to the statement.

When testing statement P you may have noticed that n(n+1) is always an even product of two factors. The only time the numerator is fully factorised by the denominator is when n=2.

Hence, the only time the fraction is prime is when n=2.
The converse: “If the fraction is prime, then n is prime” is true.

Option D

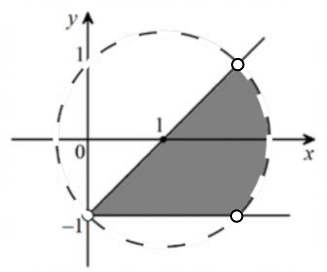
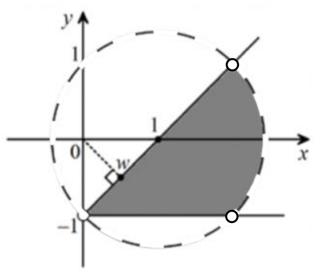
2023 Yr12 HSC Assessment Task 2	
Question 7 Solutions and Marking Guidelines	
Outcomes Addressed in this Question	
MEX12-4 uses the relationship between algebraic and geometric representations of complex numbers and complex number techniques to prove results, model and solve problems	
Solutions	**Marking Guidelines**
a i) $z+2\bar{z}=5-i+2(5+i)$ $=15+i$	1 mark Correct solution
ii) $\frac{i}{z}=\frac{i}{5-i}=\frac{(5+i)i}{25+1}$ $=\frac{-1+5i}{26}=-\frac{1}{26}+\frac{5}{26}i$	1 mark Correct solution
b) $f(2i)=(2i)^3-3(2i)^2+4(2i)-12$ $=-8i+12+8i-12$ $=0$ By factor theorem $2i$ is a root. Since the coefficients of the polynomial are real, the complex roots come in conjugate pairs. Hence $-2i$ is another root. Summing the roots we have: $2i-2i+\alpha=-\frac{-3}{1}$ $\alpha=3$	3 marks Correct solution with clear reasoning 2 marks Substantial progress towards solution 1 mark Some progress towards solution.

c) $w = \dfrac{z^2 - 1}{z^2 + 1}$ $= \dfrac{z\left(z - \frac{1}{z}\right)}{z\left(z - \frac{1}{z}\right)}$ $= \dfrac{e^{i\theta} - e^{-i\theta}}{e^{i\theta} + e^{-i\theta}}$ $= \dfrac{cos\theta + isin\theta - cos\theta + isin\theta}{cos\theta + isin\theta + cos\theta - isin\theta}$ $= \dfrac{2isin\theta}{2cos\theta}$ Which is imaginary.	3 marks Correct solution with clear reasoning 2 marks Substantial progress towards solution 1 mark Some progress towards solution.

Year 12	Mathematics Extension 2	Ass Task 2 2023 HSC
Question No. 8	Solutions and Marking Guidelines	

Outcomes Addressed in this Question

MEX12-4 uses the relationship between algebraic and geometric representations of complex numbers and complex number techniques to prove results, model and solve problems

Part / Outcome	**Solutions**	**Marking Guidelines**
(a)(i)		3 marks : correct solution Note: the open circle at (0, -1) is necessary for full marks. 2 marks : substantially correct solution 1 mark : partially correct solution
(a)(ii)	For w to lie in the region in part (i) and be of minimum modulus, w must be in the position as shown below… ie the perpendicular to $y = x - 1$, through O.  By inspection, $w = \frac{1}{2} - \frac{1}{2}i$	1 mark : correct solution
(b)(i)	$z^n - \frac{1}{z^n} = z^n - z^{-n}$ $= (\cos q + i\sin q)^n - (\cos q + i\sin q)^{-n}$ $= \cos nq + i\sin nq - (\cos(-nq) + i\sin(-nq))$ (de Moivre) $= \cos nq + i\sin nq - (\cos nq - i\sin nq)$ $= 2i\sin nq$	2 marks : correct solution 1 mark : substantially correct solution

(b)(ii)	$\left(z-\frac{1}{z}\right)^5=z^5-5z^4\frac{1}{z}+10z^3\frac{1}{z^2}-10z^2\frac{1}{z^3}+5z\frac{1}{z^4}-\frac{1}{z^5}$ $=z^5-5z^3+10z-10\left(\frac{1}{z}\right)+5\left(\frac{1}{z^3}\right)-\frac{1}{z^5}$ $=\left(z^5-\frac{1}{z^5}\right)-5\left(z^3-\frac{1}{z^3}\right)+10\left(z-\frac{1}{z}\right)$ so, $(2i\sin q)^5=2i\sin 5q-5(2i\sin 3q)+10(2i\sin q)$ $32i\sin^5 q=2i\sin 5q-10i\sin 3q+20i\sin q$ $=\frac{2i\sin 5q-10i\sin 3q+20i\sin q}{32i}$ $=\frac{\sin 5q-5\sin 3q+10\sin q}{16}$	<u>2 marks</u> : correct solution <u>1 mark</u> : substantially correct solution

Year 12	Mathematics Extension 2	Ass Task 2 2023 HSC
	Solutions and Marking Guidelines **Question 9**	

Outcomes Addressed in this Question

E2: Chooses appropriate strategies to construct arguments and proofs in both concrete and abstract settings.

Part	Solutions	Marking Guidelines
Q9 (a)	(a) The question has a $P \Rightarrow Q$ statement. (i) Converse has the form: $Q \Rightarrow P$ Hence: "If the pet is a horse, it must be a stallion." (ii) Contrapositive has the form: $\neg Q \Rightarrow \neg P$. Hence: "If the pet is not a horse, it is not a stallion." (iii) Negation has the form $\neg P \Rightarrow \neg Q$. Hence: "If not a stallion, then the pet is not a horse." Negations can also replace universal statements with "there exists". Hence the alternative: "There exists a stallion that is not a horse."	**(a)(i)1 mark:** correct statement. **(ii)1 mark:** correct statement. **(iii)1 mark:** correct statement.
(b)	(b) Assume that *a* and *b* are integers, and $$21a + 3b = 1$$ Then: $3(7a + 3b) = 1$ $$7a + b = \frac{1}{3}$$ But LHS is an integer for integers *a* and *b*. RHS is not an integer, and so we have a contradiction. Hence, the assumption is untrue; both *a* and *b* can't be integers.	**(b)2 marks:** Proper set up and proof by contradiction statement included. **1 mark:** One component of proof incomplete.

(c)	(c) (i) Let the length and width of a rectangle be *x*, *y*. Inequality will be: $xy \ \leq \left(\frac{x+y}{2}\right)^2$ (ii) <u>Method 1: Consider the difference:</u> $xy - \left(\frac{x+y}{2}\right)^2 = xy - \frac{x^2+2xy+y^2}{4}$ $= \frac{4xy - x^2 - 2xy - y^2}{4}$ $= \frac{-x^2 + 2xy - y^2}{4}$ $= -\frac{(x-y)^2}{4}$ ≤ 0 due to the square being non-negative. $\therefore \ xy \ \leq \left(\frac{x+y}{2}\right)^2$ <u>Method 2: Using the AM/GM inequality.</u> AM/GM: $\frac{a+b}{2} \geq \sqrt{ab}$ square both sides $\left(\frac{a+b}{2}\right)^2 \geq ab$ Which is equivalent to the statement in (i). *Note: Method 2 begins with a factual statement as its "axiom", then progresses towards the statement that needs to be proved.*	**(c)(i) 1 mark:** Correct answer **(ii) 2 marks:** Proper set up and proof. **1 mark:** One component of proof incomplete.