Student Name: ______________________________

Teacher: ______________________________

2024

Hurlstone Agricultural High School
Year 12 Assessment Task 2

Mathematics Extension 2

Examiners

- **Mr D Potaczala**
- **Mr G Rawson**
- **Mr G Huxley**

General Instructions

- Reading time – 3 minutes
- Working time – 40 minutes
- Write using black or blue pen
- NESA-approved calculators may be used
- A Reference sheet is provided for your use
- An A4 handwritten summary sheet may be used
- In Questions 7 to 9, show relevant mathematical reasoning and/or calculations

Total marks: 30

Section I – 6 marks (page 2–3)
- Attempt Questions 1–6
- Allow about 8 minutes for this section

Section II – 24 marks (pages 4–6)
- Attempt Questions 7–9
- Allow about 32 minutes for this section
- For each question use a separate answer booklet

Section I

6 marks
Attempt Questions 1 to 6
Allow about 8 minutes for this section

Use the multiple-choice answer sheet for Questions 1 to 6

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1. Consider the statement:
For any function $f(x)$, $f(x)$ has a stationary point of inflection at $x = c \Rightarrow f''(c)$=0.

Which of the following is correct?

A. The contrapositive statement is false and the converse statement is false.

B. The contrapositive statement is false and the converse statement is true.

C. The contrapositive statement is true and the converse statement is false.

D. The contrapositive statement is true and the converse statement is true.

2. Consider the statement:
"$\forall x \in \mathbb{N},\ x$ is even $\Rightarrow x^2(x^2+4)$ is divisible by 16."

Which of the following is the negation of the statement?

A. $\exists x \in \mathbb{N},\ x$ is not even and $x^2(x^2+4)$ is not divisible by 16.

B. $\exists x \in \mathbb{N},\ x$ is even and $x^2(x^2+4)$ is not divisible by 16.

C. $\forall x \in \mathbb{N},\ x$ is even and $x^2(x^2+4)$ is not divisible by 16.

D. $\forall x \in \mathbb{N},\ x$ is even or $x^2(x^2+4)$ is not divisible by 16.

3. Consider the proposition:

'If $2^n - 1$ is not prime, then n is not prime'

Given that each of the following statements is true, which statement disproves the proposition?

A. $2^5 - 1$ is prime

B. $2^6 - 1$ is divisible by 9

C. $2^7 - 1$ is prime

D. $2^{11} - 1$ is divisible by 23

4. Which of the following is the principal fifth root of -2?

A. $e^{i\pi}$

B. $e^{\frac{i\pi}{5}}$

C. $\sqrt[5]{2}e^{i\pi}$

D. $\sqrt[5]{2}e^{\frac{i\pi}{5}}$

5. What are the square roots of $5 - 12i$?

A. $\pm(3 - 2i)$

B. $\pm(2 + 3i)$

C. $\pm(2 - 3i)$

D. $\pm(3 + 2i)$

6. If z is a complex number such that $\frac{z-1}{z+1}$ is purely imaginary, then what is $|z|$ equal to?

A. $\frac{1}{2}$

B. $\frac{2}{3}$

C. 1

D. 2

~ END OF SECTION I ~

Section II

18 marks
Attempt Questions 7 to 9
Allow about 32 minutes for this section

***Answer each question in a new answer booklet*.**

All necessary working should be shown in every question.

Question 7 (8 marks) **Marks**

(a) If $x \leq 1$ and $y \geq 1$, prove that $x + y \geq 1 + xy$. **2**

(b) Given that a, b and c are positive real numbers, prove that **3**

$$(a + b + c)^2 \geq 3(ab + ac + bc).$$

(c) Prove, using contradiction, that $\sqrt{4n - 2}$ is irrational for all positive integers, n. **3**

~ End of question 7 ~

Question 8 (8 marks) **Marks**

(a) (i) Sketch the set of points on an Argand Diagram, for which: **1**

$$arg(z) = \frac{\pi}{3}$$

(ii) Show on your sketch from (i) why the distance given by $|z - 2i|$ is greater than or equal to 1 unit. **1**

(iii) Find the value of z for which $|z - 2i| = 1$.
Give polar coordinates for your answer. **1**

(b) Let $P(z) = 2z^3 - 13z^2 + 26z - 10$.

(i) Show that $(z - 3 + i)$ is a factor of $P(z)$. **2**

(ii) Hence or otherwise, solve $P(z) = 0$. **2**

(iii) Write $P(z)$ as a product of factors with real integer coefficients. **1**

~ End of question 8 ~

Question 9 (8 marks) **Marks**

(a) Prove by induction that, for all integers $n \geq 1$, **3**

$$(\cos\theta - i\sin\theta)^n = \cos(n\theta) - i\sin(n\theta)$$

(b) Suppose that $z = \frac{1}{2}(\cos\theta + i\sin\theta)$ where θ is real.

(i) Show that the imaginary part of the geometric series **3**

$$1 + z + z^2 + z^3 + \cdots = \frac{1}{1-z}$$

is $\frac{2\sin\theta}{5-4\cos\theta}$.

(ii) Find an expression for **2**

$$1 + \frac{1}{2}\cos\theta + \frac{1}{2^2}\cos 2\theta + \frac{1}{2^3}\cos 3\theta + \cdots$$

in terms of $\cos\theta$.

~ END OF SECTION II ~

~ END OF PAPER ~

NSW Education Standards Authority

2020 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a + b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$(x - h)^2 + (y - k)^2 = r^2$$

Financial Mathematics

$$A = P(1 + r)^n$$

Sequences and series

$$T_n = a + (n - 1)d$$

$$S_n = \frac{n}{2}\left[2a + (n - 1)d\right] = \frac{n}{2}(a + l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1}, \ r \neq 1$$

$$S = \frac{a}{1 - r}, \ |r| < 1$$

Logarithmic and Exponential Functions

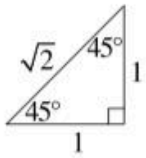
$$\log_a a^x = x = a^{\log_a x}$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$a^x = e^{x \ln a}$$

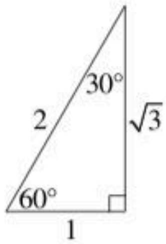
Trigonometric Functions

$\sin A = \dfrac{\text{opp}}{\text{hyp}}, \quad \cos A = \dfrac{\text{adj}}{\text{hyp}}, \quad \tan A = \dfrac{\text{opp}}{\text{adj}}$



$A = \dfrac{1}{2}ab\sin C$

$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$



$c^2 = a^2 + b^2 - 2ab\cos C$

$\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

$l = r\theta$

$A = \dfrac{1}{2}r^2\theta$

Trigonometric identities

$\sec A = \dfrac{1}{\cos A}, \ \cos A \neq 0$

$\operatorname{cosec} A = \dfrac{1}{\sin A}, \ \sin A \neq 0$

$\cot A = \dfrac{\cos A}{\sin A}, \ \sin A \neq 0$

$\cos^2 x + \sin^2 x = 1$

Compound angles

$\sin(A + B) = \sin A\cos B + \cos A\sin B$

$\cos(A + B) = \cos A\cos B - \sin A\sin B$

$\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$

If $t = \tan\dfrac{A}{2}$ then $\sin A = \dfrac{2t}{1 + t^2}$

$\cos A = \dfrac{1 - t^2}{1 + t^2}$

$\tan A = \dfrac{2t}{1 - t^2}$

$\cos A\cos B = \dfrac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$

$\sin A\sin B = \dfrac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$

$\sin A\cos B = \dfrac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$

$\cos A\sin B = \dfrac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$

$\sin^2 nx = \dfrac{1}{2}(1 - \cos 2nx)$

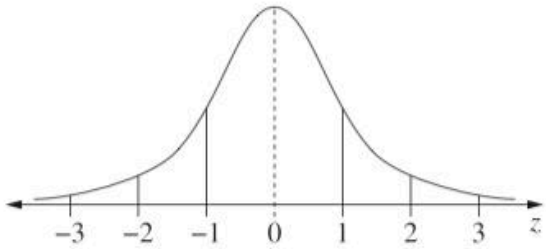
$\cos^2 nx = \dfrac{1}{2}(1 + \cos 2nx)$

Statistical Analysis

$z = \dfrac{x - \mu}{\sigma}$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between −1 and 1
- approximately 95% of scores have z-scores between −2 and 2
- approximately 99.7% of scores have z-scores between −3 and 3

$E(X) = \mu$

$\operatorname{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$

Probability

$P(A \cap B) = P(A)P(B)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A|B) = \dfrac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$

Continuous random variables

$P(X \le x) = \displaystyle\int_a^x f(x)\,dx$

$P(a < X < b) = \displaystyle\int_a^b f(x)\,dx$

Binomial distribution

$P(X = r) = {}^nC_r p^r (1 - p)^{n-r}$

$X \sim \operatorname{Bin}(n, p)$

$\Rightarrow \ P(X = x)$

$= \dbinom{n}{x} p^x (1 - p)^{n-x}, \ x = 0, 1, \ldots, n$

$E(X) = np$

$\operatorname{Var}(X) = np(1 - p)$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\dfrac{dy}{dx} = nf'(x)[f(x)]^{n-1}$
$y = uv$	$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$
$y = \dfrac{u}{v}$	$\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\dfrac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\dfrac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\dfrac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\dfrac{dy}{dx} = (\ln a) f'(x) a^{f(x)}$
$y = \log_a f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{(\ln a) f(x)}$
$y = \sin^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \cos^{-1} f(x)$	$\dfrac{dy}{dx} = -\dfrac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \tan^{-1} f(x)$	$\dfrac{dy}{dx} = \dfrac{f'(x)}{1+[f(x)]^2}$

Integral Calculus

$$\int f'(x)[f(x)]^n\,dx = \frac{1}{n+1}[f(x)]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}\,dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$$

$$\int f'(x)a^{f(x)}\,dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2-[f(x)]^2}}\,dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2+[f(x)]^2}\,dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

$$\int_a^b f(x)\,dx$$

$$\approx \frac{b-a}{2n}\Big\{f(a) + f(b) + 2\big[f(x_1) + \cdots + f(x_{n-1})\big]\Big\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$$^nP_r = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^nC_r = \frac{n!}{r!\,(n-r)!}$$

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^r + \cdots + a^n$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^2+y^2}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_1x_2 + y_1y_2\,,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$\begin{aligned} z = a + ib &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

$$\begin{aligned} \left[r(\cos\theta + i\sin\theta)\right]^n &= r^n(\cos n\theta + i\sin n\theta) \\ &= r^n e^{in\theta} \end{aligned}$$

Mechanics

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$x = a\cos(nt+\alpha) + c$$

$$x = a\sin(nt+\alpha) + c$$

$$\ddot{x} = -n^2(x-c)$$

PRO PATRIA

Student Name: ______________________________

Teacher: ______________________________

2024

Hurlstone Agricultural High School
Year 12 Mathematics Extension 2
Assessment Task 2

Section I – Multiple Choice Answer Sheet

Allow about 10 minutes for this section

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample: 2 + 4 = (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ~~●~~ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word **correct** and drawing an arrow as follows.

correct

A ~~●~~ B ~~●~~ C ○ D ○

Start Here

1. A ○ B ○ C ○ D ○

2. A ○ B ○ C ○ D ○

3. A ○ B ○ C ○ D ○

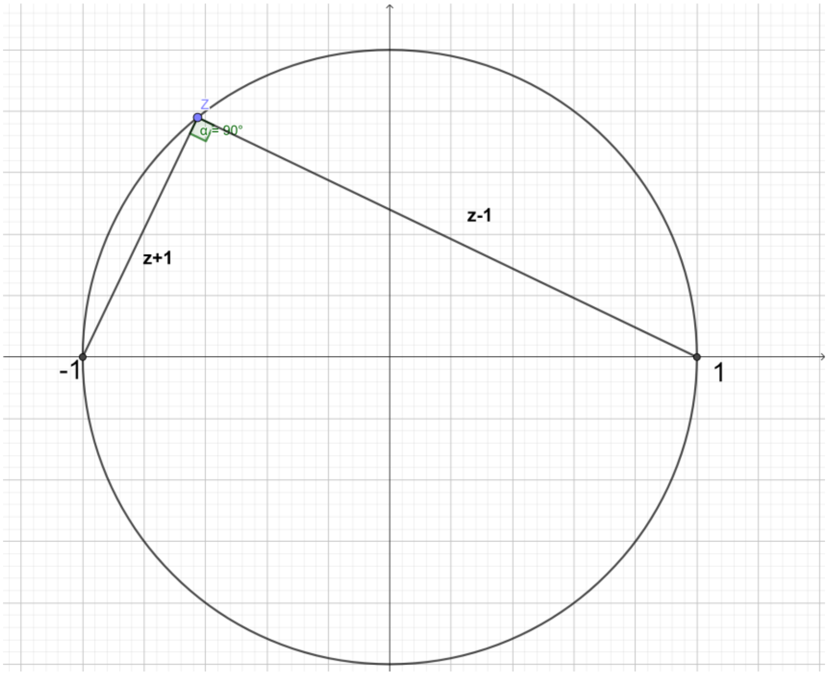
4. A ○ B ○ C ○ D ○

5. A ○ B ○ C ○ D ○

6. A ○ B ○ C ○ D ○

Multiple Choice Solutions and Marking Guidelines

Solutions	Marking Guidelines
1) Contrapositive: $f''(x) \neq 0 \Rightarrow f(x)$ does not have a stationary point of inflection at $x = c$. TRUE Converse: $f''(x) \neq 0 \Rightarrow f(x)$ has a stationary point of inflection at $x = c$. FALSE Correct answer is C **2)** The negation is B **3)** Option D Since 11 is prime and $2^{11} - 1$ is not prime as it is divisible by 23 **4)** Finding the fifth root of -2 is to solve the equation $z^5 = -2$. Using $e^{i\pi} = -1$ we have $z^5 = 2e^{i\pi}$ Hence, it is clear, with not further working required, that option D is the answer. **5)** $(x + iy)^2 = 5 - 12i$ The 4 options all give 2 and 3 as components of either real or imaginary parts, so we need to concentrate on the required relationship between the components. $2xy = -12$ and $x^2 - y^2 = 5$ give $x = \pm 3; y = \mp 2$ Answer A *Note: As an extra check: With what we know about doubling the argument when we square a number: since 5-12i is in the 4^{th} quadrant, then one of its square roots must also be in the 4^{th} quadrant.*	

6)  The question implies that the vectors of z-1 and z+1 are perpendicular. Hence the complex number z must lie on the circle with diameter joining -1 and 1. Hence $\|z\| = 1$. Answer C.	

Year 12	Mathematics Extension 2	Ass Task 2 2024 HSC
Question 7	Solutions and Marking Guidelines	

Outcomes Addressed in this Question

MEX 12-2 chooses appropriate strategies to construct arguments and proofs in both practical and abstract settings

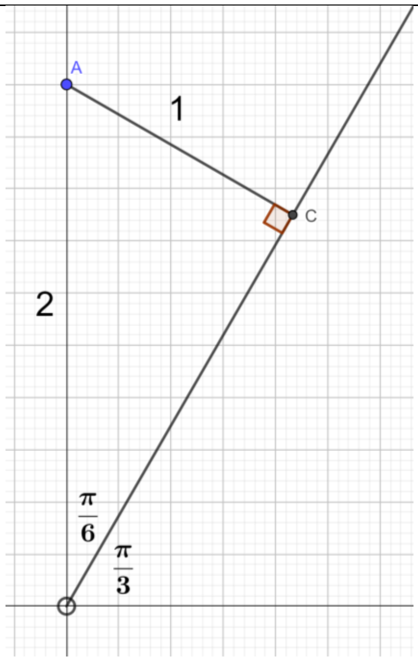
Question/ Outcome	**Solutions**	**Marking Guidelines**
(a)	$x \le 1 \quad \rightarrow \quad x - 1 \le 0$ $y \ge 1 \quad \rightarrow \quad y - 1 \ge 0$ $(x-1)(y-1) \le 0$ $xy - x - y + 1 \le 0$ $xy + 1 \le x + y$ $\therefore x + y \ge 1 + xy$	<u>2 marks</u>: correct solution <u>1 mark</u>: substantially correct solution
(b)	$(a-b)^2 \ge 0 \quad \Longrightarrow \quad a^2 + b^2 \ge 2ab$ $(b-c)^2 \ge 0 \quad \Longrightarrow \quad b^2 + c^2 \ge 2bc$ $(a-c)^2 \ge 0 \quad \Longrightarrow \quad a^2 + c^2 \ge 2ac$ $2a^2 + 2b^2 + 2c^2 \ge 2ab + 2bc + 2ac$ $a^2 + b^2 + c^2 \ge ab + bc + ac$ $(a+b+c)^2 - 2ab - 2bc - 2ac \ge ab + bc + ac$ $\therefore (a+b+c)^2 \ge 3ab + 3bc + 3ac$	<u>3 marks</u>: correct solution <u>2 marks</u>: substantially correct solution <u>1 mark</u>: partially correct solution

(c)	Assume, by way of contradiction, that $\sqrt{4n-2}$ is rational ie $\sqrt{4n-2} = \frac{p}{q}$, where p and q have highest common factor 1, and $q \neq 0$. $\therefore\ 4n-2 = \dfrac{p^2}{q^2}$ $2(2n-1) = \dfrac{p^2}{q^2}$ Since LHS is an even integer, $\frac{p^2}{q^2}$ must also be an even integer. Since HCF of p and q is 1, then HCF of p^2 and q^2 is also 1. Therefore $q^2 = 1$ such that $\frac{p^2}{q^2}$ is an even integer. $\therefore\ 4n-2 = \dfrac{p^2}{q^2} = p^2$ Hence, p^2 is even, and therefore p is also even. ie $p = 2k$ where k is an integer. $\therefore\ 4n-2 = (2k)^2$ $4n-2 = 4k^2$ $n - \dfrac{1}{2} = k^2$ $k = \sqrt{n - \frac{1}{2}}$ $\therefore$ k is *not* an integer. This is a contradiction (as $p = 2k$ where k is an integer) Hence the initial premise is false, and $\sqrt{4n-2}$ is ***not*** rational	3 marks: correct solution 2 marks: substantially correct solution (identifies ***and uses*** the factors of p^2 and q^2 and the idea of even/odd integers to establish the base premise leading up to the contradiction statement) 1 mark: partially correct solution (establishes the correct initial premise for a contradiction based proof, ***including*** the condition on p and q) Note: a common statement in solutions was along the lines of $2p^2 = b^2(2n-1)$ $\therefore\ b^2$ is even There needs to be an extra step of explanation as to *why* this is the case, eg noting that $2n - 1$ is odd, and odd x even = even

Question 8

Outcomes Addressed in this Question

MEX12-4 uses the relationship between algebraic and geometric representations of complex numbers and complex number techniques to prove results, model and solve problems

Question	Solutions	Marking Guidelines
(a)(i)(ii)		**(a)(i) 1 mark**: Correct ray from origin with angle labelled and open circle at the origin. *Note: The majority of responses did not include the open circle at the origin. This is a small but crucial detail when illustrating the boundary of an argument.* **(ii) 1 mark:** Perpendicular segment indicated, creating "exact value" right triangle with hypotenuse 2 units and angle pi/6, thus the perpendicular distance will equal 1 unit. **Aw1:** correct <u>perpendicular</u> segment indicated.
(iii)	$arg(z) = \frac{\pi}{3}$, $mod(z) = \sqrt{3}$ due to the exact values in the right triangle. $$\therefore z = \sqrt{3}\left(cos\frac{\pi}{3} + isin\frac{\pi}{3}\right)$$ *Note: Several responses to (iii) featured a circle that had no relation to the answers to (i) and (ii). Exam method requires understanding of the relationship within part (a) in this case.*	**(iii)** Correct answer. *Note: "Polar coordinates" are the modulus and argument.*

(b) **(i)**	$$P(z) = 2z^3 - 13z^2 + 26z - 10$$ $$P(3-i) = 2(3-i)^3 - 13(3-i)^2 + 26(3-i) - 10$$ $$= 2(18-26i) - 13(8-6i) + 26(3-i) - 10$$ $$= 36 - 52i - 104 + 78i + 78 - 26i - 10$$ $$= 0 + 0i$$ Since $P(3-i) = 0. (z-3+i)$ is a factor of P(z).	**(b)(i) 2 marks**: Correct solution with all relevant working shown. **1 mark**: Significant relevant progress and understanding communicated.
(ii)	All coefficients are real, so the complex roots occur in complex conjugate pairs. Use either the sum of the roots or the product of the roots to find the third root of the polynomial equation $P(z) = 0$. Either: $3 + i + 3 - i + \alpha = \frac{13}{2} \rightarrow \alpha = \frac{1}{2}$ Or: $(3+i)(3-i)\alpha = \frac{10}{2} \rightarrow 10\alpha = 5 \rightarrow \alpha = \frac{1}{2}$ $\therefore P(z) = 0$ gives $z = 3 \pm i, \frac{1}{2}$ *Note: Responses need to answer the question by clearly identifying the* ***3 values*** *that solve P(z)=0. Some sloppy working out was given the benefit of the doubt in this case but that may not always occur. Remember to give the best answer you can to increase the likelihood of marks being awarded.*	**(ii) 2 marks:** Acknowledgement of conjugate theorem and of sum/product of roots; Correct calculation of third root. **1 mark:** One component from above completed.
(iii)	The response needs to be a product of factors within the real number field, and all coefficients are to be integral. $\alpha = \frac{1}{2}$ and a leading coefficient of 2 gives one factor equal to $(2x-1)$ Hence $P(z)$ factorised over the real numbers will be: $$(2z-1)\big((z-3)-i\big)\big((z-3)+i\big)$$ $$= (2z-1)(z^2 - 6z + 9 + 1)$$ $$= (2z-1)(z^2 - 6z + 10)$$	**(iii)** Correct solution CFPA (ii)

2023 Yr12 HSC Assessment Task 2

Question 9 Solutions and Marking Guidelines

Outcomes Addressed in this Question

MEX12-4
uses the relationship between algebraic and geometric representations of complex numbers and complex number techniques to prove results, model and solve problems

Solutions	**Marking Guidelines**
a)	**3 marks** Correct solution •Includes test for n= 1 •Uses induction assumption •Applies correct trigonometric identity **2 marks** Establishes inductive step and uses assumption **1 mark** Correctly tests the base case

a)

Show true for base case $n = 1$

$$LHS = (\cos\theta - i\sin\theta)^1$$
$$RHS = \cos(1\times\theta) - i\sin(1\times\theta)$$
$$= LHS$$

Hence, true for $n = 1$.
Assume the statement is true for $n = k$ where k is some positive integer. That is:

$$(\cos\theta - i\sin\theta)^k = \cos(k\theta) - i\sin(k\theta)$$

Prove the statement true for $n = k + 1$. That is:

$$(\cos\theta - i\sin\theta)^{k+1} = \cos((k+1)\theta) - i\sin((k+1)\theta)$$

$$LHS = (\cos\theta - i\sin\theta)\,(\cos\theta - i\sin\theta)^k$$
$$= (\cos\theta - i\sin\theta)\,(\cos(k\theta) - i\sin(k\theta)) \qquad \text{(from assumption)}$$

$$= \cos(k\theta)\cos(\theta) - \sin(k\theta)\sin(\theta) - i(\sin(k\theta)\cos(\theta) + \cos(k\theta))\sin(\theta))$$
$$= \cos(k\theta + \theta) - isin(k\theta + \theta) \qquad \text{(compound angle identity)}$$

$$= \cos((k+1)\theta) - isin\big((k+1)\theta\big)$$

∴ if true for $n = k$, it is also true for $n = k + 1$.
∴ statement true by Mathematical Induction, for all positive integers.

b) i)

$$\frac{1}{1-z} = \frac{1}{1-\frac{1}{2}(\cos\theta + i\sin\theta)}$$ $$= \frac{2}{2-\cos\theta - i\sin\theta} \times \frac{2-\cos\theta + i\sin\theta}{2-\cos\theta + i\sin\theta}$$ $$= \frac{2(2-\cos\theta + i\sin\theta)}{(2-\cos\theta)^2 + \sin^2\theta}$$ $$= \frac{4-2\cos\theta + 2i\sin\theta}{4-4\cos\theta + \cos^2\theta + \sin^2\theta}$$ $$= \frac{4-2\cos\theta + 2i\sin\theta}{5-4\cos\theta}$$ Taking the imaginary part, we have: $\frac{2i\sin\theta}{5-4\cos\theta}$ Note: you can only take the imaginary part of a fraction involving complex numbers if the denominator is *realised*.	**3 marks** Correct solution **2 marks** Correctly realises the denominator **1 mark** Uses RHS part of geometric series
ii) It can be seen that $$1 + \frac{1}{2}\cos\theta + \frac{1}{2^2}\cos\theta + \frac{1}{2^3}\cos\theta + \cdots$$ is the real part of the geometric series in part (i). Hence, an expression for it can be $$\frac{4-2\cos\theta}{5-4\cos\theta}$$ 1 mark explicitly links expression to real part of i, 1 mark find correct expression.	**2 marks** Correct solution including reference to part (i) **1 mark** Makes substantial progress towards correct solution