YEAR 11

1996 4 UNIT MATHEMATICS TERM 4 ASSESSMENT

Time= 85 mins

- No equipment may be borrowed during the exam
- All necessary work must be shown. Marks may be deducted for careless or badly arranged work.

Question 1 (Start a new page)

(a) Write $1 + \dfrac{1+i}{1-i}$ in the form a + ib where a and b are real.

(b Find the two square roots of $-7 + 24i$

(c) Shade on an Argand diagram the region $|z| < |z - 2 + i|$

(d Describe,in geometric terms, the locus (in the Argand Plane) represented by $|z - 5i| = 3$

(e) Find the complex number z with the least positive argument satisfying the condition $|z - 5i| \le 3$

Question 2 (Start a new page)

(a) Define the modulus $|z|$ of a complex number z and show that every non-zero complex number z can be written in the form $r(\cos\theta + i\sin\theta)$ where $r = |z|$.

(b) Express $\dfrac{1}{1+i}$ in the form described in (a)

(c) If $z = r(\cos\theta + i\sin\theta)$,show that $\dfrac{z}{z^2 + r^2}$ is purely real and give its value

(d) The four complex numbers z_1, z_2, z_3, z_4 are represented on the compex (Argand) plane by the points A, B, C,D respectively. If $z_1 - z_2 + z_3 - z_4 = 0$ and $z_1 - i z_2 - z_3 + i z_4 = 0$, determine the possible shape (s) for the quadrilateral ABCD

Question 3 (Start a new page)

(a)(i)Find the cube roots of unity expressing them in mod-arg form. Show the roots on an argand diagram.

(ii) If w is one of the complex roots, prove the other root is w^2, and show that $1 + w + w^2 = 0$

(iii) Prove that $(1 + w - w^2)^3 - (1 - w + w^2)^3 = 0$

(iv) Prove that if n is a positive integer, then $1 + w^n + w^{2n} = 3$ or 0, depending on whether n is or is not a multiple of 3.

(b) Find the greatest value of the moduli of complex numbers z satisfying the equation

$$\left|z - \frac{4}{z}\right| = 2$$

QUESTION 4 (Start a new page)

(a)P, Q and R represent the complex numbers z_1, z_2, z_3 respectively, where $z_1 z_2 = z_3^2$. Show geometrically that OR bisects the angle POQ

(b) A variable point P on the Argand diagram represents the complex number z. α and β are fixed complex numbers. Describe and ilustrate with sketches the geometrical relations between the point P and :

(i) p_1 representing $z + \alpha$

(ii) p_2 representing βz

(c A sequence of points $z_1, z_2, \ldots\ldots\ldots$ is defined from an arbitrary point z_0 by the transformations

$$z_n = c\bar{z}_{n-1} + c - 1$$

where $c = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$.

Determine whether the sequence consists of repetitions of a finite set of points, and if so how many points there are in the set

THIS IS THE END OF THE TEST

4c 7 4 96

(a) $1+\frac{1+i}{1-i} = \frac{(1-i)+(1+i)}{1-i}$

$= \frac{2}{1-i}$

$= \frac{2}{1-i}\,\frac{1+i}{1+i}$ 1

$= \frac{2+2i}{2}$ 1

$= 1+i$

3

(b) $-7+24i = a^2-b^2+i2ab.$ 1

$\therefore a^2-b^2=-7$ 1

$ab=12$

$\therefore a^2-\frac{144}{a^2}=-7$

$a^4+7a^2-144=0.$

$(a^2+16)(a^2-9)=0$ 1

$a=\pm 3$

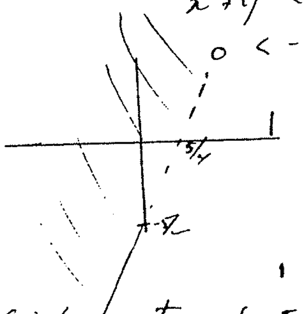
$\therefore$ roots $\pm(3+4i)$ 1

4.

(c) $z=x+iy$ $\quad |z|<|z-2+i|$

$x^2+y^2<(x-2)^2+(y+1)^2$ 1

$0<-4x+2y+5.$ 1



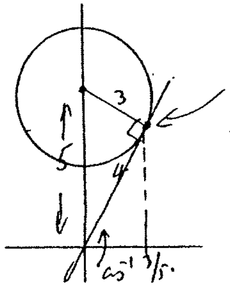
1

3

(d) Circle centre $(0,5)$ radius 2 1 1

2

(e)



$z=\left(4\times\frac{3}{5}+i\,4\times\frac{4}{5}\right)$

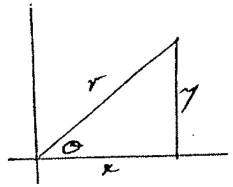
$=\left(\frac{12}{5}+\frac{16}{5}i\right)$

3.

Q2

(a) given $z=x+iy$ then $|z|=\sqrt{x^2+y^2}$ 1

$z=x+iy$
$=r(\cos\theta+i\sin\theta)$ 1



(b) $\frac{1}{1+i}$

$=\frac{1}{1+i}\,\frac{1-i}{1-i}$ 1

$=\frac{1-i}{2}$ 1

$=\frac{1}{\sqrt2}\left(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right)$ 2

(c) $\frac{r(\cos\theta+i\sin\theta)}{r^2(\cos2\theta+i\sin2\theta)+r^2}$ 1

$=\frac{1}{r}\,\frac{\cos\theta+i\sin\theta}{(1+\cos2\theta)+i2\sin\theta\cos\theta}$ 1

$=\frac{1}{r}\,\frac{\cos\theta+i\sin\theta}{2\cos^2\theta+i2\sin\theta\cos\theta}$ 1

$=\frac{1}{r}\,\frac{\sec\theta}{2}$ which is purely real 1

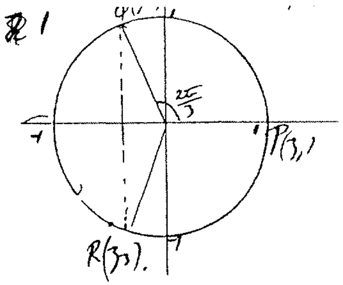
(d) If $z_1-z_2=z_4-z_3$ then AB & DC are equal & parallel. 1 1

If $z_1-z_3=i(z_2-z_4)$ then AC & BD are equal & perp. 1 1

$\therefore$ ABCD is a square. 1

Q3 a (i) $1, \ \text{cis}\,\frac{2\pi}{3}, \ \text{cis}\,\frac{4\pi}{3}$. 1

let roots be $z_1, z_2 + z_3$ represented by P, Q, R.



(ii) z_1 is real

If $z_2 = w = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$

then $z_3 = \left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$

$= \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^2$ 1

$= w^2$

+ Vice Versa. 1

& Since $1, w, w^2$ are the roots of $z^3 - 1 = 0$

then the sum of the roots $= 0$. 2

$\therefore 1 + w + w^2 = 0$.

(iii) $(1+w-w^2)^3 - (1-w+w^2)^3$

$= (-2w^2)^3 - (1-w-1-w)^3$ 2

$= -8w^6 + 8w^3$

$= 8w^3(1-w^3)$ --- $w^3 = 1$ since w is a root of $z^3 = 1$

$= 0$

(iv) let n be of form $3k, 3k+1, 3k+2$. 1

If $n = 3k$
$1 + w^n + w^{2n}$
$= 1 + w^{3k} + w^{6k}$
$= 1 + (w^3)^k + (w^3)^{2k}$
$= 1 + 1 + 1$
$= 3$

If $n = 3k+1$
$1 + w^n + w^{2n}$
$= 1 + w^{3k+1} + w^{6k+2}$
$= 1 + w.w^{3k} + w^2.w^{6k}$
$= 1 + w + w^2$
$= 0$

If $n = 3k+2$
$1 + w^n + w^{2n}$
$= 0.$ + 3

3b. Consider a △ with side lengths $|z|$, $\left|\frac{4}{z}\right|$ + $\left|z - \frac{4}{z}\right|$

Consider triangle inequality when $|z|$ is the larger side

$\therefore \left|\frac{4}{z}\right| + \left|z - \frac{4}{z}\right| \geq |z|$

$\left|\frac{4}{z}\right| + 2 \geq |z|$ 4

$|z|^2 - 2|z| \leq 4$

Complete the square

$|z| \leq \sqrt{5} + 1$

$\therefore$ greatest value is $\sqrt{5} + 1$

Q4 (a). If $z_1 z_2 = z_3$

then $\arg z_1 + \arg z_2 = 2\operatorname{Arg} z_3$

Let $\alpha = \arg z_1$

$\beta = \arg z_2$

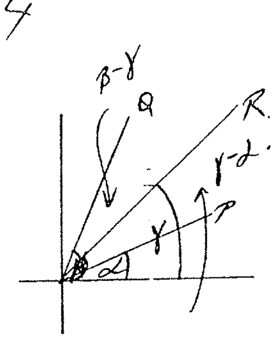
$\gamma = \arg z_3$

$\therefore$ We are given that $\alpha + \beta = 2\gamma$

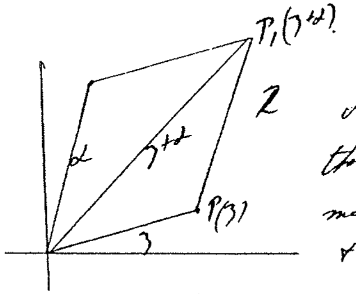
$\therefore \alpha - \gamma = \gamma - \beta$

$\therefore$ OR bisect $\boxed{POQ}$.

4



b). (i)



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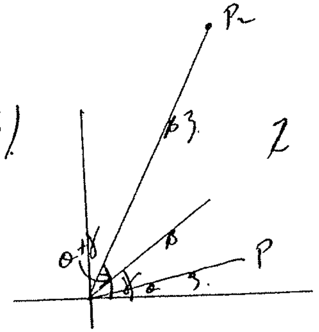
If $\alpha = a + ib$. 2

then P_1 is obtained from P by moving a units parallel to x axis & b units parallel to y axis.

(ii)



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2

P_2 is obtained from P by a rotation equal to $\arg \beta$ & an enlargement by a factor of $|\beta|$.

(c). $z_1 = c\bar{z}_0 + c - 1$ $\quad\quad \bar{z}_1 = \bar{c} z_0 + \bar{c} - 1$

$z_2 = c\bar{z}_1 + c - 1$

$= c(\bar{c} z_0 + \bar{c} - 1) + c - 1$

$= z_0$

Sequence has repetitions z_0 & z_1.

3

3 b. Consider a $\triangle$ with side lengths $|z|$ $\left|\frac{4}{z}\right|$ & $\left|z - \frac{4}{z}\right|$

Consider triangle inequality where $|z|$ is the larger side

$\therefore \left|\frac{4}{z}\right| + \left|z - \frac{4}{z}\right| \geq |z|$

$\left|\frac{4}{z}\right| + 2 \geq |z|$

$|z|^2 - 2|z| \leq 4$

Complete the square

$|z| \leq \sqrt{5} + 1$

$\therefore$ greatest value is $\sqrt{5} + 1$

4

Q4 (a). If $z_1 z_2 = z_3^2$

then $\arg z_1 + \arg z_2 = 2\operatorname{Arg} z_3$.

Let $\alpha = \arg z_1$

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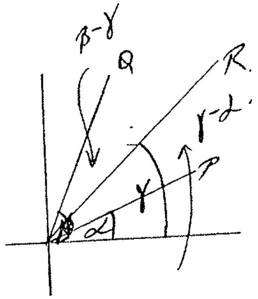
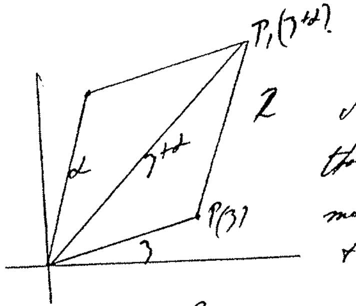
$\therefore$ OR bisects $\angle POQ$.

4



b). (i)



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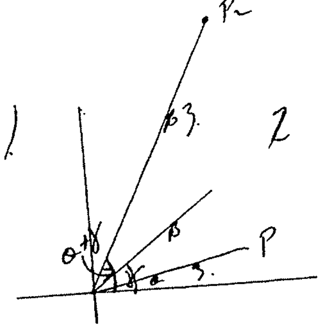
If $\alpha = a + ib$ 2

then P_1 is obtained from P by moving a units parallel to x axis & b units parallel to y axis.

(ii)



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2

P_2 is obtained from P by a rotation of equal to $\arg \beta$ & an enlargement by a factor of $|\beta|$.

(c). $z_1 = c\bar{z}_0 + c - 1$

$\bar{z}_1 = \bar{c} z_0 + \bar{c} - 1$

$z_2 = c\bar{z}_1 + c - 1$

$= c(\bar{c} z_0 + \bar{c} - 1) + c - 1$

$= z_0$

3

Sequence has repetitions z_0 & z_1.