# Year 11 - Assessment 1997

## TERM 4 – 4 UNIT

**Time allowed: 85 minutes.**
All questions are to be attempted
Each question is to be handed in separately.

## Section A ( START A NEW PAGE)

### Question 1 :

(a) If $z = -2 + 2i$ , express in the form $x + iy$ .

(i) $\bar{z}$

(ii) $\frac{1}{z}$

(iii) $z^2$

(b) Plot on the Argand diagram the points $z$, $\bar{z}$ , $\frac{1}{z}$, clearly showing the relations between their modulus and arguments.

### Question 2 :

A parallelogram ABCD is drawn with $\overrightarrow{OA} = 3 + 2i$ , $\overrightarrow{OC} = 11 + 9i$ and $\overrightarrow{AB} = 2 + 5i$.
Find the vectors $\overrightarrow{OB}$ and $\overrightarrow{OD}$.

### Question 3 :

(i) Write in mod-arg form $4\sqrt{3} - 4i$

(ii) hence describe the transformation $z_1$ to $z_2$ if $z_2 = (4\sqrt{3} - 4i)\, z_1$ .

(iii) If $z_1 = 2 + 2\sqrt{3}\, i$ , find the point $z_2$ .

## Section B (START A NEW PAGE)

### Question 1 :

(i) Describe the locus $|\, z - 2 + i \,| = 3$.

(ii) Hence find the maximum value of $|\, z \,|$ .

### Question 2 :

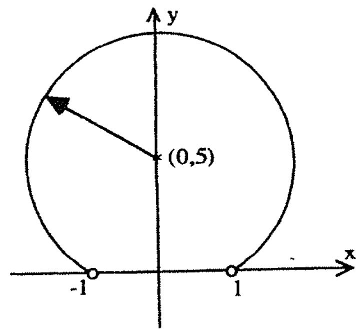
(i) Graph the locus $\mathrm{Arg}(\, z - 3 + i\,) = -\frac{\pi}{4}$.

(ii) Graph the locus $\mathrm{Im}(\, z^2\,) = 8$ .

### Question 3 :

The locus of a point z is shown. (see diagram)

Evaluate correct to 2 decimal places:
$\mathrm{Arg}\,(\, z - 1\,) - \mathrm{Arg}\,(\, z + 1\,)$.




### Question 4 :

(i) Find the Cartesian equation for the locus $|\, z - 1 + 2i\,| = 3|\, z + 3 - 4i\,|$.

(ii) Describe the locus.

## Section C : (START A NEW PAGE)

### Question 1:

(i) Find the exact square roots of $16 - 30i$ , and

(ii) Hence solve : $z^2 - (\,3 - i\,)\,z - 2 + 6i = 0$ .

### Question 2 :

(i) Solve $z^4 + 16 = 0$ .

(ii) Plot the zeros of $z^4 + 16$ on a circle on the Argand diagram if

$\mathrm{Arg}\,(z_1) < \mathrm{Arg}\,(z_2) < \mathrm{Arg}\,(z_3) < \mathrm{Arg}\,(z_4) < \pi$ .

(iii) Show graphically the vector addition of $z_1 + z_2 + z_3 + z_4$ .

(iv) Evaluate $\mathrm{Arg}(z_1 - z_3) - \mathrm{Arg}(z_2 - z_4)$ .

(v) Factorise $z^4 + 16$ over R .

## Section D (START A NEW PAGE)

### Question 1 :

(i) Given that z is a complex number with $|z| = 1$, use DeMoivres' Theorem to prove that $\cos n\theta = \dfrac{z^n + z^{-n}}{2}$.

(ii) **Hence** deduce: $\cos\theta \cos 2\theta = \frac{1}{2}(\cos 3\theta + \cos\theta)$

### Question 2 :

(i) Prove $|z|^2 = z\bar{z}$

(ii) A parallelogram OABC has diagonals intersecting at M.

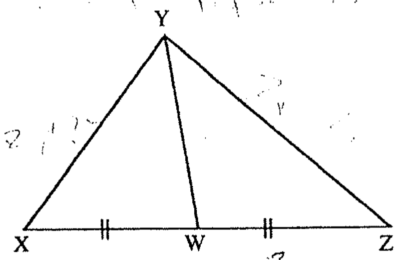
Let OA be $z_1$, OC be $z_2$ and $0 < \text{Arg}(z_1) < \text{Arg}(z_2) < \frac{\pi}{2}$.

Draw the parallelogram OABC clearly showing the point M and the vectors $z_1$, $z_2$, $z_1 + z_2$, $z_1 - z_2$

(iii) Prove that: $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + 2|z_2|^2$.

(iv) Hence prove Apollonius' Theorem for a triangle XYZ

$XY^2 + YZ^2 = 2(YW^2 + XW^2)$

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4 UNIT YR 11 1997

**1**

(i) $\bar{z} = -2-2i$ (1)

(ii) $\dfrac{1}{z} = \dfrac{1}{-2+2i} \times \dfrac{2+2i}{2+2i}$

$= \dfrac{2+2i}{8}$

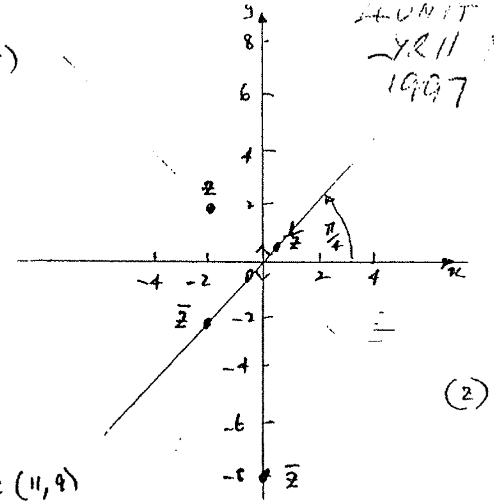
$= \dfrac{1}{4} + \dfrac{1}{4}i$ (2)

(iii) $z^2 = (-2+2i)^2$

$= 4 - 8i - 4$

$= -8i$ (2)

(b)

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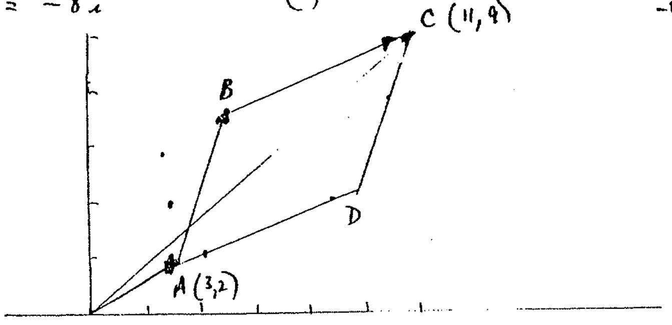



$\vec{OB} = \vec{OA} + \vec{AB}$

$= 3+2i + 2+5i$

$= 5+7i$

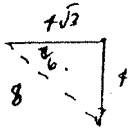
$\vec{OD} = \vec{OC} - \vec{CD}$ But $\vec{CD} = \vec{AB}$

$= 11+9i - (2+5i)$

$= 9+4i$ (2)

**13** (i) $4\sqrt{3} - 4i$

$|z| = 8$ $\operatorname{Arg} z = -\frac{\pi}{6}$




$\therefore z = 8\left[\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right]$ (2)

(ii) $z_1 \to z_2$ Modulus of $z_1$ is increased by 8 and its argument decreased by $\frac{\pi}{6}$. (2)

(iii) $z_2 = (4\sqrt{3} - 4i)(2 + 2\sqrt{3}i)$

$= 8\sqrt{3} + 24i - 8i + 8\sqrt{3}$

$= 16\sqrt{3} + 16i$ (2)

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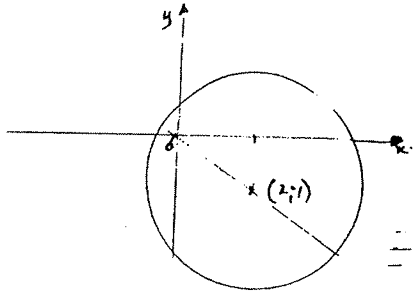
circle centre $(2, -1)$ radius 3 units (2)

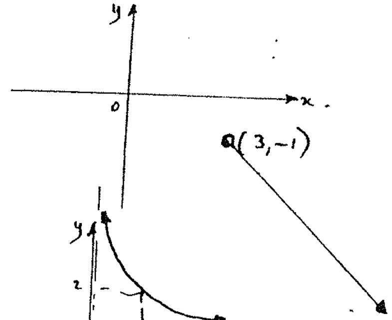
(ii) $\max |z| = \sqrt{2^2 + (-1)^2} + 3$

$= 3 + \sqrt{5}$ units.




(2).

**2** (i) $\operatorname{Arg}(z - 3 + i) = -\frac{\pi}{4}$



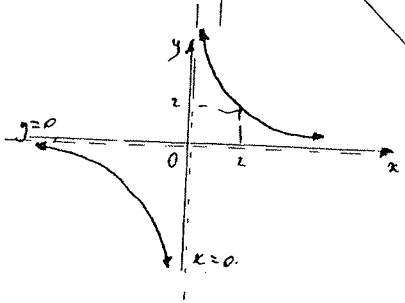

(2)

(ii) $z^2 = x^2 - y^2 + 2xyi$

$\therefore \operatorname{Im}(z^2) = 8$

$2xy = 8$

$xy = 4$

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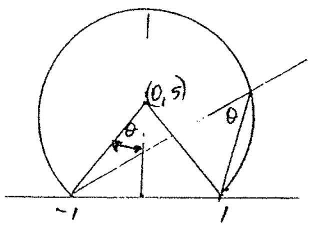
**3**




$\operatorname{Arg}(z-1) - \operatorname{Arg}(z+1) = \theta$ (shown)

$\tan\theta = \frac{1}{5}$

$\theta = 0.20$ (2 dec places) (3)

**4** $|z - 1 + 2i| = 3|z + 3 - 4i|$

$(x-1)^2 + (y+2)^2 = 9\left[(x+3)^2 + (y-4)^2\right]$

$x^2 - 2x + 1 + y^2 + 4y + 4 = 9\left[x^2 + 6x + 9 + y^2 - 8y + 16\right]$

$8x^2 + 8y^2 + 56x - 76y + 220 = 0$

$x^2 + 7x + \frac{49}{4} + y^2 - \frac{19}{2}y + \frac{361}{16} = -\frac{55}{2} + \frac{49}{4} + \frac{361}{16}$

$\left(x + \frac{7}{2}\right)^2 + \left(y - \frac{19}{4}\right)^2 = \frac{117}{16}$

Locus: circle centre $\left(-\frac{7}{2}, \frac{19}{4}\right)$ radius $\frac{3}{4}\sqrt{13}$ (4)

$(x+iy)^2 = 16-30i$

$\therefore \quad x^2-y^2 = 16$

$2xy = -30$

$y = \frac{-15}{x}$

$\therefore \quad x^2 - \frac{225}{x^2} = 16$

$x^4 - 16x^2 - 225 = 0$

$(x^2-25)(x^2+9) = 0$

$x = \pm 5$ only $x \in R$. $x^2+9=0$ No real soln.

$x=5 \quad y=-3$

$x=-5 \quad y=3$

$\therefore$ Roots are $\pm(5-3i)$ (2)

ii) $z^2 - (3-i)z - 2 + 6i = 0$

$$z = \frac{3-i \pm \sqrt{(3-i)^2 - 4(-2+6i)}}{2}$$

$$= \frac{3-i \pm \sqrt{9-6i-1+8-24i}}{2}$$

$$= \frac{3-i \pm \sqrt{16-30i}}{2}$$

$$= \frac{3-i \pm (5-3i)}{2}$$

$$= \frac{8-4i}{2} \qquad \frac{-2+2i}{2}$$

$z = 4-2i, \; -1+i$ (2)

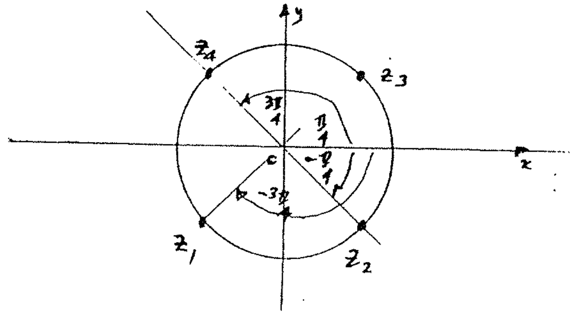
$z^4 = -16$

$z^4 = 16 \operatorname{cis}(\pi + 2\pi n)$ n integer.

$\therefore \; z = 2 \operatorname{cis}\left(\frac{\pi + 2\pi n}{4}\right)$

$z = 2\operatorname{cis}\frac{\pi}{4}, \; 2\operatorname{cis}\frac{3\pi}{4}, \; 2\operatorname{cis}-\frac{\pi}{4}, \; 2\operatorname{cis}-\frac{3\pi}{4}.$

(ii)

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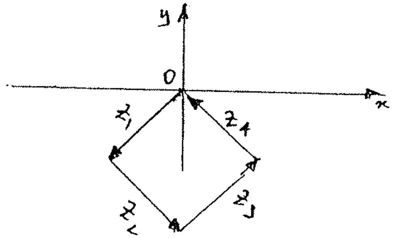


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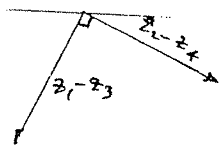



$\operatorname{Arg}(z_1-z_3) - \operatorname{Arg}(z_2-z_4) = -\frac{\pi}{2}.$

(v) $z^4+16 = \left(z-2\left[\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right]\right)\left(z-2\left[\cos-\frac{\pi}{4} - i\sin\left(-\frac{\pi}{4}\right)\right]\right)\left(z-2\left[\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\right]\right)\left(z-2\left[\cos\frac{-3\pi}{4} - \ldots\right]\right)$

$= \left[\left(z-2\cos\frac{\pi}{4}\right)^2 + 4\sin^2\frac{\pi}{4}\right]\left[\left(z-2\cos\frac{3\pi}{4}\right)^2 + 4\sin^2\frac{3\pi}{4}\right]$

$= \left[z^2 - 4\cos\frac{\pi}{4}z + 4\cos^2\frac{\pi}{4} + 4\sin^2\frac{\pi}{4}\right]\left[z^2 - 4\cos\frac{3\pi}{4}z + 4\cos^2\frac{3\pi}{4} + 4\sin^2 \ldots\right]$

$= \left[z^2 - 4\cdot\frac{1}{\sqrt{2}}z + 4\right]\left[z^2 - 4\cdot\frac{-1}{\sqrt{2}}z + 4\right]$

$= \left[z^2 - 2\sqrt{2}z + 4\right]\left[z^2 + 2\sqrt{2}z + 4\right]$

OR $\quad z^4+16 = (z^2+4)^2 - 8z^2$

$= (z^2+4+2\sqrt{2}z)(z^2+4-2\sqrt{2}z)$ (2

$(\cos\theta + i\sin\theta)^{-n} = \cos n\theta - i\sin n\theta$

∴ If $z = \cos\theta + i\sin\theta$ $\quad |z| = 1$

$z^n + z^{-n} = 2\cos n\theta$ (2)

$\cos n\theta = \frac{z^n + z^{-n}}{2}$

ii) $\cos\theta\cos 2\theta = \frac{1}{2}(z + z^{-1}) \cdot \frac{1}{2}(z^2 + z^{-2})$

$= \frac{1}{4}\left[z^3 + z^{-1} + z + z^{-3}\right]$

$= \frac{1}{2}\left[\frac{z^3 + z^{-3}}{2} + \frac{z + z^{-1}}{2}\right]$ (3)

ie $\cos\theta\cos 2\theta = \frac{1}{2}(\cos 3\theta + \cos\theta)$

(i) If $z = x + iy$ then $\bar{z} = x - iy$

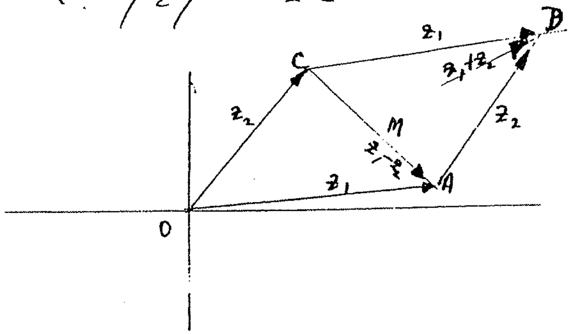
$z\bar{z} = (x+iy)(x-iy)$

$= x^2 + y^2$ (2)

$= |z|^2$

∴ $|z|^2 = z\bar{z}$

ii)

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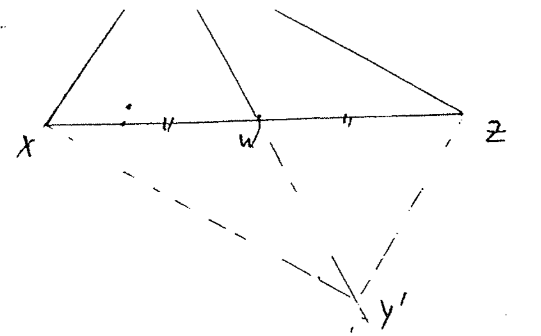
ii) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) + (z_1 - z_2)(\bar{z}_1 - \bar{z}_2)$

$= z_1\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_1 + z_2\bar{z}_2 + z_1\bar{z}_1 - z_1\bar{z}_2 - z_2\bar{z}_1 + z_2\bar{z}_2$

$= |z_1|^2 + |z_2|^2 + |z_1|^2 + |z_2|^2$

$= 2|z_1|^2 + 2|z_2|^2$ (3)




Form parallelogram $XYZY'$

From (iii) $XZ^2 + (YY')^2 = 2XY^2 + 2YZ^2$

$(2XW)^2 + (2YW)^2 = 2XY^2 + 2YZ^2$

$4XW^2 + 4YW^2 = 2XY^2 + 2YZ^2$

$XY^2 + YZ^2 = 2XW^2 + 2YW^2$ (3)

$XY^2 + YZ^2 = 2\left[XW^2 + XW^2\right]$