JAMES RUSE AGRICULTURAL HIGH SCHOOL
1998 4 UNIT MATHEMATICS
TERM 4 ASSESSMENT

Time: 85 minutes.
No equipment may be borrowed during the examination.
All necessary working must be shown.

Question 1

(a) If $z = 2-3i$ and $\omega = -1-i$, find

(i) $|z|$

(ii) $\arg \omega$

(iii) $\omega \bar{z}$

(b) Find $\sqrt{-5-12i}$ in the form $a + ib$.

(c) If $z = 1$ is one of the roots of the equation $iz^2 + (ia - 1)z + (i-b) = 0$, where a and b are real, find:

(i) the values of a and b.
(ii) the other root.

(d) If $z = 4\lambda + 3i(1-\lambda)$ and λ is real,

(i) prove that the point P which represents z on the Argand plane lies on a straight line.

(ii) find the least value of $|z|$.

Question 2 (Start a new page)

(a) Express in mod–arg form:

(i) $\dfrac{5+i}{2+3i}$

(ii) $i\bar{z}$ if $z = 4(\cos\theta + i\sin\theta)$.

(b) Find all n for which $(2 - 2\sqrt{3}\,i)^n$ is real.

(c) What single transformation maps z to:

(i) $\bar{z}$
(ii) iz

(d) Show that there is only one value of z for which both $\arg(z+i) = \dfrac{\pi}{4}$ and $|z+1| = 2$, and that this value is real.

Question 3 (Start a new page)

(a) Given $z = x + iy$ and $P(x,y)$ moves in the complex plane, graph the locus of P and state its Cartesian equation if:

(i) $| z - i | = | z + 1 |$

(ii) $\arg\left(\dfrac{z+2}{z-2}\right) = \dfrac{\pi}{2}$

(iii) $\operatorname{Im}\left(z + \dfrac{1}{z} \right) = 0$

(b) Using suitable diagram(s) on the Argand plane, prove the following inequalities from **geometric considerations** alone. Give clear reasons.

(i) $| z_1 + z_2 | \leq | z_1 | + | z_2 |$.

(ii) $| z - 1 | \leq \big| |z| - 1 \big| + |z| \, | \arg z |$, where $-\pi < \arg z \leq \pi$.

Question 4 (Start a new page)

(a) If $\alpha = \cos\dfrac{2\pi}{7} + i \sin\dfrac{2\pi}{7}$, $\beta = \alpha + \alpha^2 + \alpha^4$ and $\gamma = \alpha^3 + \alpha^5 + \alpha^6$,

(i) prove that $\beta + \gamma = -1$ and $\beta\gamma = 2$.

(ii) Hence show that: $\sin\dfrac{2\pi}{7} + \sin\dfrac{4\pi}{7} + \sin\dfrac{8\pi}{7} = \dfrac{\sqrt{7}}{2}$.

(b) In the Argand plane, consider the triangle ABC with vertices at $A(-1,0)$, $B(1,0)$ and C represents the complex number z, for $-\pi < \arg z \leq \pi$.
Let L be a straight line through the origin and parallel to the bisector of $\angle ACB$.
Prove that both values of $\sqrt{z^2 - 1}$ lie on the line L, clearly labelling all important features in your diagram.

(c) The points $P_1, P_2, P_3, \ldots\ldots, P_n$ representing the complex numbers $z_1, z_2, z_3, \ldots\ldots, z_n$ lie below the straight line $y = \sqrt{3}\,x$.

(i) The points $Q_1, Q_2, Q_3, \ldots\ldots, Q_n$ representing $\dfrac{1}{z_1}, \dfrac{1}{z_2}, \dfrac{1}{z_3}, \ldots\ldots, \dfrac{1}{z_n}$ possess a similar property. State this property and the equation of the corresponding straight line.

(ii) Show that $z_1 + z_2 + \ldots\ldots\ldots\ldots + z_n \neq 0$, for **any** set of n points, and similarly $\dfrac{1}{z_1} + \dfrac{1}{z_2} + \dfrac{1}{z_3} \ldots\ldots\ldots\ldots \dfrac{1}{z_n} \neq 0$.

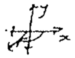
END of PAPER

1998 4U Term 4 Assessment Answers.

QUESTION 1.

[4]

a) (i) $|z| = \sqrt{2^2+3^2} = \underline{\sqrt{13}}$ 1 ①

(ii) $\arg \omega = \underline{-\frac{3\pi}{4}}$ 1  ①

(iii) $\omega \bar{z} = (-1-i)(2+3i)$ 1

$= -2-3i-2i+3$

$= \underline{1-5i}$ 1 ②

[3]

b) $\sqrt{-5-12i} = x+iy$

$-5-12i = (x+iy)^2$

$\begin{cases} x^2-y^2 = -5 \\ 2xy = -12 \end{cases}$ 1 $\Rightarrow y = \frac{-6}{x}$ ③

$\therefore x^2 - \frac{36}{x^2} = -5$

$x^4+5x^2-36=0$ 1 $\therefore x = 2$ or $x=-2$

$\Downarrow \quad \Downarrow$

$y=-3 \quad y=3$

$\therefore$ roots $\underline{2-3i,\ -2+3i}$ 1

[5]

c) (i) $i + ia - 1 + i - b = 0$ 1 ③

$\begin{cases} -1-b = 0 \\ 2+a = 0 \end{cases}$ $\therefore b=-1$ 1

$\underline{a=-2}$ 1

(ii) $\alpha+\beta = -a + \frac{1}{i} = 2-i$ 1 ②

$\therefore \alpha = 1 \quad \therefore$ other root $\underline{+1-i}$ 1

[3]

d) (i) $z = 4\lambda + 3i(1-\lambda)$ where $z = x+iy$

$\begin{cases} x = 4\lambda \\ y = 3(1-\lambda) \end{cases}$ 1 $\lambda = \frac{x}{4}$

$y = 3 - \frac{3x}{4}$ 1 ②

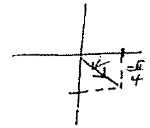
$\therefore P$ lies on the straight line $\underline{3x+4y=12}$

(ii) $d = \frac{|0+0-12|}{\sqrt{3^2+4^2}} = \underline{2\frac{2}{5}}$ ①

1

QUESTION 2.

[6]

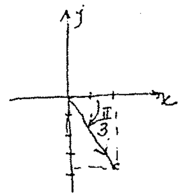
a) (i) $\frac{5+i}{2+3i} = \frac{(5+i)(2-3i)}{(2+3i)(2-3i)} = \frac{13-13i}{13} = 1-i$ 1  ③

$= \underline{\sqrt{2}\left(\cos\frac{-\pi}{4} + i\sin\frac{-\pi}{4}\right)}$ 1

(ii) $i\bar{z} = i(4\cos\theta - 4i\sin\theta)$ 1 $= 4(\sin(-\theta) + i\cos(-\theta))$ 1 ③

$= \underline{4\left(\cos\left(\frac{\pi}{2}+\theta\right) + i\sin\left(\frac{\pi}{2}+\theta\right)\right)}$ 1

[3]

b) 

$(2-2\sqrt{3}\,i)^n = \left[4\left(\cos\frac{-\pi}{3} + i\sin\frac{-\pi}{3}\right)\right]^n$

$= 4^n\left(\cos\frac{-n\pi}{3} + i\sin\frac{-n\pi}{3}\right)$ ③

Real for $\sin\frac{-n\pi}{3} = 0$

$\therefore \frac{n\pi}{3} = k\pi$ 1 $\therefore \underline{n=3k}$, k integer

[2]

c) $z \to \bar{z}$ $\underline{\text{reflexion on } X \text{ axis}}$ 1 ②

$iz \to i\bar{z}$ $\underline{\text{rotation of } \frac{\pi}{2}, \text{ anticlockwise}}$ 1

[4] if from diagram give 4 marks

d) $\begin{cases} \arg(z+i) = \frac{\pi}{4} & \Rightarrow y = x-1,\ x>0 \\ |z+1| = 2 & \Rightarrow (x+1)^2+y^2 = 4 \end{cases}$ 1 1 ④

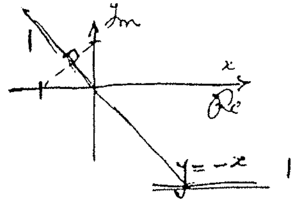
$\therefore 2x^2+2=4 \quad \therefore$ 1 $x = \pm 1$

but $x>0 \quad \therefore$ $\underline{x=1 \text{ is the only solution}}$

$\therefore \underline{z=1}$ 1

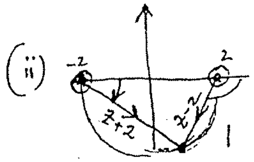
QUESTION 3.

a) (i)



$|z-i| = |z+i|$ ②

(ii)



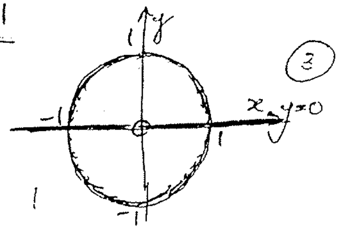
$\arg\left(\frac{z+2}{z-2}\right) = \frac{\pi}{2}$ ③

$x^2+y^2=4, \; y<0$

(iii) $\mathrm{Im}\left(z+\frac{1}{z}\right)=0$

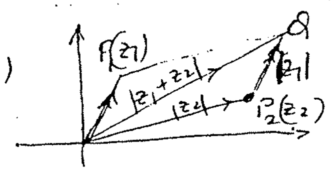
Either z real $\neq 0$ or $\frac{1}{z} = \bar{z}$

$\therefore \; y=0, \; x\neq 0$, $x^2+y^2=1$



③

b) (i)



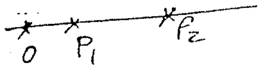
OP_1QP_2 parm by construction

In $\triangle OQP_2$ $OQ < OP_2 + P_2Q$

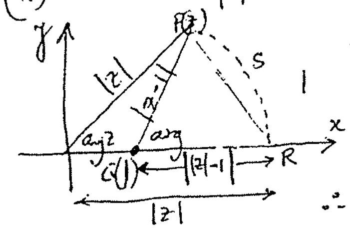
"each side of a $\triangle$ is less than the sum of the other two sides"

$\therefore |z_1+z_2| < |z_1| + |z_2|$

But if O, P_1, P_2 are in a straight line then the equality stands $|z_1+z_2| = |z_1|+|z_2|$



(ii) Case 1. $|z|>1$



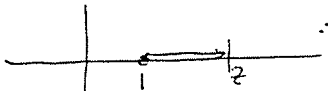
$|z-1| < \big||z|-1\big| + |PR|$

but arc $SR = |z|\,|\arg z|$

$l = r\theta$, $|PR| <$ arc

$\therefore |z-1| < \big||z|-1\big| + |z|\,|\arg z|$ ④

Case 2. z real with $\theta = 0$



$\therefore |z-1| = \big||z|-1\big|$

$\therefore$ equality applies since $|\arg z| = 0$

$|z-1| = \big||z|-1\big| + |z$ $|z-1| = \big||z|-1\big| + |z|\,|\arg z|$

Bonus mark if student considers $\boxed{|z|<1}$ and $\boxed{|z|=1 \text{ chord less}}$

QUESTION 4.

a) $z^7 = 1 = \cos 2n\pi + i\sin 2n\pi \Rightarrow \alpha = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$

$\beta = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7} + \cos\frac{4\pi}{7} + i\sin\frac{4\pi}{7} + \cos\frac{8\pi}{7} + i\sin\frac{8\pi}{7}$

$\gamma = \cos\frac{6\pi}{7} + i\sin\frac{6\pi}{7} + \cos\frac{10\pi}{7} + i\sin\frac{10\pi}{7} + \cos\frac{12\pi}{7} + i\sin\frac{12\pi}{7}$

$\alpha^7 = 1$

$\beta + \gamma = \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \alpha^6 (+\alpha^7 - \alpha^7)$ ②

$= (\text{sum of roots of } z^7 - 1 = 0) - 1$

$= 0 - 1 = -1$ $\qquad \Sigma \text{ roots} = \frac{-b}{a} = \frac{0}{1} = 0$

[7]

$\beta\gamma = (\alpha + \alpha^2 + \alpha^4)(\alpha^3 + \alpha^5 + \alpha^6)$

$= \alpha^4 + \alpha^6 + 1 + \alpha^5 + 1 + \alpha + 1 + \alpha^2 + \alpha^3$

$= \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \alpha^6 + 3$

$= -1 + 3 = 2$

$\alpha^7 = 1$, $\alpha^8 = \alpha$, $\alpha^9 = \alpha^2$, $\alpha^{10} = \alpha^3$ ②

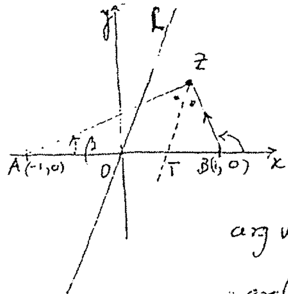
(ii) $\sin\frac{2\pi}{7} + \sin\frac{4\pi}{7} + \sin\frac{8\pi}{7} = \mathrm{Im}(\beta)$

To find β & γ: $x^2 + x + 2 = 0 \therefore x = \frac{-1 \pm \sqrt{7}\,i}{2}$ ③

$\therefore \beta = \frac{-1+\sqrt{7}\,i}{2}$, $+\sqrt{7}$ since $\sin\frac{4\pi}{7} > \sin\frac{8\pi}{7}$ $\therefore$ sum of three sines is >0.

$\therefore \sin\frac{2\pi}{7} + \sin\frac{4\pi}{7} + \sin\frac{8\pi}{7} = \frac{\sqrt{7}}{2}$

b)



$\arg(z-1) = \alpha$, $\arg(z+1) = \beta$

$\angle TZB = \frac{\alpha-\beta}{2}$ (external angle of $\triangle AZB$) $\therefore \angle ZTB = \frac{\alpha+\beta}{2}$ (angle sum of $\triangle TZB$)

[4]

$\therefore \angle LOX = \frac{\alpha+\beta}{2}$ ④

$\arg\sqrt{z^2-1} = \frac{1}{2}\left[\arg(z-1) + \arg(z+1)\right]$

$\therefore \arg(\sqrt{z^2-1}) = \frac{1}{2}(\alpha+\beta) = \angle LOX$

$\therefore$ both values of $\sqrt{z^2-1}$ lie on line L.

c) (i) Since $\arg\frac{1}{z} = -\arg z$ the points representing $\frac{1}{z_1}, \frac{1}{z_2} \cdots \frac{1}{z_n}$ lie above the line $y = -\sqrt{3}\,x$ ②

[5]

(ii) By rotating the full plane $\frac{\pi}{6}$ anticlockwise, all points will have $x > 0$ $\therefore$ let $\omega = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$

$\therefore \omega z_1 + \omega z_2 + \cdots + \omega z_n \neq 0$ since $\mathrm{Re}(\omega z_i) > 0$

$\therefore \mathrm{Re}(\omega z_1 + \omega z_2 + \cdots + \omega z_n) > 0$

$\therefore \omega(z_1 + z_2 + \cdots + z_n) \neq 0$ since $z = 0$ iff $x=0, y=0$

$\therefore z_1 + z_2 + \cdots + z_n \neq 0$ ③

Similarly for $\frac{1}{z_1}, \frac{1}{z_2}, \ldots \frac{1}{z_n}$ with $\omega = \mathrm{cis}\left(\frac{-\pi}{6}\right)$