JAMES RUSE AGRICULTURAL HIGH SCHOOL
YEAR 11 TERM 4 ASSESSMENT
4 UNIT 1999

Time: 85 minutes

QUESTION 1 : START A NEW PAGE (6 MARKS)

(a) Given $z_1 = 3 + i$ and $z_2 = 4 - 2i$

find (i) $|z_1|$

(ii) $\overline{z_2}$

(iii) $\dfrac{z_1}{z_2}$ in the form $a + bi$.

(b) Graph on the Argand diagram the position vectors $z_1, \overline{z_2}$ and z_3 where $z_3 = \dfrac{z_1}{z_2}$

QUESTION 2 : START A NEW PAGE (14 MARKS)

(i) Solve the equation $z^3 = 1$ for the roots P_0, P_1 and P_2 expressed in both mod – arg form and the form $a + ib$.

(ii) Show these roots on the Argand diagram clearly indicating their relative positions.

(iii) Deduce that if ω is a non – real cube root of unity then $1 + \omega + \omega^2 = 0$

(iv) Given $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ and if $e = a + b$,

$f = a\omega + b\omega^2$ and $g = a\omega^2 + b\omega$ where ω is a non – real cube root of unity

simplify $e^3 + f^3 + g^3$ in terms of a and b.

(v) Given $(2 - 3i)^3 = -46 - 9i$ find another value of z such that $z^3 = -46 - 9i$

QUESTION 3 : START A NEW PAGE (20 Marks)

(a) (i) Graph the region : $|z - 4 + i| \leq 1$

(ii) Graph the locus : $\text{Arg}(z - 3i) = \dfrac{-\pi}{4}$

(b) A complex root of $z^2 + (3 - i)z + k = 0$ is $2 + i$.

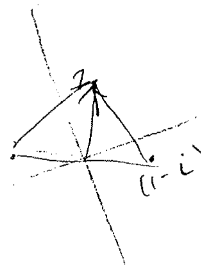
Find (i) the value of the other root

(ii) the value of k

(c) Given that $z^2 = 8 - 6i$ (i) find the values of z

(ii) show that one of the values of z is a solution to the equation

$$z^4 - 15z^2 - 6z + 110 = 0$$



(d) If $\dfrac{z - 1 + i}{z + 1 - i} = ki$ where $k \in R$ find with the aid of a diagram the value of $|z|$, giving reasons.

QUESTION 4: START A NEW PAGE (20 MARKS)

(a) Describe the locus of :

(i) $|z| = |z-3|$

(ii) $\text{Arg}(z-3) - \text{Arg}(z+3) = \frac{\pi}{2}$

(iii) What value of z satisfies both (i) and (ii) ?

(b) Simplify $(\sqrt{3}+i)^{12} + (\sqrt{3}-i)^{12}$

(c) If $z = \cos\theta + i\sin\theta$ then

(i) show that $\cos n\theta = \dfrac{z^n + z^{-n}}{2}$ and $\sin n\theta = \dfrac{z^n - z^{-n}}{2i}$

(ii) find an expression in z as a fraction in simplest terms for

$$1 + \cos\theta + \cos 2\theta + \ldots\ldots\ldots\ldots + \cos n\theta$$

(iii) find an expression in z for $\dfrac{\cos\frac{n\theta}{2}\sin\frac{(n+1)}{2}\theta}{\sin\frac{\theta}{2}}$

hence deduce a relationship between (ii) and (iii) .

END OF EXAM

Y11 - 4U - 1994 TERM 4

1 (a) (i) $|z_1| = \sqrt{3^2+1^2}$

$= \sqrt{10}$ units ①

(ii) $\bar{z}_2 = 4+2i$ ①

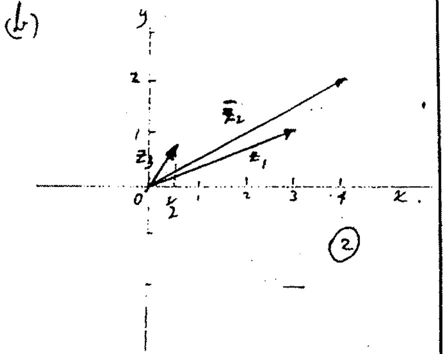
(iii) $\dfrac{z_1}{z_2} = \dfrac{3+i}{4-2i} \cdot \dfrac{4+2i}{4+2i}$

$= \dfrac{12+7i-2}{16+4}$

$= \dfrac{10+7i}{20}$

$= \dfrac{1}{2} + \dfrac{7}{20}i$ ②

(b)



②

2 (i) $z^3 = 1$

$z^3 = 1\left[\cos 2\pi k + i\sin 2\pi k\right]$

$z = \cos\dfrac{2\pi k}{3} + i\sin\dfrac{2\pi k}{3}$

$P_0 = \operatorname{cis} 0$

$\therefore k=0 \quad P_0 = 1+0i$ ②

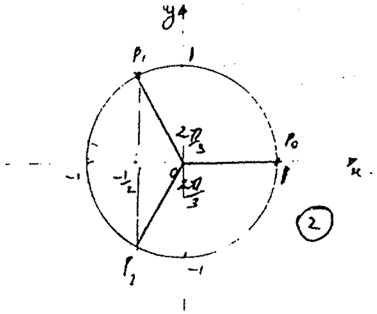
$k=1 \quad P_1 = \cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}$

$= -\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i$ ②

$k=2 \quad P_2 = \cos\dfrac{4\pi}{3} + i\sin\dfrac{4\pi}{3}$

$= -\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i$ ②

(ii)



②

(iii) $w^3 = 1$

$w^3 - 1 = 0$

$(w-1)(w^2+w+1) = 0$ ①

If w is non real $w \neq 1$

$\therefore w$ satisfies $w^2+w+1=0$

(iv) $f = aw + bw^2 \qquad g = aw^2 + bw$

$\therefore f^3 = a^3w^3 + 3a^2bw^4 + 3ab^2w^5 + b^3w^6$

$= a^3 + 3a^2bw + 3ab^2w^2 + b^3$

$g^3 = a^3w^6 + 3a^2bw^5 + 3ab^2w^4 + b^3w^3$

$= a^3 + 3a^2bw^2 + 3ab^2w + b^3$

$\therefore e^3+f^3+g^3 = a^3 + 3a^2b + 3ab^2 + b^3 + a^3 + 3a^2bw + 3ab^2w^2 + a^3 + 3a^2bw^2 + 3ab^2w + b^3$

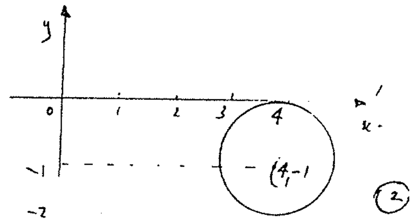
$= 3a^3 + 3b^3 + 3\left[a^2b\right]\left[1+w+w^2\right] + 3ab^2\left[1+w+w^2\right]$

$= 3\left[a^3+b^3\right]$ ③

(v) $z = (2-3i)\left(-\dfrac{1}{2} \pm \dfrac{\sqrt{3}}{2}i\right)$

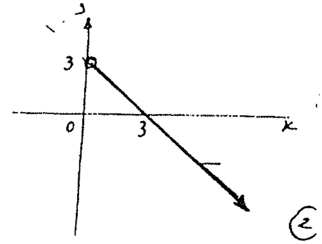
$z_1 = \dfrac{3\sqrt{3}}{2} - 1 + \left(\sqrt{3} + \dfrac{3}{2}\right)i \quad$ OR $\quad z_2 = -1 - \dfrac{3\sqrt{3}}{2} + \left(\dfrac{3}{2} - \sqrt{3}\right)i$ ②

3 (a) (i)



②

(ii)



②

(b) (i) $z + 2 + i = -3 + i$

other root $z = -5$ ②

(ii) $k = -5(2+i)$

$= -10 - 5i$ ②

(c) (i) $z^2 = 8 - 6i$

Let $z = x + iy \qquad x, y \in R$.

$(x+iy)^2 = 8 - 6i$

$x^2 - y^2 + 2xyi = 8 - 6i$

$\therefore x^2 - y^2 = 8$

$xy = -3 \Rightarrow y = \dfrac{-3}{x}$

$\therefore x^2 - \left(\dfrac{-3}{x}\right)^2 = 8$

$x^4 - 8x^2 - 9 = 0$

$(x^2-9)(x^2+1) = 0$

$x = 3$ And $x = -3$ And $x^2+1=0$ has no real soln $x \in R$

$y = -1 \qquad y = 1$

$\therefore z_1 = 3 - i \qquad z_2 = -3 + i$ ④

(ii) $P(z) = z^4 - 15z^2 - 6z + 110$

$P(3-i) = (8-6i)^2 - 15(8-6i) - 6(3-i) + 110$

$= 64 - 96i - 36 - 120 + 90i - 18 + 6i + 110$

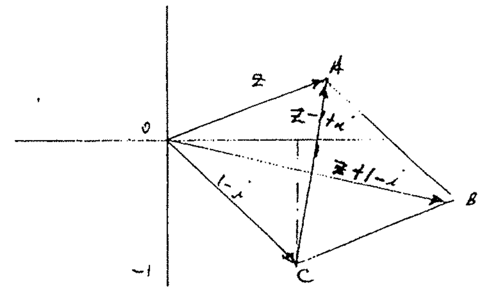
$= 0$

$P(-3+i) = (8-6i)^2 - 15(8-6i) - 6(-3+i) + 110$ ④

$= 36 - 12i$

$\neq 0$ $\qquad$ $3-i$ is the only soln

(d)



$$\frac{z-1+i}{z+1-i} = ki$$

$z-1+i = (z+1-i)ki$

ie vector $z-1+i \perp$ vector $z+1-i$ and modulus changed.

ie Diagonals OABC are at right angles in parallelogram OABC

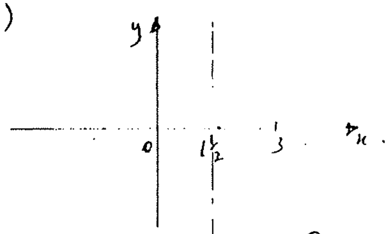
$\therefore$ OABC is a rhombus

$\therefore |z| = |1-i|$

$= \sqrt{2}$ units.

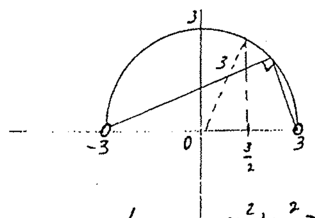
④

4(a) (i)



Locus $x = 1\frac{1}{2}$ ②

(ii)



Locus $x^2+y^2=9$

$\{0 < y \leq 3\}$

$\{-3 < x < 3\}$

②

(iii) $z = 3 \text{ cis } \frac{\pi}{3}$

$= 3\left[\frac{1}{2} + \frac{\sqrt{3}}{2}i\right]$

$= \frac{3}{2} + \frac{3\sqrt{3}}{2}i$ ②

(b) $(\sqrt{3}+i)^{12} + (\sqrt{3}-i)^{12} = \left[2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\right]^{12} + \left[2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)\right]^{12}$

$= 2^{12}(\cos 2\pi + i\sin 2\pi) + 2^{12}(\cos 2\pi - i\sin 2\pi)$

$= 2^{13}$

$= 8192$

③

(c) $z = \cos\theta + i\sin\theta$

$z^n = (\cos\theta + i\sin\theta)^n$

$z^n = \cos n\theta + i\sin n\theta$

$z^{-n} = (\cos\theta + i\sin\theta)^{-n}$

$= \cos(-n\theta) + i\sin(-n\theta)$

$z^{-n} = \cos n\theta - i\sin n\theta.$

$\therefore z^n + z^{-n} = 2\cos n\theta$

$$\cos n\theta = \frac{z^n + z^{-n}}{2}$$

And $\frac{z^n - z^{-n}}{2} = 2i\sin n\theta$

$$\sin n\theta = \frac{z^n - z^{-n}}{2i}$$

(ii) $1 + \cos\theta + \cos 2\theta + \ldots \cos n\theta$

$= 1 + \frac{z+z^{-1}}{2} + \frac{z^2+z^{-2}}{2} + \ldots \frac{z^n+z^{-n}}{2}$

$= 1 + \frac{1}{2}\left[2 + z^2 + \ldots z^n\right] + \frac{1}{2}\left(z^{-1} + z^{-2} + \ldots z^{-n}\right)$

$= 1 + \frac{z}{2}\left(\frac{z^n-1}{z-1}\right) + \frac{z^{-1}}{2}\left(\frac{z^{-n}-1}{z^{-1}-1}\right)$

$= 1 + \frac{z}{2}\frac{(z^n-1)}{(z-1)} + \frac{z^{-n}-1}{2(1-z)}$

$$= \frac{2z-2+z^{n+1}-z-z^{-n}+1}{2(z-1)}$$

$$= \frac{z^{n+1}-z^{-n}+z-1}{2(z-1)}$$

④

(iii) $$\frac{\cos\frac{n\theta}{2}\cdot\sin\left(\frac{n+1}{2}\right)\theta}{\sin\frac{\theta}{2}} = \frac{z^{\frac{n}{2}}+z^{-\frac{n}{2}}}{2}\cdot\frac{\dfrac{z^{\frac{n+1}{2}}-z^{\frac{-n-1}{2}}}{2i}}{\dfrac{z^{\frac{1}{2}}-z^{-\frac{1}{2}}}{2i}}$$

$$= \frac{\left(z^{\frac{n}{2}}+z^{-\frac{n}{2}}\right)}{2}\cdot\frac{\left(z^{\frac{n}{2}+1}-z^{-\frac{n}{2}}\right)}{z-1}$$

$$= \frac{z^{n+1}-1+z-z^{-n}}{2(z-1)} \qquad (3)$$

$$\therefore\ 1+\cos\theta+\cos 2\theta+\cos n\theta = \frac{\cos\frac{n\theta}{2}\cdot\sin\left(\frac{n+1}{2}\right)\theta}{\sin\frac{\theta}{2}}$$

$$\sum_{k=0}^{n}\cos n\theta = \frac{\cos\frac{n\theta}{2}\cdot\sin\left(\frac{n+1}{2}\right)\theta}{\sin\frac{\theta}{2}}. \qquad (1)$$