Name:	
Class:	

JAMES RUSE AHS

GESTA NON VERBA

YEAR 12

ASSESSMENT TEST 2
TERM 1, 2016

MATHEMATICS EXTENSION 2

***Time Allowed** – 120 Minutes*
(Plus 5 minutes Reading Time)

General Instructions:

- *All* questions may be attempted
- *All* questions are of equal value
- Reference Sheet will be supplied
- Department of Education approved calculators and templates are permitted
- In every Question, show all relevant mathematical reasoning and/or calculations.
- Marks may not be awarded for careless or badly arranged work

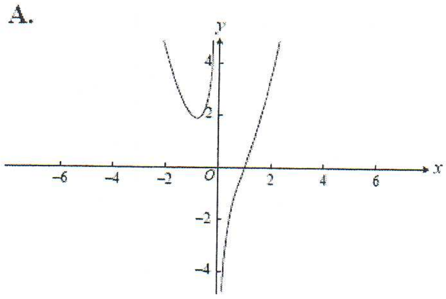
Question 1-5 are to be completed on multiple choice sheet then Question 6,7,8,9 to be completed on separate sheets of paper and handed in in separate bundles.
Each question must show your Candidate Number (in the top right hand corner).

For the following questions colour the most correct answer on your multiple choice answer sheet.

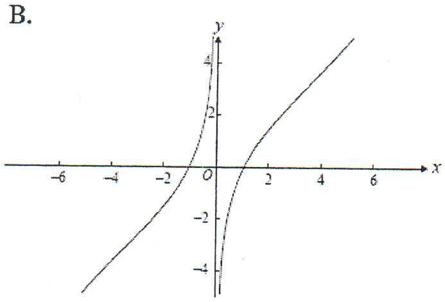
1. Let $f(x)=\dfrac{x^k+a}{x}$, where k and a are real constants.

   If k is an odd integer which is greater than 1 and $a<0$, a possible graph of $f(x)$ could be

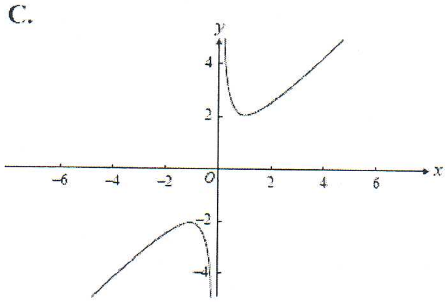
   A.

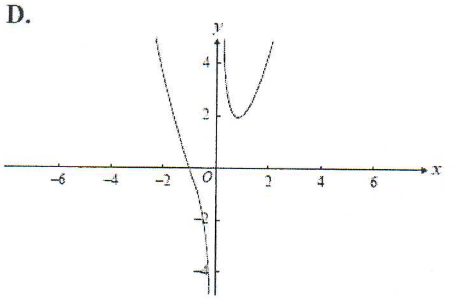
   B.

   C.

   D.

2. The coordinates of the foci of the hyperbola $\dfrac{x^2}{4}-y^2=1$ are given by:

   A. $(0,\ \pm\sqrt{5}\ )$ B. $(\pm\sqrt{5}\,,\ 0)$ C. $(0,\ \pm2\sqrt{5}\,)$ D. $(\pm2\sqrt{5}\,,\ 0)$

3. The polynomial $P(z)=z^3+(1+i)z^2+(1+i)z+1$ has a real zero $z=-1$ and a complex zero $z=\alpha$. By considering the sum and the product of the roots, the third root is:

   A. $\dfrac{1}{\alpha}$

   B. $\overline{\alpha}$

   C. $-\alpha$

   D. $(1-\alpha)$

4. $\int x \sin 2x \, dx =$

A. $-2x\cos 2x + \sin 2x + C$

B. $-\frac{x}{2}\cos 2x - \frac{1}{4}\sin 2x + C$

C. $\frac{x}{2}\cos 2x + \frac{1}{4}\sin 2x + C$

D. $-\frac{x}{2}\cos 2x + \frac{1}{4}\sin 2x + C$

5. The graph of $y = \dfrac{1}{ax^2 + bx + c}$ has asymptotes at x = -5, x = 3 and y = 0.

Given that the graph has one stationary point with a y – coordinate of $-\frac{1}{8}$, it follows that

A. $a = 1,\ b = 2,\ c = -15$

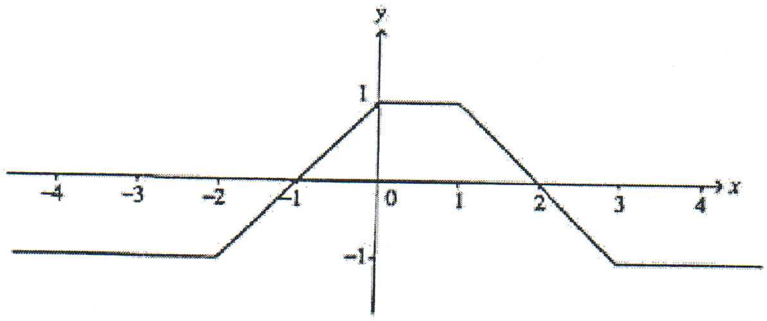
B. $a = \frac{1}{2},\ b = 1,\ c = -\frac{15}{2}$

C. $a = -\frac{1}{2},\ b = -1,\ c = 15$

D. $a = \frac{1}{2},\ b = -1,\ c = -\frac{15}{2}$

Question 6 (16 marks) Marks

a) The diagram is a sketch of the function $y = f(x)$.



On separate diagrams sketch:

(i) $y = \dfrac{1}{f(x)}$ 2

(ii) $y = f(x) - |f(x)|$ 2

(iii) $y = \sin^{-1}(f(x))$ 2

(iv) $y = e^{f(x)}$ 2

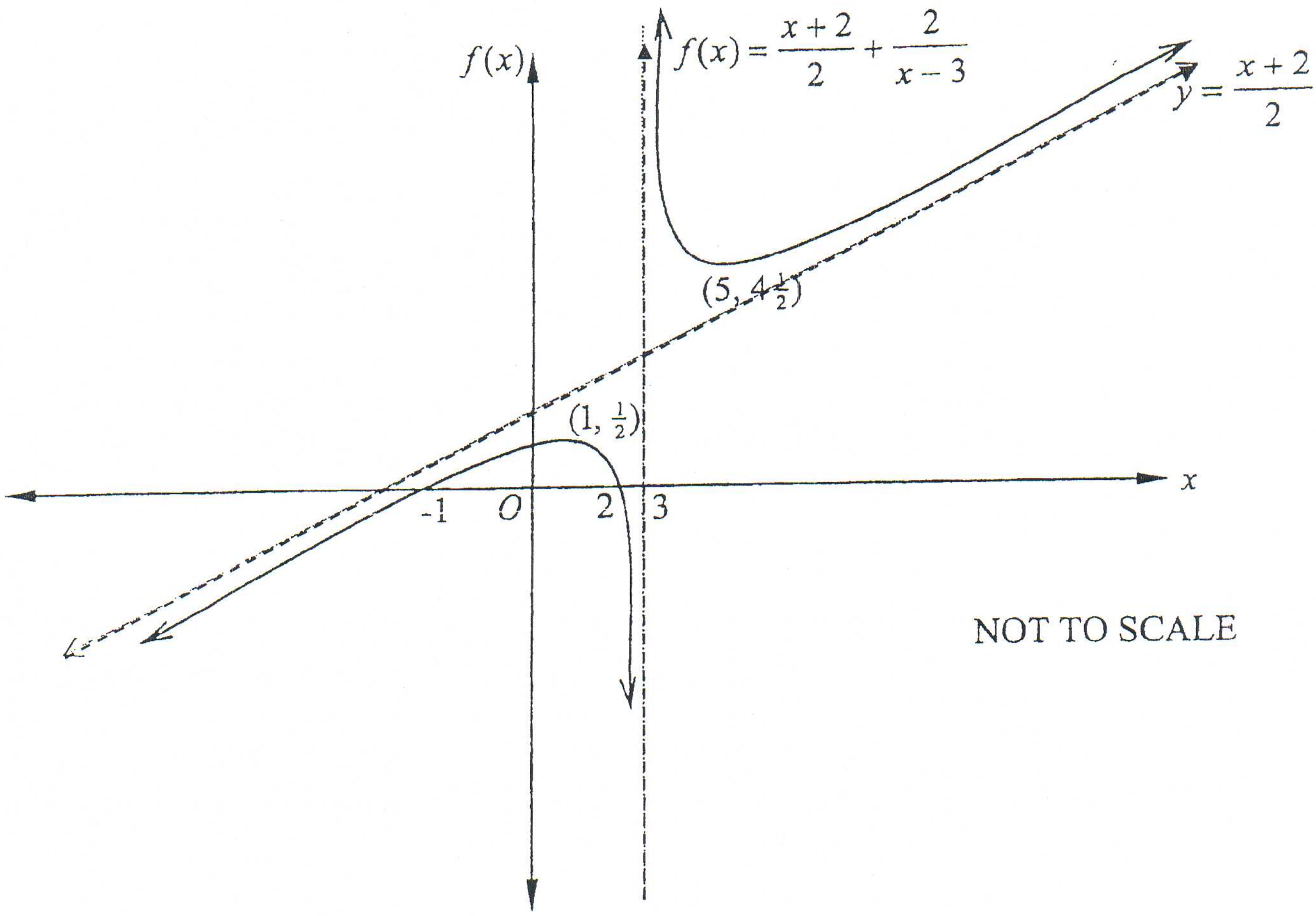
b) Consider the function $f(x) = \begin{cases} \dfrac{e^x - 1}{x}, & x \neq 0 \\ 1 & , x = 0 \end{cases}$

(i) Use differentiation to show that $e^{-x} + x - 1 \geq 0$ for all values of x. Hence show that $f(x)$ is an increasing function for $x \neq 0$. 3

(ii) Show that $f(x)$ is continuous at $x = 0$. 2

(iii) Sketch the graph of $y = f(x)$. 1

c) The function defined by $f(x)=\dfrac{x+2}{2}+\dfrac{2}{x-3}$ is drawn below.



Draw a separate sketch of $y=[f(x)]^2$. 2

Question 7 (16 marks) Marks

a) Find $\int \frac{\sin 2x}{\left(1+\sin^2 x\right)^2}\,dx$ 1

b) Consider $I = \int_{-1}^{1} \frac{x^2 e^x}{e^x + 1}\,dx$ and $J = \int_{-1}^{1} \frac{x^2}{e^x + 1}\,dx.$

(i) Use the substitution $u = -x$ in I to show that $I = J$. 3

(ii) Hence evaluate I. 1

c) Evaluate $\int_0^{\frac{\pi}{4}} \frac{1-\tan x}{1+\tan x}\,dx$ 3

d) (i) Use the result $\int_0^a f(x)dx = \int_0^a f(a-x)dx$, to show that

$$\int_0^{\frac{\pi}{4}} \ln(1+\tan x)dx = \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan x}\right)dx.$$ 2

(i) Hence, evaluate $\int_0^{\frac{\pi}{4}} \ln\left(1+\tan x\right)dx.$ 2

e) Using a suitable substitution find the exact value of $\int_0^1 \sqrt{2-\sqrt{x}}\,dx$. 4

Question 8 (16 marks) Marks

a) The locus of a point is defined by the equation $|z-2| = 2\,\text{Re}\left(z-\frac{1}{2}\right)$.

(i) If $z = x+iy$, explain why $x \geq \frac{1}{2}$. 1

(ii) Show that the locus satisfies the equation $3x^2 - y^2 = 3$. 2

(iii) Describe the locus that satisfies (i) and (ii). 1

(iv) Sketch the locus showing its asymptotes and vertex. 1

(v) Find all possible values for arg z for this locus. 1

b) The equation $3x^3 - 9x^2 + 6x + 2 = 0$ has roots α, β and γ.

(i) Find the cubic equation with roots α^2, β^2 and γ^2. 2

(ii) Hence evaluate $\alpha^3\beta\gamma + \alpha\beta^3\gamma + \alpha\beta\gamma^3$. 2

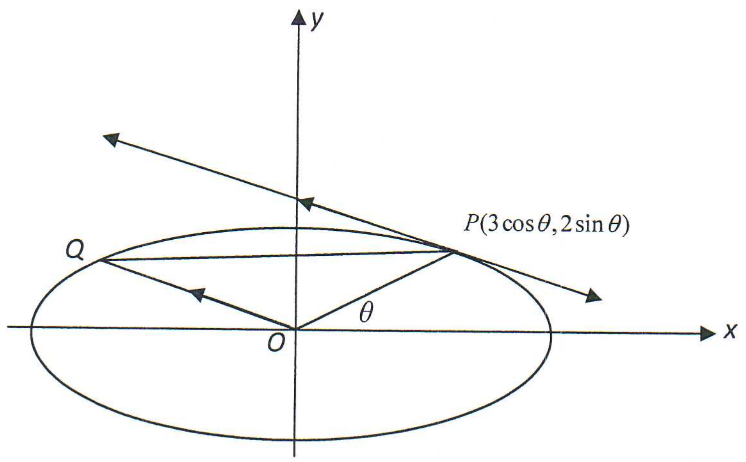
c) Given that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$,

(i) By letting $x = \frac{2}{3}\cos\theta$, use the above formula to find the three roots of the equation $27x^3 - 9x + 1 = 0$. 3

(ii) Hence, find the value of $\cos\frac{\pi}{9}\cos\frac{3\pi}{9}\cos\frac{5\pi}{9}\cos\frac{7\pi}{9}$. 3

Question 9 (16 marks) Marks

The point $P(3\cos\theta, 2\sin\theta)$ lies on the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ with centre O.



(i) Find the equation of the tangent to this ellipse at P. 2

(ii) A line passing through the origin O parallel to the tangent at P meets the ellipse at Q as shown on the diagram. Show that Q is $(-3\sin\theta,\ 2\cos\theta)$. 2

(iii) Show that the distance between the tangent at P and the line OQ is $\frac{6}{OQ}$. 3

(iv) Hence prove that the area of ΔOPQ is independent of the position of P. 1

(v) Find the length of PQ and state the maximum and minimum values of this length for $-\pi \le \theta \le \pi$. 3

(vi) The tangents at P and Q to the ellipse intersect at T. Find the coordinates of T, and verify that T lies on the curve $4x^2+9y^2=72$. 2

(v) PD is a diameter of the ellipse. The tangent at P meets the vertical through D at V. Show that the ratio of the area of ΔPDV to the area of the ellipse is $2:\pi|\tan\theta|$. 3

Questions continue over page…

Question 10 (16 marks) Marks

a) i) Show that, if $(r\cos\theta, r\sin\theta)$ and $\left(s\cos\left(\theta+\frac{\pi}{2}\right), s\sin\left(\theta+\frac{\pi}{2}\right)\right)$ lie on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ with centre O, then $\frac{1}{r^2}+\frac{1}{s^2}=\frac{1}{a^2}-\frac{1}{b^2}$. 2

(ii) Deduce that, if P and Q are points on the hyperbola such that OP is perpendicular to OQ, then $\frac{1}{OP^2}+\frac{1}{OQ^2}$ is independent of the position of P and Q. Give a sketch of the hyperbola and points O, P, Q related in this way. 3

b) Find the point(s) on the curve $xy\,(x+y)+16=0$ where the gradient of the tangent to the curve is -1. 4

c) $I_n=\int_0^1 x^n \ln(1+x)dx,\ n=0, 1, 2, \ldots$

(i) Show that $\int \ln(1+x)dx=(1+x)\ln(1+x)-x+c$. 1

(ii) Show that $(n+1)I_n=2\ln 2-\frac{1}{n+1}-nI_{n-1}$, $n=1, 2, \ldots$ 2

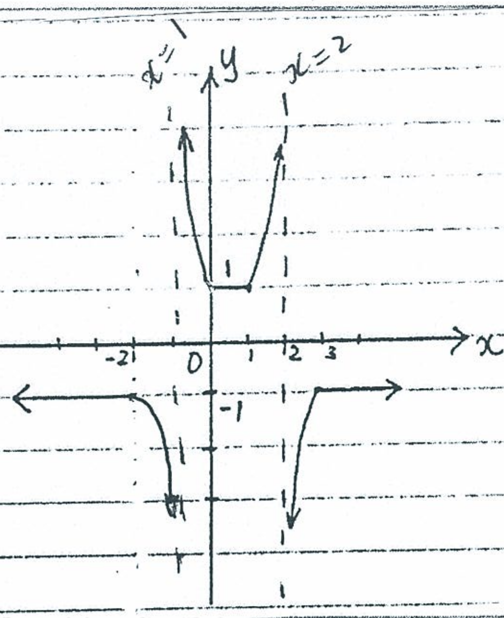
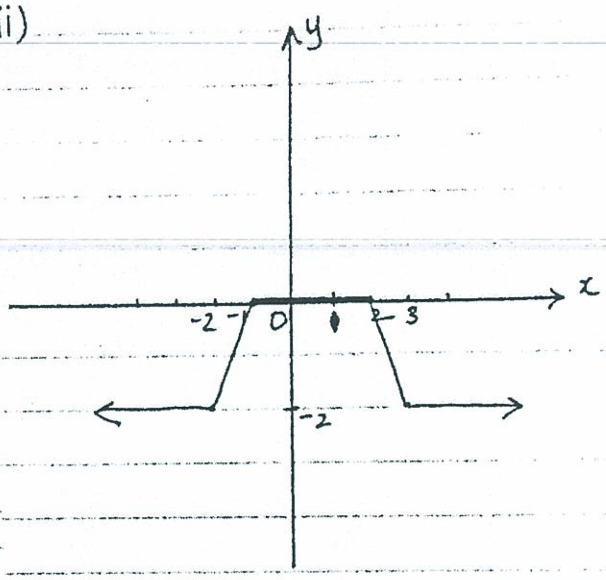
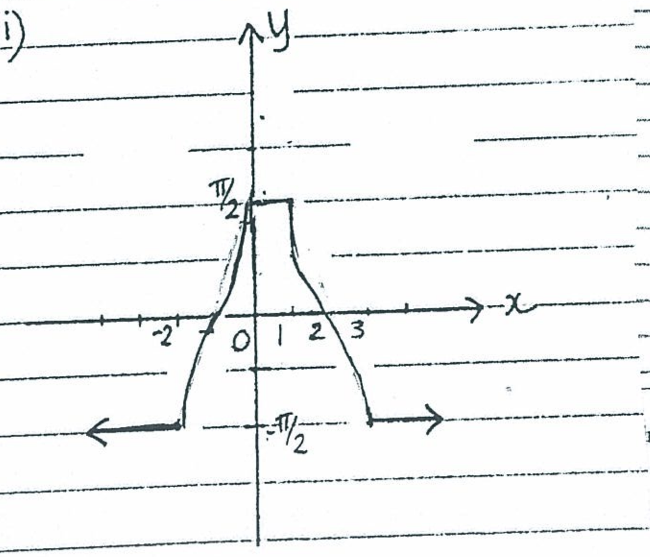
(iii) Evaluate $3I_2$ and $4I_3$. 2

(iv) Show that $(n+1)\,I_n=\begin{cases}\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots-\frac{1}{n+1}, & n \text{ odd}\\ 2\ln 2-\left(\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots+\frac{1}{n+1}\right), & n \text{ even}\end{cases}$ 2

Ext 2 **MATHEMATICS: Question** M/C .

Suggested Solutions	Marks	Marker's Comments
1. crosses x-axis when $x^k + a = 0$ $x^k = -a$ but a is negative so has to be graph A	A	
2. $a = 2$ $b = 1$ $1 = 4(e^2 - 1)$ $\frac{1}{4} = e^2 - 1$ $e^2 = \frac{5}{4}$ $e = \pm\frac{\sqrt{5}}{2}$ $S(\pm\sqrt{5}, 0)$	B	
3. $\alpha\beta\gamma = -1$ $-1 \times \alpha\beta = -1$ $\underline{\alpha\beta = 1}$ $\Rightarrow \beta = \frac{1}{\alpha}$	A	
4. $\int x \sin 2x \, dx = \int x \frac{d}{dx}\left(-\frac{1}{2}\cos 2x\right) dx$ $= -\frac{1}{2} x \cos 2x + \frac{1}{2}\int \cos 2x \, dx$ $= -\frac{1}{2} x \cos 2x + \frac{1}{4}\sin 2x + c$	D	
5. $y = \frac{1}{d(x+5)(x-3)}$ $\frac{dy}{dx} = \frac{-(2x+2)c}{d(x^2+2x-15)^2}$ stat point when $x = -1$ $y = -\frac{1}{8}$ when $x = -1 \Rightarrow y = \frac{1}{d(-16)}$ $\Rightarrow d = \frac{1}{2}$ $\left(a = \frac{1}{2}\ b = 1\ c = -\frac{15}{2}\right)$	B	

MATHEMATICS: Question.......Ex2 Q6

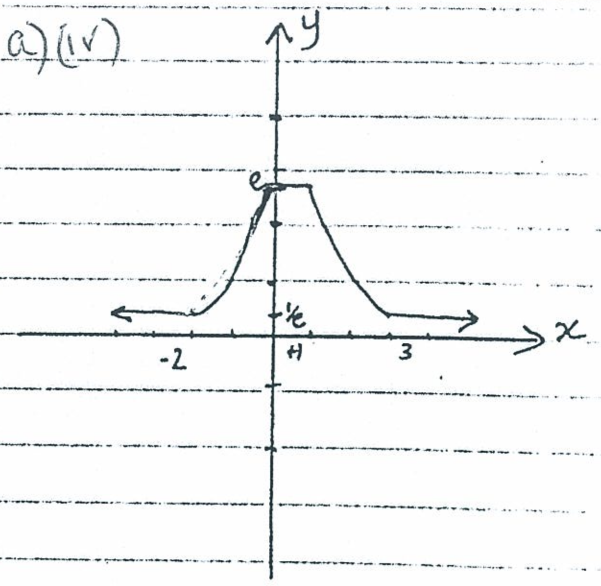
Suggested Solutions	Marks	Marker's Comments
a) (i) 	2	1 Asymptotes and curved lines 1 straight lines
(ii) 	2	1 graph $-1 \le x \le 2$ 1 complete graph
(iii) 	2	1 straight sections 1 curved sections (shape of inverse sine function)

(2)

MATHEMATICS: Question EXT2 Q6.

Suggested Solutions | **Marks** | **Marker's Comments**

a)(iv)



(2)

(1) straight sections
(1) curved sections (shape e^{x} and e^{-x})

6b) $f(x) = \begin{cases} \dfrac{e^x - 1}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

(i) Consider the function $g(x) = e^{-x} + x - 1$.

$g(0) = 0$ and $g'(x) = -e^{-x} + 1$

$\therefore g'(0) = 0$

Also, $g''(x) = e^{-x} > 0$ for all x and $g(0) = 0$

$\therefore g(x)$ has a minimum of 0 when $x = 0$.

$\therefore e^{-x} + x - 1 \geq 0$ for all x, with equality only if $x = 0$.

For $x \neq 0$, $f'(x) = \dfrac{d}{dx}\left(\dfrac{e^x - 1}{x}\right)$

$= \dfrac{e^x . x - (e^x - 1) . 1}{x^2}$

$= \dfrac{1 + xe^x - e^x}{x^2}$

$= \dfrac{e^x}{x^2}(e^{-x} + x - 1)$

> 0 as $e^{x} > 0$, $x^2 > 0$, $e^{-x} + x - 1 > 0$

Hence $f(x)$ is an increasing function.

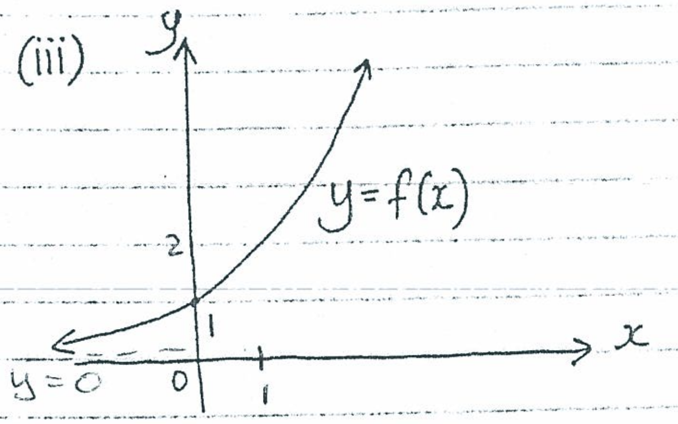
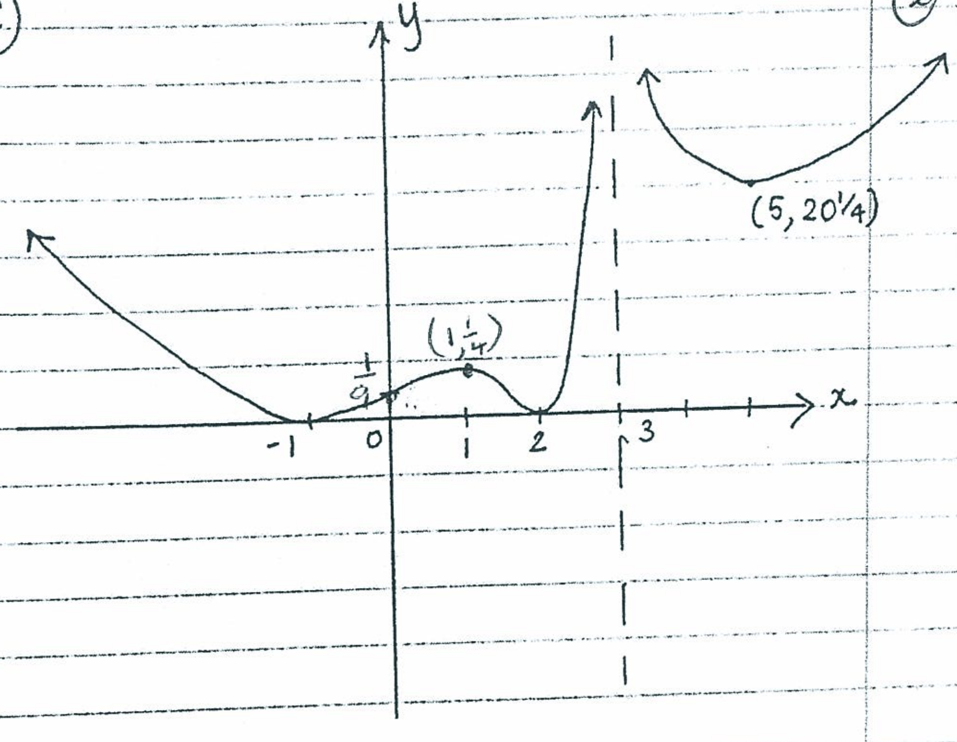
(3)

(1) full + complete proof of $e^{-x} + x - 1 \geq 0$ by showing minimum at $(0,0)$

(1) $f'(x) = \dfrac{1 + xe^{x} - e^{x}}{x^2}$

(1) explanation that $f'(x) > 0$

MATHEMATICS: Question EXT 2 Q6

Suggested Solutions	Marks	Marker's Comments
(ii) Let $h(x) = e^x$. Then $h'(x) = e^x$. $\lim_{x\to 0} \frac{e^x - 1}{x} = \lim_{x \to 0} \frac{e^x - e^0}{x - 0}$ $= h'(0)$ $= 1$ $= f(0)$ $\therefore f(x)$ is continuous at $x = 0$.	2	1 Apply concept $\lim_{x\to 0} f(x) = f(0) = 1$ 1 complete proof $\lim_{x\to 0}\left[\frac{e^x-1}{x}\right] = 1$
(iii) 	1	Must show increasing graph with increasing gradient through $y=1$ Asymptote $y=0$ x scale should be labelled
6c) 	2	1 Rounded T.P's at $x=-1$ and $x=2$. 1 Rest of graph. T P's must be labelled with coordinates y intercept should be shown

Ext2 Mathematics T2 2016

Question 7

a) $\int \frac{\sin 2x}{(1+\sin^2 x)^2}\,dx$

Let $u = \sin^2 x$

$\frac{du}{dx} = 2\sin x\cos x$

$= \sin 2x$

$\therefore \int \frac{\sin 2x}{(1+\sin^2 x)^2}\,dx = \int \frac{du}{(1+u)^2}$

$= -\frac{1}{1+u} + C$

$= -\frac{1}{1+\sin^2 x} + C$

Note: Other answers include:

* $\frac{-2}{3-\cos 2x} + C$
* $\frac{-1}{2-\cos^2 x} + C$

b) (i) Let $u = -x$

$\frac{du}{dx} = -1$

$I = \int_{-1}^{1} \frac{x^2 e^x}{e^x+1}\,dx$

$= \int_{+1}^{-1} \frac{u^2 e^{-u}}{e^{-u}+1} \cdot -du$ — 1

$= \int_{-1}^{1} \frac{u^2}{e^u} \cdot \frac{e^u}{1+e^u}\,du$

$= \int_{-1}^{1} \frac{u^2}{1+e^u}\,du$ — 1

$= \int_{-1}^{1} \frac{x^2}{1+e^x}\,dx$ — 1

$= J$

* Students should state (change of variable) in the last step

(ii) $I + J = \int_{-1}^{1} \left[\frac{x^2 e^x}{e^x+1} + \frac{x^2}{e^x+1}\right] dx$

$2I = \int_{-1}^{1} x^2 \frac{(e^x+1)}{e^x+1} dx$

$= \int_{-1}^{1} x^2 dx$

$= \left[\frac{1}{3}x^3\right]_{-1}^{1}$

$= \frac{1}{3}(1-(-1)) = \frac{2}{3}$ — 1

$\therefore I = 1/3.$

Since $\int_0^a f(a-x)\,dx = \int_0^a f(x)\,dx$

c) $\int_0^{\pi/4} \frac{1-\tan x}{1+\tan x} dx$ $\xrightarrow{\text{or}}$ $\int_0^{\frac{\pi}{4}} \frac{1-\tan(\frac{\pi}{4}-x)}{1+\tan(\frac{\pi}{4}-x)} dx$ — 1

$= \int_0^{\pi/4} \frac{1-\frac{\sin x}{\cos x}}{1+\frac{\sin x}{\cos x}} dx$

$= \int_0^{\pi/4} \frac{\cos x - \sin x}{\cos x + \sin x} dx$

$= \left[\ln(\cos x + \sin x)\right]_0^{\pi/4}$

$= \ln(1/\sqrt{2} + 1/\sqrt{2}) - \ln(1+0)$

$= \ln 2/\sqrt{2}$ — 1

$= 1/2 \ln 2$ or $\ln\sqrt{2}$

$= \int_0^{\frac{\pi}{4}} \frac{1-\frac{1-\tan x}{1+\tan x}}{1+\frac{1-\tan x}{1+\tan x}} dx$

$= \int_0^{\frac{\pi}{4}} \frac{1+\tan x - 1 + \tan x}{1+\tan x + 1 - \tan x} dx$

$= \int_0^{\frac{\pi}{4}} \tan x \, dx$

$= \left[-\ln|\cos x|\right]_0^{\frac{\pi}{4}}$

$= -\ln\frac{1}{\sqrt{2}} + \ln 1$

$= \ln\sqrt{2}$ — 1

Scanned by CamScanner

d)(i) $\int_0^{\pi/4} \ln(1+\tan x)\,dx$

$= \int_0^{\pi/4} \ln\left(1+\tan\left(\frac{\pi}{4}-x\right)\right) dx$ since $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

$= \int_0^{\pi/4} \ln\left(1+\frac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\tan x}\right) dx$ —— 1

$= \int_0^{\pi/4} \ln\left(1+\frac{1-\tan x}{1+\tan x}\right) dx$

$= \int_0^{\pi/4} \ln\left(\frac{1+\tan x+1-\tan x}{1+\tan x}\right) dx$

$= \int_0^{\pi/4} \ln\left(\frac{2}{1+\tan x}\right) dx$ —— 1

(ii) $\int_0^{\pi/4} \ln(1+\tan x)\,dx$

$= \int_0^{\pi/4} \ln\left(\frac{2}{1+\tan x}\right) dx$

$= \int_0^{\pi/4} \ln 2\,dx - \int_0^{\pi/4} \ln(1+\tan x)\,dx$ —— 1

$\therefore 2\int_0^{\pi/4} \ln(1+\tan x)\,dx = \int_0^{\pi/4} \ln 2\,dx$

$= \left[x \ln 2\right]_0^{\pi/4}$

$= \frac{\pi}{4}\ln 2 - 0$

$\therefore \int_0^{\pi/4} \ln(1+\tan x)\,dx = \frac{\pi}{8}\ln 2$ —— 1

e) $\int_0^1 \sqrt{2-\sqrt{x}}\,dx$

let $u = 2-\sqrt{x}$

when $x=1$, $u=1$

$x=0$, $u=2$

$\therefore \sqrt{x} = 2-u$

$x = 4-4u+u^2$

$\frac{dx}{du} = -4+2u$ — 1

$$\therefore I = \int_2^1 \sqrt{u}\,(-4+2u)\,du \text{ — } 1$$

$$= \int_2^1 2u^{3/2} - 4u^{\frac{1}{2}}\,du$$

$$= 2\left[\frac{u^{5/2}}{5/2}\right]_2^1 - 4\left[\frac{u^{3/2}}{3/2}\right]_2^1 \text{ — } 1$$

$$= 2\left(\frac{2}{5} - \frac{8\sqrt{2}}{5}\right) - 4\left(\frac{2}{3} - \frac{4\sqrt{2}}{3}\right)$$

$$= \left(\frac{4}{5} - \frac{8}{3}\right) + \left(\frac{16\sqrt{2}}{3} - \frac{16\sqrt{2}}{5}\right)$$

$$= \frac{32\sqrt{2}}{15} - \frac{28}{15} \text{ — } 1$$

e) $\int_0^1 \sqrt{2-\sqrt{x}}\, dx$

$$\left.\begin{aligned} &\text{Let } x = 4\sin^4\theta \\ &\frac{dx}{d\theta} = 16\sin^3\theta\cos\theta\, d\theta \\ &\text{At } x=0,\ \theta=0 \\ &\quad\ x=1,\ \theta=\pi/4 \end{aligned}\right\} \text{— } 1$$

$$\int_0^1 \sqrt{2-\sqrt{x}}\, dx = \int_0^{\pi/4} \sqrt{2-2\sin^2\theta}\,.\,16\sin^3\theta\cos\theta\, d\theta$$

$$= \int_0^{\pi/4} \sqrt{2}\sqrt{1-\sin^2\theta}\,.\,16\sin^3\theta\cos\theta\, d\theta$$

$$= \sqrt{2}\int_0^{\pi/4} \cos\theta\,.\,16\sin^3\theta\cos\theta\, d\theta$$

$$= 16\sqrt{2}\int_0^{\pi/4} \sin^3\theta\cos^2\theta\, d\theta \text{ — } 1$$

$$= 16\sqrt{2}\int_0^{\pi/4} \sin\theta\cos^2\theta(1-\cos^2\theta)\, d\theta$$

$$= 16\sqrt{2}\int_0^{\pi/4} \cos^2\theta\sin\theta\, d\theta$$

$$- 16\sqrt{2}\int_0^{\pi/4} \cos^4\theta\sin\theta\, d\theta$$

$$= 16\sqrt{2}\left[-\frac{1}{3}\cos^3\theta + \frac{1}{5}\cos^5\theta\right]_0^{\pi/4} \text{ — } 1$$

$$= 16\sqrt{2}\left(-\frac{1}{3}\left(\frac{1}{\sqrt{2}}\right)^3 + \frac{1}{5}\left(\frac{1}{\sqrt{2}}\right)^5\right) + 16\sqrt{2}\left(\frac{1}{3}-\frac{1}{5}\right.$$

$$= 16\sqrt{2}\left(-\frac{1}{3}\times\frac{1}{2\sqrt{2}} + \frac{1}{5}\times\frac{1}{4\sqrt{2}}\right) + 16\sqrt{2}\left(\frac{2}{15}\right)$$

$$= \left(\frac{-16}{6} + \frac{16}{20}\right) + \frac{32\sqrt{2}}{15}$$

$$= -\frac{28}{15} + \frac{32\sqrt{2}}{15} = \frac{-28+32\sqrt{2}}{15} \text{ — } 1$$

2) $\int_0^1 \sqrt{2-\sqrt{x}}\,dx$

let $u^2 = x$

$\therefore \frac{dx}{du} = 2u$

$dx = 2u\,du$

when $x=1,\ u=1$

$x=0,\ u=0$

$= \int_0^1 \sqrt{2-u}\; 2u\,du$ —— 1

$= 2\int_0^1 u\sqrt{2-u}\,du$

1st Approach

let $k = 2-u$ when $u=1,\ k=1$

$u=0,\ k=2$

$\frac{dk}{du} = -1$

$\therefore I = 2\int_2^1 (2-k)\sqrt{k}\,(-dk)$ —— 1

$= 2\int_1^2 2k^{\frac{1}{2}} - k^{3/2}\,dk$

$= 2\left[\frac{2k^{3/2}}{3/2} - \frac{k^{5/2}}{5/2}\right]_1^2$ —— 1

$= 2\left[\frac{4k^{3/2}}{3} - \frac{2k^{5/2}}{5}\right]_1^2$

$= 2\left[\left(\frac{8\sqrt{2}}{3} - \frac{8\sqrt{2}}{5}\right) - \left(\frac{4}{3} - \frac{2}{5}\right)\right]$

$= 2\left[\frac{40\sqrt{2} - 24\sqrt{2}}{15} - \left(\frac{20-6}{15}\right)\right]$

$= 2\left(\frac{16\sqrt{2}}{15} - \frac{14}{15}\right)$

$= \frac{32\sqrt{2} - 28}{15}$ —— 1

2nd Approach:

Let $u = 2\sin^2\theta$ when $u=1,\ \theta = \frac{\pi}{4}$ $\quad \frac{du}{d\theta} = 4\sin\theta\cos\theta$

$u=0,\ \theta=0$

$$\therefore I = 2\int_0^{\frac{\pi}{4}} \sqrt{2-2\sin^2\theta}\ .\ 2\sin^2\theta\ .\ 4\sin\theta\cos\theta\ d\theta$$

$$= 16\sqrt{2}\int_0^{\frac{\pi}{4}} \sin^3\theta\cos^2\theta\ d\theta \quad \text{—— } 1$$

$\downarrow$

The rest follows from the original solution when we made the substitution $x = 4\sin^4\theta$ initially

3rd Approach:

$I = 2\int_0^1 u\sqrt{2-u}\,du$ Integration by parts

$$= 2\int_0^1 \frac{d}{dx}\left(\frac{-(2-u)^{3/2}}{3/2}\right) u\ du$$

$$= \frac{4}{3}\left[-u(2-u)^{3/2}\right]_0^1 + \frac{4}{3}\int_0^1 (2-u)^{3/2}\,du \quad \text{—— } 1$$

$$= \frac{-4}{3} + \frac{4}{3}\left[\frac{-(2-u)^{5/2}}{5/2}\right]_0^1 \quad \text{—— } 1$$

$$= \frac{-4}{3} - \frac{8}{15}\left[(2-u)^{5/2}\right]_0^1$$

$$= \frac{-4}{3} - \frac{8}{15}\left(1 - 4\sqrt{2}\right)$$

$$= \frac{-20}{15} - \frac{8}{15} + \frac{32\sqrt{2}}{15}$$

$$= \frac{32\sqrt{2} - 28}{15} \quad \text{—— } 1$$

4th Approach :

$$I = 2\int_0^1 u\sqrt{2-u} + 2\sqrt{2-u} - 2\sqrt{2-u}\, du$$

$$= 2\int_0^1 (u-2)\sqrt{2-u} + 2\sqrt{2-u}\, du \quad \text{—} \quad 1$$

$$= 2\int_0^1 2(2-u)^{\frac{1}{2}} - (2-u)^{\frac{3}{2}}\, du$$

$$= 2\left[\frac{-2(2-u)^{\frac{3}{2}}}{\frac{3}{2}} + \frac{(2-u)^{\frac{5}{2}}}{\frac{5}{2}}\right]_0^1 \quad \text{—} \quad 1$$

$$= 4\left[\left(\frac{1}{5} - \frac{2}{3}\right) - \left(\frac{4\sqrt{2}}{5} - \frac{4\sqrt{2}}{3}\right)\right]$$

$$= 4\left[\frac{-7}{15} + \frac{8\sqrt{2}}{15}\right]$$

$$= \frac{32\sqrt{2} - 28}{15} \quad \text{—} \quad 1$$

2016 Extension 2 Term 1 Task Q 8

Part (a)

(i)

Note that $\forall\, z \in \mathbb{C}, \quad |z| \geq 0$

$|z-2| \geq 0 \;\Rightarrow\; Re\left(z - \frac{1}{2}\right) \geq 0$

$z = x + iy \;\Rightarrow\; Re\left(x + iy - \frac{1}{2}\right) \geq 0$

$x - \frac{1}{2} \geq 0$

$x \geq \frac{1}{2}$ **(1 mark)**

Comments:

Students needed to demonstrate they knew $|z-2| \geq 0$ and that, therefore, the real component was also greater than zero.

(ii)

$|x - 2 + iy| = 2\left(x - \frac{1}{2}\right)$

$(x-2)^2 + y^2 = 4\left(x^2 - x + \frac{1}{4}\right)$ **(1 mark)**

$x^2 - 4x + 4 + y^2 = 4x^2 - 4x + 1$

$3x^2 - y^2 = 3$ **(1 mark)**

(iii)

Locus is an hyperbola with vertex at (1,0), domain restricted (noting above result) to $x \geq \frac{1}{2}$. It's therefore the right-hand branch only. **(1 mark)**

Comments:

The mark was not awarded unless the domain restriction was clearly stated.

(iv)

Asymptotes: $y = \pm\sqrt{3}x$

(1 mark)

Comments:

The asymptotes had to be clearly shown and labelled for the mark to be awarded. A mark for carry-forward error was awarded if the student had failed to show the domain restriction in part (iii), and had proceeded to otherwise correctly graph the entire hyperbola.

(v)

To find the boundaries of the argument range, find the angles of inclination of the asymptotes:

$\tan\theta = \pm\sqrt{3}$

$\theta = \pm\frac{\pi}{3}$

$-\frac{\pi}{3} < \arg z < \frac{\pi}{3}$ **(1 mark)**

Comments:

The student needed to write the argument as a strict inequality.

If the student had carried forward the error of not restricting the domain in part (iii), then a mark was awarded for finding the domain for the left-hand arm of the hyperbola: $\frac{2\pi}{3} < \arg z < \frac{4\pi}{3}$

Part (b)

(i)

Method 1:

Let $m = x^2,\ x = \sqrt{m}$

$3x^3 - 9x^2 + 6x + 2 = 0$

$3m\sqrt{m} - 9m + 6\sqrt{m} + 2 = 0$ **(1 mark)**

$\sqrt{m}(3m + 6) = 9m - 2$

$m(9m^2 + 36m + 36) = 81m^2 - 36m + 4$ **(1 mark)**

$9m^3 - 45m^2 + 72m - 4 = 0$

(It was unnecessary to change the variable back)

Method 2:

Find each of the following:

$\alpha^2 + \beta^2 + \gamma^2 = 5$

$\alpha^2\beta^2\gamma^2 = \frac{4}{9}$

$\alpha^2\beta^2 + \beta^2\gamma^2 + \alpha^2\gamma^2 = 8$ **(1 mark)**

$x^3 - 5x^2 + 8x - \frac{4}{9} = 0$ **(1 mark)**

(ii)

$\alpha^3\beta\gamma + \alpha\beta^3\gamma + \alpha\beta\gamma^3 = \alpha\beta\gamma(\alpha^2 + \beta^2 + \gamma^2)$ **(1 mark)**

$\alpha\beta\gamma = -\frac{2}{3}$

$\alpha^2 + \beta^2 + \gamma^2 = 5$

$\therefore \alpha^3\beta\gamma + \alpha\beta^3\gamma + \alpha\beta\gamma^3 = -\frac{2}{3} \times 5$

$= -\frac{10}{3}$ **(1 mark)**

Comments:

If Method 2 was used in part (ii) and an error was made in finding the sum or product of the roots, and that error was brought forward here, then a mark for carry forward error was awarded.

Part (c)

(i)

Let $x = \frac{2}{3}\cos\theta$

$27\left(\frac{2}{3}\cos\theta\right)^3 - 9\left(\frac{2}{3}\cos\theta\right) + 1$

$= 27\left(\frac{8}{27}\cos^3\theta\right) - 6\cos\theta + 1$ **(1 mark)**

$= 8\cos^3\theta - 6\cos\theta + 1$

$= 2\cos 3\theta + 1$, since $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

$\therefore 2\cos 3\theta + 1 = 0$

$\cos 3\theta = -\frac{1}{2}$ **(1 mark)**

$3\theta = \pm\frac{2\pi}{3} + 2k\pi,\ k = 0, 1, 2$

$\theta = \pm\frac{2\pi}{9} + \frac{2k\pi}{3}$

$\theta = \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{8\pi}{9}$

$x = \frac{2}{3}\cos\frac{2\pi}{9}, \frac{2}{3}\cos\frac{4\pi}{9}, \frac{2}{3}\cos\frac{8\pi}{9}$ (or equivalent values, noting that $cos\theta$ is an even function) **(1 mark)**

Comments:

Many students made computational errors when finding the angle.

(ii)

$\left(\cos\frac{\pi}{9}\right)\left(\cos\frac{3\pi}{9}\right)\left(\cos\frac{5\pi}{9}\right)\left(\cos\frac{7\pi}{9}\right) = \left(-\cos\frac{8\pi}{9}\right)\left(\frac{1}{2}\right)\left(-\cos\frac{4\pi}{9}\right)\left(-\cos\frac{2\pi}{9}\right)$ **(1 mark)**

$= -\frac{1}{2}\left(\cos\frac{2\pi}{9}\right)\left(\cos\frac{4\pi}{9}\right)\left(\cos\frac{8\pi}{9}\right)$

From the equation $27x^3 - 9x + 1 = 0$,

Product of the roots $= -\frac{1}{27}$

$\therefore \frac{2}{3}\cos\frac{2\pi}{9} \times \frac{2}{3}\cos\frac{4\pi}{9} \times \frac{2}{3}\cos\frac{8\pi}{9} = -\frac{1}{27}$ **(1 mark)**

$\left(\cos\frac{2\pi}{9}\right)\left(\cos\frac{4\pi}{9}\right)\left(\cos\frac{8\pi}{9}\right) = -\frac{1}{8}$

$\therefore \left(\cos\frac{\pi}{9}\right)\left(\cos\frac{3\pi}{9}\right)\left(\cos\frac{5\pi}{9}\right)\left(\cos\frac{7\pi}{9}\right) = -\frac{1}{2} \times -\frac{1}{8}$

$= \frac{1}{16}$ **(1 mark)**

Comments:

Students were required to show the equivalence of the product of the roots derived directly from the equation, and the roots found in part (i).

Note that if the roots were determined incorrectly in part (i) and the logic of part (ii) was **obviously** otherwise sound, then a mark was awarded for carry-forward error.

(i) GRADIENT at $P(3\cos\theta, 2\sin\theta)$:

$$\frac{dy}{dx} = \frac{-4x}{9y} = \frac{-4}{9}\cdot\frac{3\cos\theta}{2\sin\theta} = -\frac{2}{3}\frac{\cos\theta}{\sin\theta}$$

1 slope must be in parametric form

EQ of tangent at P:

$$y - 2\sin\theta = \frac{-2\cos\theta}{3\sin\theta}(x - 3\cos\theta)$$

$$3\sin\theta\, y - 6\sin^2\theta = 6\cos^2\theta - 2\cos\theta\, x$$

$$3\sin\theta\, y + 2\cos\theta\, x = 6(\cos^2\theta + \sin^2\theta) = 6$$

1 EQ of tangent at P

$$\frac{x}{3}\cos\theta + \frac{y}{2}\sin\theta = 1$$

☆ or $2\cos\theta x + 3\sin\theta y - 6 = 0$

(ii) EQ of OQ: $y = -\frac{2\cos\theta}{3\sin\theta}x$ meets ellipse

1 m for EQ of OQ (not for slope of OQ)

$$\therefore \frac{x^2}{9} + \frac{4}{4\times 9}\frac{\cos^2\theta}{\sin^2\theta}x^2 = 1$$

$$x^2(1 + \cot^2\theta) = 9$$

$$x^2 = 9\sin^2\theta$$

But $x<0$ $\quad \therefore x = -3\sin\theta,\ y = -\frac{2}{\cancel{3}}\frac{\cos\theta}{\sin\theta}\cancel{3}\sin\theta$

$$\therefore Q = [-3\sin\theta, 2\cos\theta]$$

must show $x<0$

1 correct coordinates for Q

(iii) Perpendicular distance of origin to tangent at P (from eq ☆)

$$= \frac{|(2\cos\theta)0 + (3\sin\theta)0 - 6|}{\sqrt{4\cos^2\theta + 9\sin^2\theta}}$$

must show clearly ⊥ dist from which pt to which line

1

$$= \frac{|-6|}{\sqrt{4\cos^2\theta + 9\sin^2\theta}}$$

1 simplifying expression

$$= \frac{6}{\sqrt{(0-3\sin\theta)^2 + (0-2\cos\theta)^2}}$$

must show

1 $OQ = \sqrt{(0-3\sin\theta)^2 + (0-2\cos\theta)^2}$

alternatively can use from Q $(-3\sin\theta, 2\cos\theta)$ to tangent

$$= \frac{6}{OQ} \quad \#$$

Alternatively :

Eq OQ : $y = -\frac{2\cos\theta}{3\sin\theta}x$

$\therefore 2\cos\theta\, x + 3\sin\theta\, y = 0$

Perp. distance from P to OQ :

$$= \frac{|2\cos\theta(3\cos\theta) + 3\sin\theta(2\sin\theta) + 0|}{\sqrt{(2\cos\theta)^2 + (3\sin\theta)^2}}$$

$$= \frac{|6(\cos^2\theta + \sin^2\theta)|}{\sqrt{4\cos^2\theta + 9\sin^2\theta}}$$

$$= \frac{6}{\sqrt{(0-2\cos\theta)^2 + (0--3\sin\theta)^2}}$$

$$= \frac{6}{OQ}$$

iv) Area $\triangle OPQ = \frac{1}{2} \cdot \frac{6}{OQ} \cdot OQ = 3$ square unit

$\therefore$ independent of position of P

v) $PQ = \sqrt{(3\cos\theta + 3\sin\theta)^2 + (2\sin\theta - 2\cos\theta)^2}$ 1

$= \sqrt{9\cos^2\theta + 18\cos\theta\sin\theta + 9\sin^2\theta + 4\cos^2\theta - 8\cos\theta\sin\theta + 4\sin^2\theta}$

$= \sqrt{13 + 10\cos\theta\sin\theta}$ 1

$PQ = 13 + 5\sin 2\theta$

max $PQ = 18$ when $\sin 2\theta = 1$

min $PQ = 8$ when $\sin 2\theta = -1$ 1

vi) slope of tangent at $Q = -\frac{4}{9}\frac{x}{y} = -\frac{4}{9} \cdot \frac{-3\sin\theta}{2\cos\theta}$

$= \frac{2\sin\theta}{3\cos\theta}$ $\therefore$ parallel to OP

$\therefore$ OPQT is a parm (2 pairs of opposite sides parallel)

$\therefore T = [3(\cos\theta - \sin\theta),\ 2(\sin\theta + \cos\theta)]$

★ Using this method must show eq. of line OQ clearly & $P(\cos\theta, 2\sin\theta)$

1 (using formula)

many students fudging since answer was given.

1

1 many students did not show the details how to find OQ & lost this mark

1 many did not mention independent of P & some forgot $\frac{1}{2}$ for area $\triangle$.

Some students use calculus to find max / min

both max / min must be correct to get this mark

Most students try to find eq of tangent at Q & solve simultaneously with tangent at P but failed

1 both x, y must be correct

$4x^2 + 9y^2$

$= 4[3(\cos\theta - \sin\theta)]^2 + 9[(2(\sin\theta + \cos\theta))]^2$

$= 36[\cos^2\theta + \sin^2\theta - 2\sin\cancel{\theta}\cos\theta + \sin^2\theta +$

$= \quad 2\sin\cancel{\theta}\cos\theta + \cos^2\theta .]$

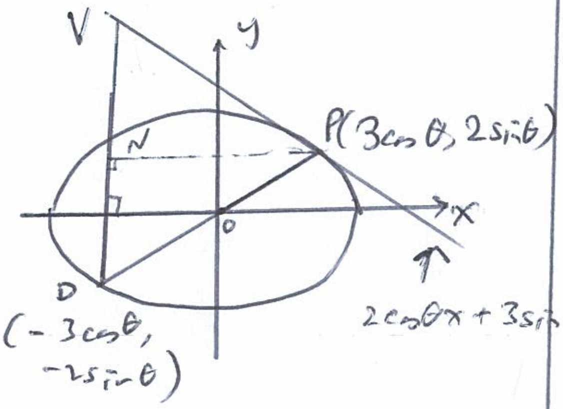
$= 36 \times 2 = 72 \qquad (\sin^2\theta + \cos^2\theta = 1)$

/ Well Done

v) Area of Ellipse $= 2 \times 2 \times 3\pi = 6\pi$

/ Students get 1 m for area of ellipse. Well done

$D = [-3\cos\theta, -2\sin\theta]$



To find the coordinates of $V = ?$

V lies on tangent at P

when $x = -3\cos\theta$, $2\cos\theta(-3\cos\theta) + 3\sin\theta\, y - 6 = 0$

$$y = \frac{6 + 6\cos^2\theta}{3\sin\theta} = \frac{2 + 2\cos^2\theta}{\sin\theta}$$

$V = \left[-3\cos\theta, \dfrac{2 + 2\cos^2\theta}{\sin\theta}\right]$

/ Students get 1 m for the coordinates of $V = (-3\cos\theta, \frac{2+2\cos^2\theta}{\sin\theta})$ or $VD = \frac{2+2\cos^2\theta}{\sin\theta} + 2\sin\theta$

$\therefore VD = \left|\dfrac{2 + 2\cos^2\theta}{\sin\theta} + 2\sin\theta\right| = \left|\dfrac{2 + 2(\cos^2\theta + \sin^2\theta)}{\sin\theta}\right|$

$VD = \left|\dfrac{4}{\sin\theta}\right|$

Construct $PN \perp VD$

$PN = 6|\cos\theta|$

$\therefore$ Area $\triangle PVD = \frac{1}{2} \times 6|\cos\theta|\left|\dfrac{4}{\sin\theta}\right| = \dfrac{12}{|\tan\theta|}$

$\therefore$ Area $\triangle PVD$: Area Ellipse $=$

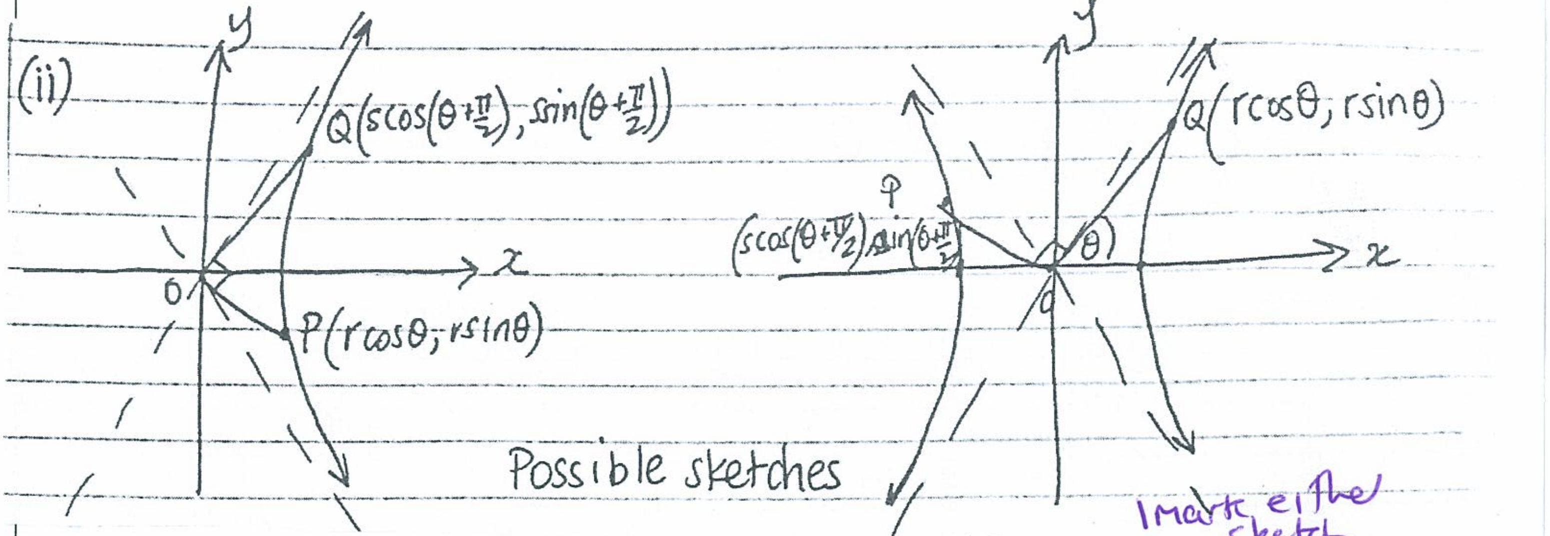
/

$= \dfrac{12}{|\tan\theta|} : 6\pi = 2 : \pi|\tan\theta|$

MATHEMATICS Extension 2: Question ..10...

Suggested Solutions	Marks	Marker's Comments
(a) Sub $(r\cos\theta, r\sin\theta)$ and $(s\cos(\theta+\frac{\pi}{2}), s\sin(\theta+\frac{\pi}{2}))$ into curve		
$\frac{r^2\cos^2\theta}{a^2} - \frac{r^2\sin^2\theta}{b^2} = 1$		
i.e $\frac{\cos^2\theta}{a^2} - \frac{\sin^2\theta}{b^2} = \frac{1}{r^2}$ — (1)		
Now $s\cos(\theta+\frac{\pi}{2}) = -\sin\theta$ $s\sin(\theta+\frac{\pi}{2}) = \cos\theta$	1	Had to change $\cos(\theta+\frac{\pi}{2})$ to gain first mark.
$\frac{s^2\sin^2\theta}{a^2} - \frac{s^2\cos^2\theta}{b^2} = 1$		
$\frac{\sin^2\theta}{a^2} - \frac{\cos^2\theta}{b^2} = \frac{1}{s^2}$ —(2)		
(1)+(2) $\frac{1}{r^2} + \frac{1}{s^2} = \frac{\sin^2\theta+\cos^2\theta}{a^2} - \frac{\sin^2\theta+\cos^2\theta}{b^2}$		
$= \underline{\frac{1}{a^2} - \frac{1}{b^2}}$	1	TOO many students didn't explain how they got last step
(ii) $P(r\cos\theta, r\sin\theta)$ and $Q(s\cos(\theta+\frac{\pi}{2}), s\sin(\theta+\frac{\pi}{2}))$		
$\frac{1}{OP^2} + \frac{1}{OQ^2} = \frac{1}{(r\cos\theta)^2 + (r\sin\theta)^2} + \frac{1}{(s\sin\theta)^2 + (s\cos\theta)^2}$ $= \frac{1}{r^2} + \frac{1}{s^2}$ $= \frac{1}{a^2} - \frac{1}{b^2}$ from (i)	1	Students needed to clearly explain what OP and OQ were not just state $= r$ and s.
which is independent of the position of P and Q.	1	

Ext 2 MATHEMATICS: Question 10

Suggested Solutions	Marks	Marker's Comments
(ii)  Possible sketches		1 mark either sketch.
(b) $x^2y + xy^2 + 16 = 0$ $x^2\frac{dy}{dx} + 2xy + 2xy\frac{dy}{dx} + y^2 = 0$	2	1 for 2 correct terms 2 for correct expression
$\frac{dy}{dx} = \frac{-y(y+2x)}{x(x+2y)}$		
$M_T = -1$ $\therefore -1 = \frac{-y(y+2x)}{x(x+2y)}$ $x^2 + 2xy = y^2 + 2xy$		poor algebra many students lost negative sign.
$x^2 = y^2$ $\underline{x = \pm y}$	1	
if $x = y$ $x^3 + x^3 + 16 = 0$ $2x^3 = -16$ $x = -2$ $y = -2$ $y = -x$ $-x^3 + x^3 + 16 = 0$ no solution	1	needed to look at both cases

EXT 2 **MATHEMATICS: Question** 10

Suggested Solutions	Marks	Marker's Comments
c) (i) $\int \ln(1+x)\frac{d}{dx}(1+x)\,dx$		Number of different methods
$= (1+x)\ln(1+x) - \int 1+x \times \frac{1}{1+x}\,dx$		
$= \underline{(1+x)\ln(1+x) - x + C}$	1	
(ii) $\int_0^1 \overset{u}{x^n} \overset{v}{\frac{d}{dx}\left[(1+x)\ln(1+x) - x\right]}\,dx$		
$= \left[x^n\left((1+x)\ln(1+x) - x\right)\right]_0^1 - n\int_0^1 x^{n-1}\left[(1+x)\ln(1+x) - x\right]dx$	1	1 for correct v those who chose x^n made little progress. Why is part i) there.?
$= 2\ln 2 - 1 - n\int_0^1 x^{n-1}\ln(1+x) + x^n\ln(1+x) - x^n\,dx$		
$= 2\ln 2 - 1 - nI_{n-1} - nI_n + n\int_0^1 x^n\,dx$		
$= 2\ln 2 - 1 - nI_{n-1} - nI_n + \left[\frac{nx^{n+1}}{n+1}\right]_0^1$		
$\therefore I_n = 2\ln 2 - 1 - nI_{n-1} - nI_n + \frac{n}{n+1}$	1	needed to see $-1+\frac{n}{n+1}$ to get 2nd mark. Many students made 2 errors
$(n+1)I_n = 2\ln 2 - \frac{1}{n+1} - nI_{n-1}$		

4/5

Ext2 MATHEMATICS: Question 10

Suggested Solutions	Marks	Marker's Comments
(iii) when $n=0$ $I_0 = 2\ln 2 - 1$ $n=1$ $2I_1 = 2\ln 2 - \frac{1}{2} - I_0$ $n=2$	1	1 mark for I_0 and some progress to calculate $3I_2$
$3I_2 = 2\ln 2 - \frac{1}{3} - 2I_1$ $= 2\ln 2 - \frac{1}{3} - \left[2\ln 2 - \frac{1}{2} - (2\ln 2 - 1)\right]$ $= 2\ln 2 - \frac{1}{3} - \frac{1}{2}$ $= 2\ln 2 - \frac{5}{6}$		• Many students made careless errors with respective sign.
$4I_3 = 2\ln 2 - \frac{1}{4} - 3I_2$ $= 2\ln 2 - \frac{1}{4} - \left[2\ln 2 - \frac{5}{6}\right]$ $= -\frac{1}{4} + \frac{5}{6}$ $= \frac{7}{12}$	1	2nd mark for correct $4I_3$.

5/5

Ext 2 MATHEMATICS: Question 10		
Suggested Solutions	**Marks**	**Marker's Comments**
$(n+1)I_n = \left(2\ln 2 - \frac{1}{n+1}\right) - \left(2\ln 2 - \frac{1}{n}\right) + \cdots (-1)^{n-1}\left(2\ln - \frac{1}{2}\right) + (-1)^n\left(2\ln 2 - 1\right)$	1	1 mark for building up pattern.
If n is odd even number of terms so $2\ln 2 - 2\ln 2 + 2\ln 2 - 2\ln 2 \cdots = 0$ $\therefore (n+1)I_n = -\frac{1}{n+1} + \frac{1}{n} \cdots -\frac{1}{2} + 1$ If n is even odd number of terms $2\ln 2 - 2\ln 2 + 2\ln 2 - \ln 2 \cdots = 2\ln 2$		students needed clearly explain why $2\ln 2$ when ~~even~~ and no $2\ln 2$ when odd.
$(n+1)I_n = 2\ln 2 - \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \cdots + \frac{1}{n+1}\right)$	1	very students did this. Some proved by induction which wasn't needed but needed to prove for both odd and even