Name:	
Class:	

YEAR 12

ASSESSMENT TASK 2
TERM 1, 2017

MATHEMATICS EXTENSION 2

Time Allowed – 120 Minutes
(Plus 5 minutes Reading Time)

General Instructions:

- *All* questions may be attempted
- *All* questions are of equal value
- Reference Sheet will be supplied
- Department of Education approved calculators and templates are permitted
- In every Question, show all relevant mathematical reasoning and/or calculations.
- Marks may not be awarded for careless or badly arranged work

Question 1-5 are to be completed on multiple choice sheet then Question 6-10 to be completed on separate sheets of paper and handed in in separate bundles.
Each question must show your Candidate Number (in the top right hand corner).

QUESTION ONE

The value of $\displaystyle\int_0^{\frac{\pi}{4}} \frac{e^{\tan x}}{\cos^2 x}\,dx$ is which of the following?

(A) $e-1$

(B) $e+1$

(C) e

(D) 0

QUESTION TWO

Which of the following is the equation of the hyperbola with foci $(6,5)$ and $(-4,5)$, and eccentricity $\dfrac{5}{4}$?

(A) $\dfrac{(x-1)^2}{16} - \dfrac{(y-5)^2}{9} = 1$

(B) $\dfrac{(x-1)^2}{16} - \dfrac{(y-5)^2}{9} = -1$

(C) $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$

(D) $\dfrac{x^2}{16} - \dfrac{y^2}{9} = -1$

QUESTION THREE

The zeros of the polynomial $z^4 + az^3 + bz^2 + cz + d$ are all complex numbers that lie on the unit circle. The coefficients a, b, c and d are real numbers. What is the sum of the reciprocals of the roots of the polynomial?

(A) a

(B) $-a$

(C) c

(D) $-c$

Examination continues on next page ...

QUESTION FOUR

Suppose that n is a positive integer and that $z + \dfrac{1}{z} = 2\cos\theta$, for $0 < \theta < \pi$. What is the value of $z^n + \dfrac{1}{z^n}$?

(A) $2(\cos\theta)^n$

(B) $2^n \cos n\theta$

(C) $2(\cos\theta)^n$

(D) $2\cos n\theta$

QUESTION FIVE

The function P satisfies the equation $P(2+x) = P(2-x)$ for all real numbers x. Also, $P(x) = 0$ has four distinct real roots. What is the sum of these roots?

(A) 8

(B) 6

(C) 4

(D) 2

Examination continues on next page ...

QUESTION SIX (Start a new answer booklet.) Marks

(a) Evaluate $\displaystyle\int_0^1 \frac{x^2+x}{\sqrt{x+1}}\,dx$. **2**

(b) Find $\displaystyle\int \tan^3\theta\,d\theta$. **2**

(c) Use the substitution $u=\sqrt{1-x}$ to find $\displaystyle\int_0^1 x^2\sqrt{1-x}\,dx$. **3**

(d) Define the integral J_n as $J_n = \displaystyle\int (\log_e x)^n\,dx$. **4**

(i) Use integration to show that $J_n = x(\log_e x)^n - nJ_{n-1}$.

(ii) Hence show that $\displaystyle\int_1^e (\log_e x)^2\,dx = e-2$.

(e) Evaluate $\displaystyle\int \frac{5x^2-4x+5}{(x-1)(x^2+2)}\,dx$ by use of partial fractions. **3**

(f) Using $t=\tan\frac{x}{2}$, evaluate $\displaystyle\int \frac{1}{1+\sin x}\,dx$. Leave your answer in terms of $\tan\frac{x}{2}$. **2**

Examination continues on next page . . .

QUESTION SEVEN (Start a new answer booklet.) Marks

(a) If α, β, γ are the roots of the equation $3x^3 - 5x^2 - 4x + 3 = 0$, find the cubic equation having roots $\alpha - 1$, $\beta - 1$, $\gamma - 1$. 3

(b) The cubic equation $x^3 + px + q = 0$ has three non-zero roots α, β, γ. 3

Find, in terms of the constants p and q, the values of

(i) $\alpha^2 + \beta^2 + \gamma^2$

(ii) $\alpha^3 + \beta^3 + \gamma^3$

(c) For the ellipse with equation $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$:

(i) Sketch the ellipse, showing the foci S and S', and the directrices. 4

(ii) Prove that the tangent to the ellipse at the point $P(4\cos\theta, 3\sin\theta)$ has the equation $\dfrac{x\cos\theta}{4} + \dfrac{y\sin\theta}{3} = 1$. 3

(iii) If the ellipse meets the y-axis at B and B', and the tangents at B and B' meet the tangent at P at the points Q and Q', prove that $BQ \cdot B'Q' = 16$. 3

Examination continues on next page . . .

QUESTION EIGHT (Start a new answer booklet.) Marks

(a) (i) Express $\sqrt{3}+i$ in modulus-argument form. 2

(ii) Hence simplify $(\sqrt{3}+i)^{12}$ using de Moivre's theorem. 2

(b) The triangle OPQ lies in the complex plane such that O is the origin, $OP \perp PQ$, $OP = \overset{2}{\cancel{3}}$ and $PQ = \overset{11}{\cancel{26}}$. Let X be a point on PQ such that $PX = 1$ and let z be the complex number representing X. 3

By considering z^3, show that $\angle XOP = \frac{1}{3}\angle QOP$.

(c) (i) Let z be a complex number such that $z \neq 0$. Show that $\dfrac{z}{\overline{z}} - \dfrac{\overline{z}}{z}$ is purely imaginary. 2

(ii) Hence sketch the locus of $\dfrac{z}{\overline{z}} - \dfrac{\overline{z}}{z}$. (Note, you may wish to consider z in the form of $z = r(\cos\theta + i\sin\theta)$, where $r = |z|$ and $\theta = \arg(z)$.) 2

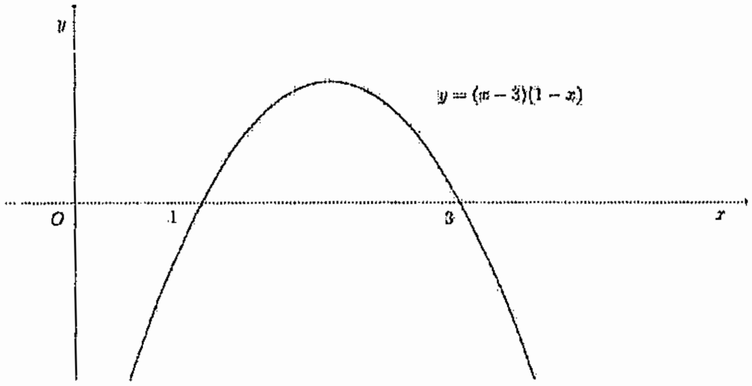
(d) The polynomial $P(z) = z^4 - 4z^3 + 24z^2 - 40z + 100$ has two non-real double zeros. Let α be one of these zeros so that $P(\alpha) = 0$.

(i) Show that $P(\overline{\alpha}) = 0$. 2

(ii) Let $\alpha = x+iy$. By considering the relationship between the zeros of a polynomial and the coefficients of its terms, find x and y, and hence solve $P(z) = 0$. 3

Examination continues on next page . . .

QUESTION NINE (Start a new answer booklet.) Marks



(a) The diagram above shows the graph of $y = f(x)$ where $f(x) = (x-3)(1-x)$.

On separate diagrams, and without the use of calculus, sketch the following graphs, clearly indicating any significant features.

(i) $y = |f(x)|$ 2

(ii) $y = \dfrac{1}{f(x)}$ 2

(iii) $y = e^{f(x)}$ 2

(iv) $y^2 = f(x)$ 2

(b) The hyperbolae $H_1 : x^2 - \dfrac{y^2}{8} = 1$ and $H_2 : y^2 - x^2 = 1$ are known to intersect. 3

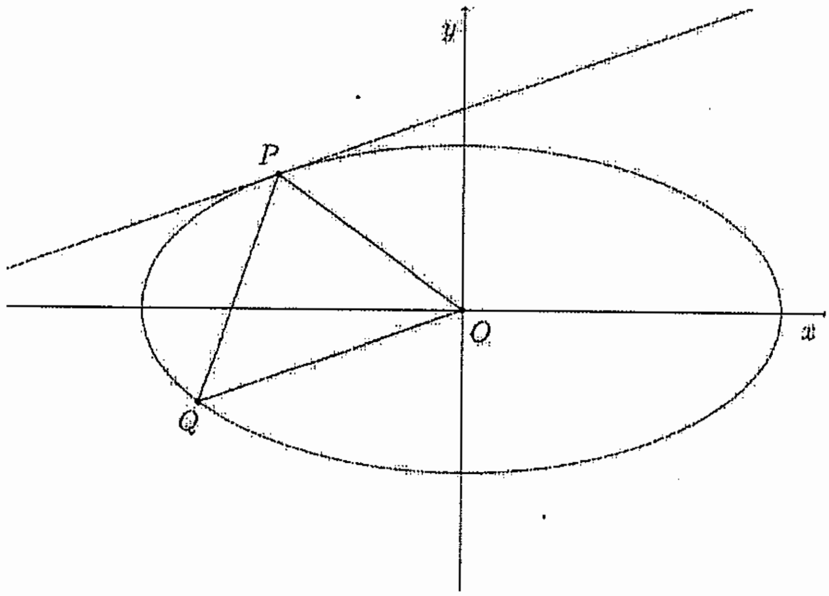
Let C be the family of curves

$$C : (x^2 - \frac{y^2}{8} - 1) + k(y^2 - x^2 - 1) = 0$$

that pass through the points of intersection of H_1 and H_2, for k a real number.

For what values of k will the family of curves C be hyperbolic and with its foci on the x–axis?

Examination continues on next page ...



(c) The diagram above is the graph of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Let $P = (a\cos\alpha, b\sin\alpha)$ and $Q(a\cos\beta, b\sin\beta)$ be two points on the ellipse such that OQ is parallel to the tangent at P, and where O is the origin.

(i) By considering the gradient of the tangent at P and the gradient of OQ, show that $\cot\alpha = -\tan\beta$. **2**

(ii) Hence show that the area of triangle OPQ is $\dfrac{|ab|}{2}$. **3**

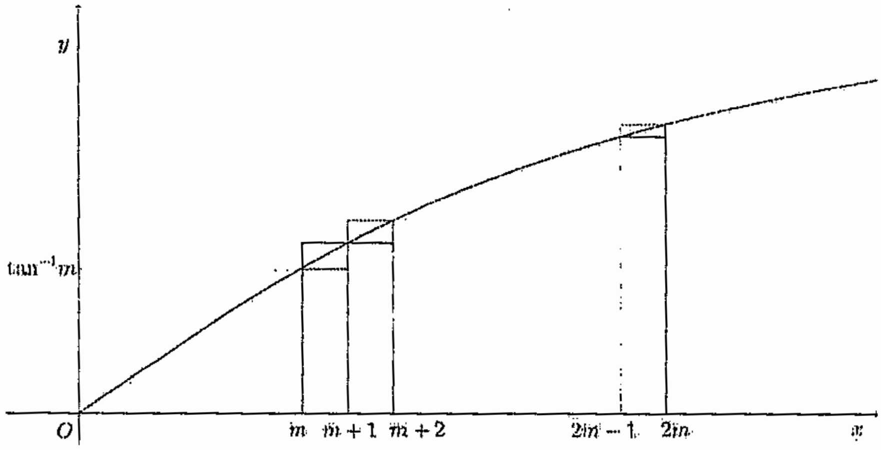
(Note, you may use the fact that for any two points $U = (x_1, y_1)$ and $V = (x_2, y_2)$ in the plane, $\dfrac{1}{2}|x_1y_2 - x_2y_1|$ gives the area of the triangle OUV.)

Examination continues on next page ...

QUESTION TEN (Start a new answer booklet.) Marks

(a) The points $P, Q,$ and R lie in the complex plane and represent the complex numbers $z_1, z_2,$ and z_3 respectively. Show that if triangle PQR is equilateral, then 4

$$z_1{}^2 + z_2{}^2 + z_3{}^2 = z_1 z_2 + z_2 z_3 + z_3 z_1.$$



(b) The diagram above shows the graph of $y = \tan^{-1} x$ for $x \geq 0$. A construction of m upper, and m lower, rectangles has been made between $x = m$ and $x = 2m$, each of width one unit.

Let $S_m = \displaystyle\sum_{k=1}^{m} \tan^{-1}(m+k) = \tan^{-1}(m+1) + \tan^{-1}(m+2) + \cdots + \tan^{-1}(2m)$.

(i) By considering the sums of the areas of the upper and lower rectangles, along with the integral $\displaystyle\int_m^{2m} \tan^{-1} x\, dx$, prove that 4

$$2m\tan^{-1}(2m) - m\tan^{-1} m + \frac{1}{2}\ln\left(\frac{1+m^2}{1+4m^2}\right) < S_m$$

and that

$$S_m < (2m+2)\tan^{-1}(2m+2) - (m+1)\tan^{-1}(m+1) + \frac{1}{2}\ln\left(\frac{1+m^2}{1+4m^2}\right).$$

(ii) Hence find 2

$$S_{1000} = \tan^{-1}(1001) + \tan^{-1}(1002) + \cdots + \tan^{-1}(2000)$$

to three significant figures.

Examination continues on next page . . .

(c) A polynomial $P(z)$ is said to be *reducible* over the integers $\mathbb{Z}$ if it can be written as a product of polynomials $Q(z)R(z) = P(z)$, where $Q(z)$ and $R(z)$ are both integer polynomials (that is, polynomials whose coefficients are all integers). If such a factorisation cannot be performed, we say that the polynomial is *irreducible.*

Define the polynomial $P(z)$ as

$$P(z) = z^n + z + p$$

where z is a complex number, n is a positive integer and $p > 2$ is a prime number.

(i) Suppose that $P(z)$ may be factorised into a product of two integer polynomials as 1

$$P(z) = Q(z)R(z)$$

Show that either $Q(0) = \pm 1$ and $R(0) = \pm p$ or $R(0) = \pm 1$ and $Q(0) = \pm p$.

(ii) Without loss of generality, take the case where $Q(0) = \pm 1$ and $R(0) = \pm p$. 2
Let the roots of $Q(z) = 0$ be $\alpha_1, \alpha_2, \ldots, \alpha_k$ where k is some integer less than n.
Given that the product of the roots of a monic polynomial of degree k is given by

$$\alpha_1 \alpha_2 \cdots \alpha_k = (-1)^k a_0$$

where a_0 is the constant term of the polynomial, show that there is at least one root α_j of $Q(z)$ such that $|\alpha_j| \leq 1$.

(iii) Hence, using the triangle inequality $|z + w| \leq |z| + |w|$ or otherwise, show that $P(z)$ is *irreducible.* 3

END OF EXAMINATION

MULTIPLE CHOICE 1. A 2. A 3. B . 4. D 5. A

2017 Task 2 Term 1

MATHEMATICS Extension 2: Question...6...		
Suggested Solutions	**Marks Awarded**	**Marker's Comments**
$\int \frac{x^2+x}{\sqrt{x+1}}\,dx = \int_0^1 \frac{x(x+1)}{\sqrt{x+1}}\,dx$ $= \int_0^1 x\sqrt{x+1}\,dx$ let $u = x+1$ $du = dx$ $x=0 \quad u=1$ $x=1 \quad u=2$ $= \int_1^2 (u-1)u^{1/2}\,du.$	1	students could use integration by parts but were less successful
$= \int_1^2 u^{3/2} - u^{1/2}$ $= \left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right]_1^2$ $= \frac{8\sqrt{2}}{5} - \frac{4\sqrt{2}}{3} - \frac{2}{5} + \frac{2}{3}$ $= \frac{4}{15}(1+\sqrt{2})$	1	2nd mark was given for this Answer should have been simplified

1

J:\Maths\marking templates\Suggested Mk solns template.doc

MATHEMATICS Extension 2: Question........ 2		
Suggested Solutions	**Marks Awarded**	**Marker's Comments**
(b) $\int \tan^3\theta = \int \tan\theta \cdot \tan^2\theta \, d\theta$		
$= \int \tan\theta(\sec^2\theta - 1)\,d\theta$	1	
$= \int \tan\theta\sec^2\theta \, d\theta - \int \frac{\sin\theta}{\cos\theta}\,d\theta$		well done
$= \frac{1}{2}\tan^2\theta + \log_e\lvert\cos x\rvert + C$	1	
OR $\frac{1}{2}\sec^2\theta + \log_e\lvert\cos x\rvert + c.$		
(c) let $u = \sqrt{1-x}$		
$u^2 = 1-x$		
$x = 1-u^2$		
$x^2 = (1-u^2)^2$		
$dx = -2u\,du.$	1	
$x=0 \quad u=1$		
$x=1 \quad u=0$		
$\int_0^1 x^2\sqrt{1-x}\,dx = \int_1^0 (1-u^2)^2 u(-2u)\,du$	1	Many students were careless and lost terms

<table>
<tr><th colspan="3">MATHEMATICS Extension 2: Question 6....</th></tr>
<tr><th>Suggested Solutions</th><th>Marks Awarded</th><th>Marker's Comments</th></tr>
<tr><td>

$2\int_0^1 u^2 - 2u^4 + u^6\,du$

$= 2\left[\frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7}\right]_0^1$

$= 2\left(\frac{1}{3} - \frac{2}{5} + \frac{1}{7}\right)$

$= \frac{16}{105}$

</td><td>1</td><td>

$x^2\sqrt{1-x}$ is positive so answer should be positive

depending on how many terms were lost CFE were not necessarily given.

</td></tr>
<tr><td>

(c) $J_n = \int (\log_e x)^n\,dx$

$= \int (\log_e x)^n \frac{d}{dx}(x)\,dx$

$= x(\log_e x)^n - \int x \times n(\log_e x)^{n-1} \cdot \frac{1}{x}\,dx$

$= (x\log_e x)^n - n\int (\log_e)^{n-1}\,dx$

$= x(\log_e x)^n - nJ_{n-1}$

</td><td>1</td><td>

had to see this term somewhere.

</td></tr>
</table>

MATHEMATICS Extension 2: Question……..

Suggested Solutions	Marks Awarded	Marker's Comments

MATHEMATICS Extension 2: Question…….. 4

Suggested Solutions	Marks Awarded	Marker's Comments
(ii) $J_2 = \left[x(\ln x)^2\right]_1^e - 2J_1$	1	
$= e(\ln e)^2 - 1(\log_e 1)^2 - 2J_1$		
$J_1 = \left[x\ln x\right]_1^e - \int_1^e dx$	1	
$= e\ln e - 1\ln 1 - \left[x\right]_1^e$		Students made careless errors and skipped steps.
$= e\ln e - (e-1)$	1	
$= \underline{1}$		
$J_2 = \underline{e-2}$	3	
(e) $\dfrac{5x^2-4x+5}{(x-1)(x^2+2)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+2}$		
~~A+B = 5~~ ~~C−B = …~~		
$5x^2-4x+5 = A(x^2+2)+(Bx+C)(x-1)$		
if $x=1$		
$3A = 6$		
$A = 2$		

5

MATHEMATICS Extension 2: Question.....6		
Suggested Solutions	**Marks Awarded**	**Marker's Comments**
if $x=0$ $5 = 2A - C$ $5 = 4 - C$ $C = -1$ if $x = -1$ $14 = 3A + 2B - 2C$ $= 6 + 2B + 2$ $2B = 6$ $\underline{B = 3}$	1	
$\therefore \int \frac{2}{x-1} dx + \int \frac{3x-1}{x^2+2} dx$		
$= 2\int \frac{dx}{x-1} + \frac{3}{2}\int \frac{2x\,dx}{x^2+2} - \int \frac{dx}{x^2+2}$	1	
$= 2\ln(x-1) + \frac{3}{2}\ln(x^2+2) - \frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + C.$	1	No carryforward error if did not have a $\tan^{-1}$ integration

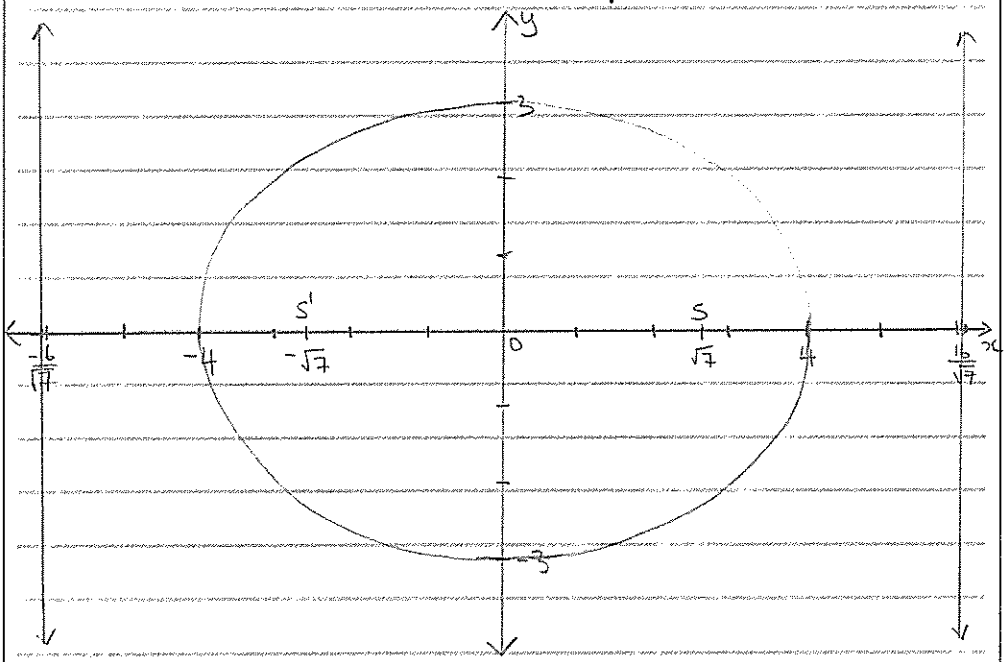
MATHEMATICS Extension 2: Question……..

Suggested Solutions	Marks Awarded	Marker's Comments
$I = \int \frac{1}{1+\sin x}\,dx$		
$t = \tan\frac{x}{2} \quad \sin x = \frac{2t}{1+t^2}$		
$\frac{2}{1+t^2}\,dt = dx$		
$I = \int \frac{1}{1+\frac{2t}{1+t^2}} \cdot \frac{2}{1+t^2}\,dt$	1	
$= \int \frac{2}{1+2t+t^2}$		
$= \int \frac{2}{(t+1)^2}$		
$= -2(t+1)^{-1} + c$		Many students lost mark for not substituting back
$= \frac{-2}{1+\tan\frac{x}{2}} + c.$	1	

J:\Maths\marking templates\Suggested Mk solns template.doc

MATHEMATICS Extension 2: Question 7		
Suggested Solutions	**Marks**	**Marker's Comments**
a) $x = \alpha - 1$		
$\therefore \alpha = x + 1$	1	
$\therefore$ equation will be:		
$3(x+1)^3 - 5(x+1)^2 - 4(x+1) + 3 = 0$		
$3(x^3+3x^2+3x+1) - 5(x^2+2x+1)$ $-4x - 4 + 3 = 0$	1	
$3x^3 + 9x^2 + 9x + 3 - 5x^2 - 10x - 5$ $-4x - 4 + 3 = 0$		
$3x^3 + 4x^2 - 5x - 3 = 0$	1	
b) $x^3 + px + q = 0$		some students had incorrect sum & product of roots. ie. $\alpha+\beta+\gamma = -p$, $\alpha\beta+\alpha\gamma+\beta\gamma = q$
$\therefore \alpha + \beta + \gamma = 0$		
$\alpha\beta + \alpha\gamma + \beta\gamma = p$		
$\alpha\beta\gamma = -q$		
i) $(\alpha+\beta+\gamma)^2 = \alpha^2 + \alpha\beta + \alpha\gamma + \alpha\beta + \beta^2 + \beta\gamma + \alpha\gamma + \beta\gamma + \gamma^2$		
$= \alpha^2 + \beta^2 + \gamma^2 + 2(\alpha\beta + \alpha\gamma + \beta\gamma)$		
$\therefore \alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$		
$= 0^2 - 2(p)$		
$= -2p$	1	
ii) $\alpha^3 + p\alpha + q = 0$ — ①		students who used $(\alpha+\beta+\gamma)^3$ made more mistakes with the expansion.
$\beta^3 + p\beta + q = 0$ — ②	1	
$\gamma^3 + p\gamma + q = 0$ — ③		
① + ② + ③		
$\alpha^3 + \beta^3 + \gamma^3 + p(\alpha+\beta+\gamma) + 3q = 0$		
$\therefore \alpha^3 + \beta^3 + \gamma^3 = -p(0) - 3q$		
$= -3q$	1	
c) i) $b^2 = a^2(1 - e^2)$		
$9 = 16 - 16e^2$		
$e^2 = \frac{7}{16}$		
$e = \frac{\sqrt{7}}{4}$	1	

MATHEMATICS Extension 2: Question 7

Suggested Solutions	Marks	Marker's Comments
$S = (\pm\sqrt{7}, 0)$	1	
directrices: $x = \dfrac{\pm 16}{\sqrt{7}}$	1	
	1	Need to show - x & y intercepts - focus - directrices Please make sure you have a graph that is at least $\frac{1}{3}$ of the page.
ii) $P(4\cos\theta, 3\sin\theta)$ $x = 4\cos\theta$ $\quad y = 3\sin\theta$ $\dfrac{dx}{d\theta} = -4\sin\theta$ $\quad \dfrac{dy}{d\theta} = 3\cos\theta$ $\dfrac{dy}{dx} = \dfrac{dy}{d\theta} \times \dfrac{d\theta}{dx}$ $= -\dfrac{3\cos\theta}{4\sin\theta}$	1	differentiating to find gradient
$\therefore$ equation of tangent is: $y - 3\sin\theta = -\dfrac{3\cos\theta}{4\sin\theta}(x - 4\cos\theta)$	1	
$4y\sin\theta - 12\sin^2\theta = -3x\cos\theta + 12\cos^2\theta$ $3x\cos\theta + 4y\sin\theta = 12\sin^2\theta + 12\cos^2\theta$ $3x\cos\theta + 4y\sin\theta = 12(\sin^2\theta + \cos^2\theta)$ $3x\cos\theta + 4y\sin\theta = 12$ $\dfrac{x\cos\theta}{4} + \dfrac{y\sin\theta}{3} = 1$	1	need to show how you got the 12, ie. $\sin^2\theta + \cos^2\theta = 1$

MATHEMATICS Extension 2: Question 7

Suggested Solutions	Marks	Marker's Comments
iii) tangent at B: $y=3$ B': $y=-3$ when $y=3$, $\frac{x\cos\theta}{4}+\frac{3\sin\theta}{3}=1$ $\frac{x\cos\theta}{4}=1-\sin\theta$ $x=\frac{4(1-\sin\theta)}{\cos\theta}$ $\therefore$ Q is $\left(\frac{4(1-\sin\theta)}{\cos\theta}, 3\right)$ $BQ=\left\|\frac{4(1-\sin\theta)}{\cos\theta}\right\|$ when $y=-3$ $\frac{x\cos\theta}{4}-\frac{3\sin\theta}{3}=1$ $\frac{x\cos\theta}{4}=1+\sin\theta$ $x=\frac{4(1+\sin\theta)}{\cos\theta}$ $\therefore$ Q' is $\left(\frac{4(1+\sin\theta)}{\cos\theta}, -3\right)$ $B'Q'=\left\|\frac{4(1+\sin\theta)}{\cos\theta}\right\|$ $\therefore BQ.B'Q'=\left\|\frac{4(1-\sin\theta)}{\cos\theta}\right\|\times\left\|\frac{4(1+\sin\theta)}{\cos\theta}\right\|$ $=\left\|\frac{16(1-\sin^2\theta)}{\cos^2\theta}\right\|$ $=\left\|\frac{16\cos^2\theta}{\cos^2\theta}\right\|$ $=16$		1 for Q & Q' 1 for distance of BQ & B'Q' 1 for showing $BQB'Q'=16$ need to see how $\frac{16(1-\sin^2\theta)}{\cos^2\theta}$ $=16\frac{\cos^2\theta}{\cos^2\theta}$ cannot just say $\frac{(1-\sin^2\theta)}{\cos^2\theta}$ is 1, they are different terms! Need to show why they are equal.

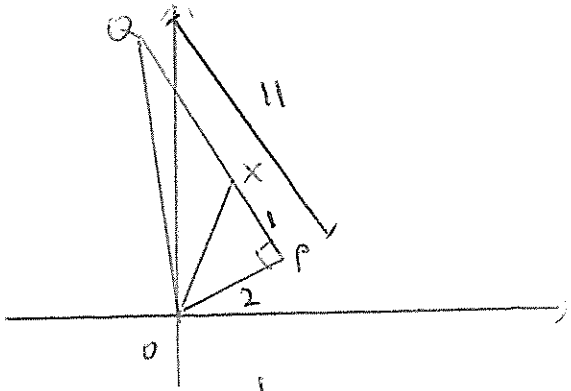
Mathematics Ext2 Question 8.

Suggested Solutions	Marks	Marker's Comments
a) i) let $z = \sqrt{3} + i$ $\lvert z\rvert = \sqrt{(\sqrt{3})^2 + 1^2} = 2$ $\operatorname{Arg}(z) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$	1	For either just the argument or the modulus
$\therefore z = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$	1	
$\boxed{\underline{\text{Note}}: z = 2\operatorname{cis}\frac{\pi}{6} \text{ is insufficient!}}$		
ii) $(\sqrt{3} + i)^{12} = 2^{12}\left(\cos\frac{12\pi}{6} + i\sin\frac{12\pi}{6}\right)$ (De Moivre's) $= 2^{12}(\cos 2\pi + i\sin 2\pi)$	1	
$= 2^{12}$ $= 4096$	1	

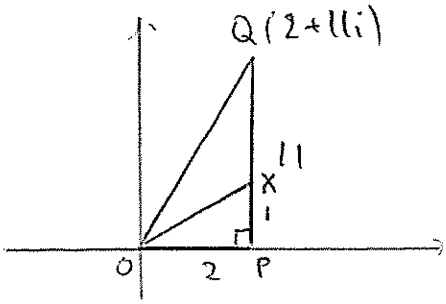
Mathematics Ext2 Question 8

Suggested Solutions	Marks	Marker's Comments

b)



↓ is congruent to



$$z^3 = (2+i)^3$$
$$= 2^3 + 3.2^2 i + 3.2 i^2 + i^3 \quad \left(\text{Binomial Expansion}\right)$$
$$= 8 + 12i - 6 - i$$
$$= 2 + 11i$$
$$= Q \ldots \qquad (1)$$

$\therefore \operatorname{Arg}(z^3) = \operatorname{Arg}(X) = \operatorname{Arg}(2+11i) = \angle QOP$ — (1)

$\therefore 3\operatorname{Arg} z = \angle QOP$

$\operatorname{Arg} z = \angle XOP$

$\therefore \angle XOP = \frac{1}{3}\angle QOP$ — (1)

Marker's comments:

Note: 1/3 if:

* $\angle QOP = \tan^{-1}\frac{11}{2}$ and $\angle XOP = \tan^{-1}\frac{1}{2}$

or

* $\angle XOP = \operatorname{Arg}(z) - \operatorname{Arg}(\overrightarrow{OP})$ and $\angle QOX = \operatorname{Arg}(\overrightarrow{OQ}) - \operatorname{Arg}(z)$

But not 2/3 for both!!!

You receive $\boxed{\text{NO MARKS}}$ for recognising that $|z^3| = |\overrightarrow{OQ}|$!!!

Mathematics _____ Question _____

Suggested Solutions	Marks	Marker's Comments
c) i) let $z = x+iy$, $\therefore \bar{z} = x-iy$		
$\therefore \dfrac{z}{\bar{z}} - \dfrac{\bar{z}}{z} = \dfrac{x+iy}{x-iy} - \dfrac{x-iy}{x+iy}$		
$= \dfrac{(x+iy)^2 - (x-iy)^2}{x^2+y^2}$	1	
$= \dfrac{4xyi}{x^2+y^2}$	1	
which is imaginary.		
<u>or</u>		
$\dfrac{z}{\bar{z}} - \dfrac{\bar{z}}{z} = \dfrac{z^2 - \bar{z}^2}{z\bar{z}}$	1	
$= \dfrac{(z+\bar{z})(z-\bar{z})}{\lvert r\rvert^2}$		
$= \dfrac{Re(z) \times Im(z)\, i}{\lvert r\rvert^2}$	1	
Which is imaginary		

Mathematics ____ Question ____

Suggested Solutions	Marks	Marker's Comments

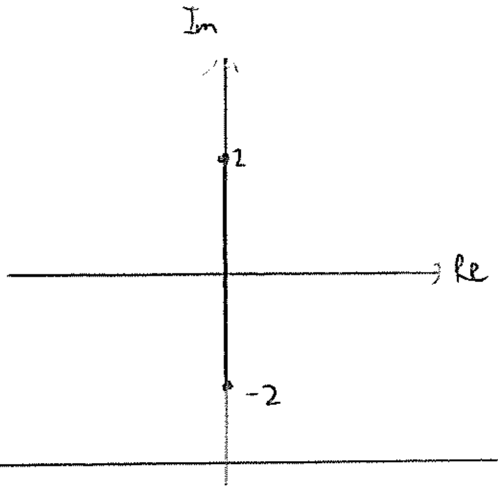
ii) let $z = r(\cos\theta + i\sin\theta)$

$$\therefore \frac{4xyi}{x^2+y^2} = \frac{4(r\cos\theta)(r\sin\theta)i}{r^2\cos^2\theta + r^2\sin^2\theta}$$

$$= \frac{4\cos\theta\sin\theta i}{1}$$

$$= 2i\sin 2\theta$$ — 1

$\therefore$ locus is between -2 and 2 on the Imagnory axis.



— 1

Note: Students receive 1 for drawing a locus on the Imagnory axis without the restricted domain.

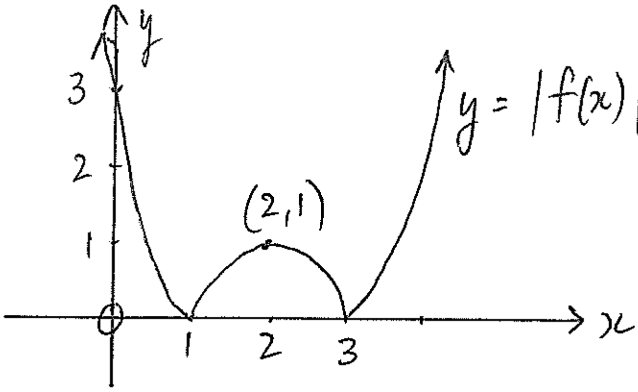
Mathematics ____ Question ____		
Suggested Solutions	Marks	Marker's Comments
d) i) $P(\alpha)=0$ (Factor theorem) $\therefore \alpha^4-4\alpha^3+24\alpha^2-40\alpha+100=0$ $\overline{\alpha^4-4\alpha^3+24\alpha^2-40\alpha+100}=\overline{0}$	1	
$\overline{\alpha^4}-4\overline{\alpha^3}+24\overline{\alpha^2}-40\overline{\alpha}+100=0$ $\overline{\alpha}^4-4\overline{\alpha}^3+24\overline{\alpha}^2-40\overline{\alpha}+100=0$	1	
$\therefore \overline{\alpha}$ is also a root. $\therefore P(\overline{\alpha})=0$ (factor theorem) or Polynomial with real coefficients.		
$\therefore$ If α is a complex root, then $\overline{\alpha}$ is also a root. $\therefore P(\overline{\alpha})=0$ (factor theorem)	1	mark only!!!

Mathematics ____ Question ____

Suggested Solutions	Marks	Marker's Comments
ii) let the roots be $\alpha, \alpha, \bar{\alpha}, \bar{\alpha}$,		
let $\alpha = x+iy$, $\therefore \bar{\alpha} = x - iy$		
$\therefore 2\alpha + 2\bar{\alpha} = 4$ (sum of roots)	1	
$\therefore 4x = 4$		
$x = 1$	1	
$\alpha^2\bar{\alpha}^2 = 100$ (product of roots)		
$\alpha\bar{\alpha} = \pm 10$		
$\therefore x^2+y^2 = \pm 10$		
Since $x=1$, $1+y^2 = \pm 10$		
$y^2 = -1 \pm 10$		
$y = 9$ only (since $y \in \mathbb{R}$)		
$\therefore y = \pm 3$		
$\therefore$ roots are $1+3i$ and $1-3i$	1	

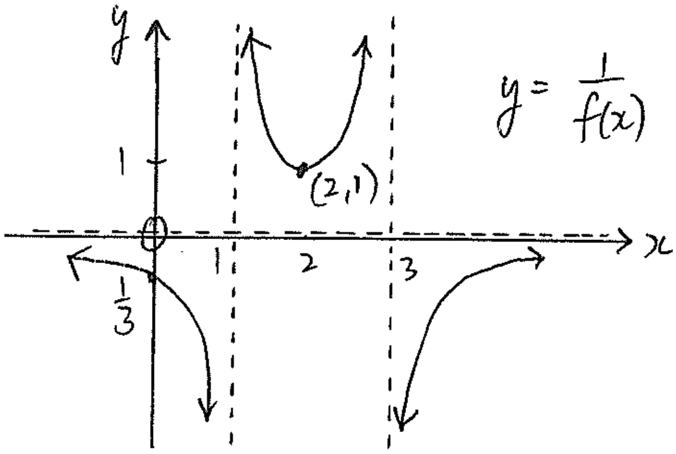
Task 2 : 2017 **MATHEMATICS Extension 2: Question** 9 JRAHS

Suggested Solutions	Marks	Marker's Comments
a) (i)		



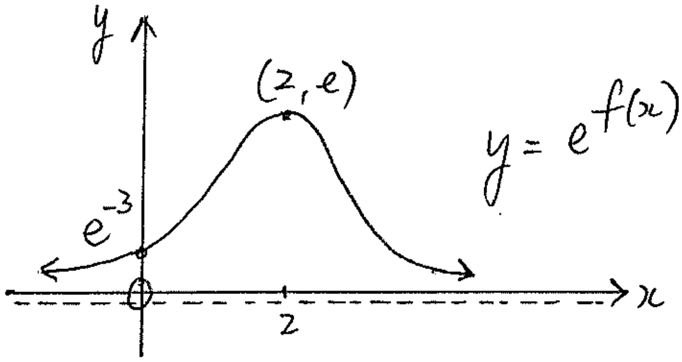
Marks	Marker's Comments
1	correct shape
1	intercepts + turning points +

(ii)



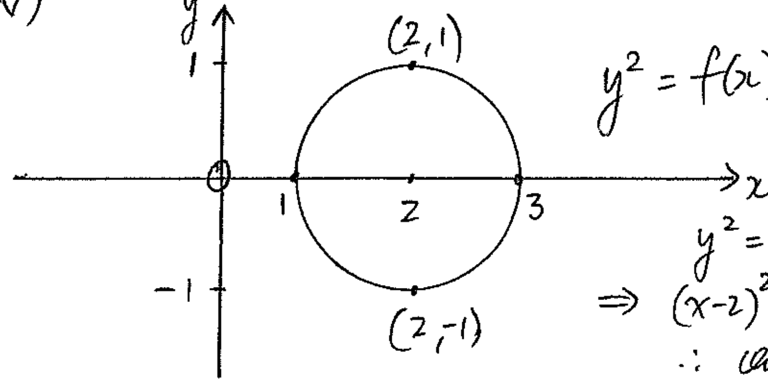
Marks	Marker's Comments
1	correct shape
1	asymptotes + intercepts

(iii)



Marks	Marker's Comments
1	correct shape
1	intercepts, + turning points + asymptotes

(iv)



$y^2 = -x^2+4x-3$

$\Rightarrow (x-2)^2+y^2 = 1$

$\therefore$ centre $(2,0)$

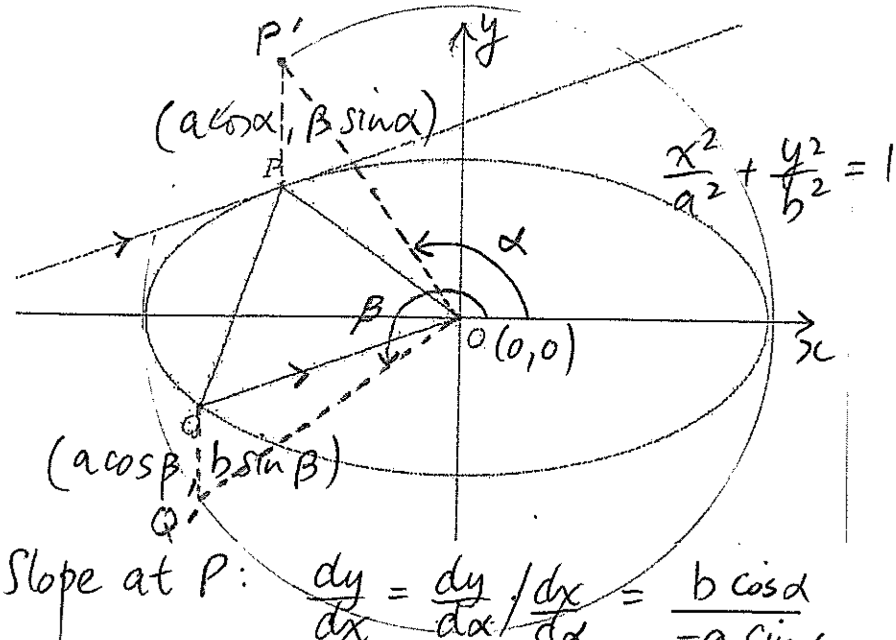
radius 1 unit

Marks	Marker's Comments
1	shape
1	turning points + vertical tangents

Task 2 : 2017 **MATHEMATICS Extension 2: Question** 9 JRAHS

Suggested Solutions	Marks	Marker's Comments
b) Consider curve C: $(x^2 - y^2/8 - 1) + k(y^2 - x^2 - 1) = 0$ $(1-k)x^2 - (\frac{1}{8} - k)y^2 = 1+k$ $\frac{x^2}{\left(\frac{1+k}{1-k}\right)} - \frac{y^2}{\left(\frac{1+k}{\frac{1}{8}-k}\right)} = 1$ which is of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.	1	For expressing equation in the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ or $mx^2 - ny^2 = 3$
Hence hyperbolic and focii on x-axis, $\frac{1+k}{1-k} > 0$ and $\frac{1+k}{\frac{1}{8}-k} > 0$	1	For Both inequalities for Focii on x-axis
$\Rightarrow (1+k)(1-k) > 0$ and $(1+k)(\frac{1}{8} - k) > 0$ $\Rightarrow -1 < k < 1$ and $-1 < k < \frac{1}{8}$ (intersection) Hence Solution Set: $\{k : -1 < k < \frac{1}{8}\}$	1	For final correct solution
<u>OR</u> From $(1-k)x^2 - (\frac{1}{8} - k)y^2 = 1+k$ $1-k > 0$ and $(\frac{1}{8} - k) > 0$ and $1+k > 0$ i.e. $k < 1$ and $k < \frac{1}{8}$ and $k > -1$ yielding: $-1 < k < \frac{1}{8}$		If $b^2 = a^2(e^2-1)$, then $e > 1$, $\Rightarrow k < 1$ max 1 Note: $e > 1$ $\not\Rightarrow$ focii on x-axis

Task 2 : 2017 **MATHEMATICS Extension 2: Question** 9 JRAHS

Suggested Solutions	Marks	Marker's Comments
C (i)		
		
Slope at P: $\frac{dy}{dx} = \frac{dy}{d\alpha} / \frac{dx}{d\alpha} = \frac{b\cos\alpha}{-a\sin\alpha}$		
$= -\frac{b}{a}\cot\alpha$	1	For correct gradient
$m_{OQ} = \frac{b\sin\beta - 0}{a\cos\beta - 0} = \frac{b}{a}\tan\beta$		
But OQ is parallel to tangent at P		
$\Rightarrow m_{OQ} = m(\text{at } P)$	1	Correctly finding gradient of OQ and equating to gradient at P
i.e $\frac{b}{a}\tan\alpha = -\frac{b}{a}\cot\alpha$		
or $\cot\alpha = -\tan\beta$		
$\longrightarrow$		

Task 2 : 2017 **MATHEMATICS Extension 2: Question** 9 JRAHS

Suggested Solutions	Marks	Marker's Comments
C (ii) $\lvert OPQ\rvert = \frac{1}{2}\lvert x_1 y_2 - y_1 x_2\rvert$		
$= \frac{1}{2}\lvert a\cos\alpha\, b\sin\beta - a\cos\beta \cdot b\sin\alpha\rvert$		
$= \frac{1}{2}\lvert ab\rvert \lvert\sin(\beta-\alpha)\rvert$	1	Correctly for reversing the trig expansion
But from (i) $\cot\alpha = -\tan\beta$		
$\Rightarrow \dfrac{\cos\alpha}{\sin\alpha} = -\dfrac{\sin\beta}{\cos\beta}$		
$\Rightarrow \cos\alpha\cos\beta + \sin\alpha\sin\beta = 0$		
$\Rightarrow \cos(\beta-\alpha) = 0$	1	For using (ii) to show $\beta - \alpha = \frac{\pi}{2}$
$\Rightarrow \beta - \alpha = \frac{\pi}{2}$ or $-\frac{\pi}{2}$		
Note: $\cot\alpha = -\tan\beta = \tan(-\beta) \Rightarrow \alpha - \beta = \frac{\pi}{2}$ $\therefore \lvert OPQ\rvert = \frac{1}{2}\lvert ab\rvert \lvert\pm 1\rvert = \frac{1}{2}\lvert ab\rvert$	1	For confirmation that $\lvert\sin(\beta-\alpha)\rvert = 1$
$\lvert\sin(\beta-\alpha)\rvert = \sqrt{\sin^2(\beta-\alpha)} = \sqrt{1-\cos^2(\beta-\alpha)} = \sqrt{1-0} = 1$		

Question 10 1/6

*badly done, handwriting through the entire question was hard to read ⍨

(a)

$$\arg(z_3 - z_1) - \arg(z_2 - z_1) = \pi/3$$

$$\arg(z_1 - z_2) - \arg(z_3 - z_2) = \pi/3$$

$$\arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right) = \arg\left(\frac{z_1 - z_2}{z_3 - z_2}\right) = \pi/3 \quad \ldots\ldots (1)$$

$PQ = QR = PR$ (all sides of an equilateral triang

$$\therefore |z_1 - z_2| = |z_3 - z_2| = |z_3 - z_1|$$

so $$\left|\frac{z_3 - z_1}{z_2 - z_1}\right| = \left|\frac{z_1 - z_2}{z_3 - z_2}\right| \quad \ldots\ldots (2)$$

now if $|z| = |w|$ and $\arg(z) = \arg(w)$ then $z = w$.

so $$\frac{z_3 - z_1}{z_2 - z_1} = \frac{z_1 - z_2}{z_3 - z_2}$$

$$\therefore (z_3 - z_1)(z_3 - z_2) = (z_1 - z_2)(z_2 - z_1)$$

$$z_3^2 - z_2 z_3 - z_1 z_3 + z_1 z_2 = z_1 z_2 - z_1^2 - z_2^2 + z_1 z_2$$

$$\therefore z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_1 z_3$$

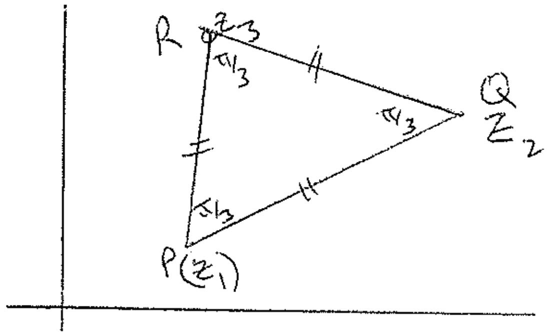
* A lot of students mixed up vectors with modulus, this was a fatal error.
* Students picked up the LHS. and suddenly made it equal to zero with no reasons why.
* A lot of basic errors, and then lots of fudging.

* A clear diagram would have helped more students do better.

OR

2/6

$\vec{PQ} = z_2 - z_1$
$\vec{PR} = z_3 - z_1$
$\vec{QR} = z_3 - z_2$ (1)



$PQ = PR = QR$ (all sides of an equilateral triangle) (

$\therefore \frac{PQ}{PR} = \frac{PR}{QR}$

$$\frac{z_2 - z_1}{z_3 - z_1} = \frac{z_3 - z_1}{z_3 - z_2} \quad (1)$$

$$z_2z_3 - z_2^2 - z_1z_3 + z_1z_2 = z_3^3 - z_3z_1 - z_1z_3 + z_1^2$$

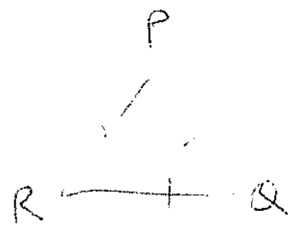
$$z_2z_3 + z_1z_2 + z_1z_3 = z_1^2 + z_2^2 + z_3^2 \quad (1)$$

ie $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_1z_3 + z_2z_3$

OR

$\triangle PQR$ is equilateral (data) $\vec{PQ} = z_2 - z_1$ $\vec{PR} = z_3$
$\therefore$ All angles are $\pi/3$ and all sides are equal. $\vec{QR} = z_3 -$
ie $R\hat{P}Q = P\hat{R}Q = R\hat{Q}P = \pi/3$

$\vec{PR} \operatorname{cis} \pi/3 = \vec{PQ}$ $\therefore \operatorname{cis} \pi/3 = \frac{\vec{PQ}}{\vec{PR}}$
$\vec{RQ} \operatorname{cis} \pi/3 = \vec{RP}$
$\vec{PQ} \operatorname{cis} \pi/3 = \vec{RQ}$ $\operatorname{cis} \pi/3 = \frac{\vec{RP}}{\vec{RQ}}$



$$\therefore \frac{z_2 - z_1}{z_3 - z_1} = \frac{z_1 - z_3}{z_2 - z_3}$$

$$(z_2 - z_1)(z_2 - z_3) = (z_1 - z_3)(z_3 - z_1)$$
$$z_2^2 - z_2z_3 - z_1z_2 + z_1z_3 = z_1z_3 - z_1^2 - z_3^2 + z_1z_3$$

$$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$$

(b)

lower rectangles $< \int_m^{2m} \tan^{-1} x \, dx <$ upper rectangles

upper rectangles: $1 \times \tan^{-1}(m+1) + 1 \times \tan^{-1}(m+2) + \dots + 1 \times \tan^{-1}(2m)$
$= \tan^{-1}(m+1) + \tan^{-1}(m+2) + \dots + \tan^{-1}(2m)$
$= S_m$

$\int_m^{2m} \tan^{-1} x \, dx \Rightarrow$ let $u = \tan^{-1} x$ $\quad \frac{dv}{dx} = 1$

$\frac{du}{dx} = \frac{1}{1+x^2} \quad v = x$

$$I = \left[x \tan^{-1} x\right]_m^{2m} - \int_m^{2m} \frac{x}{1+x^2} \, dx$$

$$= 2m \tan^{-1}(2m) - m \tan^{-1}(m) - \frac{1}{2} \int_m^{2m} \frac{2x}{1+x^2} \, dx$$

$$= 2m \tan^{-1}(2m) - m \tan^{-1}(m) - \left[\tfrac{1}{2} \ln(1+x^2)\right]_m^{2m}$$

$$= 2m \tan^{-1}(2m) - m \tan^{-1}(m) - \tfrac{1}{2} \ln(1+4m^2) + \tfrac{1}{2} \ln(1+m^2)$$

$$= 2m \tan^{-1}(2m) - m \tan^{-1}(m) + \tfrac{1}{2} \ln\left(\frac{1+m^2}{1+4m^2}\right)$$

$\therefore$ so $\int_m^{2m} \tan^{-1} x \, dx < S_m$

ie $2m \tan^{-1}(2m) - m \tan^{-1}(m) + \frac{1}{2} \ln\left(\frac{1+m^2}{1+4m^2}\right)$

now for the lower rectangles:

area of lower rectangles $= 1 \times \tan^{-1}(m) + 1 \times \tan^{-1}(m+1) + \dots + 1 \times \tan^{-1}(2m-1)$

$$= \tan^{-1}(m) + S_m - \tan^{-1}(2m)$$

so $\tan^{-1}(m) + S_m - \tan^{-1}(2m) < 2m \tan^{-1}(2m) - m \tan^{-1}(m) + \frac{1}{2} \ln\left(\frac{1+x^2}{1+4x^2}\right)$

$S_m < 2m \tan^{-1}(2m) + \tan^{-1}(2m) - m \tan^{-1}(m) - \tan^{-1}(m) + \frac{1}{2} \ln\left(\frac{1+x^2}{1+4x^2}\right)$

$S_m < (2m+1) \tan^{-1}(2m) - (m+1) \tan^{-1}(m) + \frac{1}{2} \ln\left(\frac{1+x^2}{1+4x^2}\right)$

now, $y = \tan^{-1}x$ is an increasing function
so $\tan^{-1}(2m+2) > \tan^{-1}(2m)$
$m > 0$, $2m+2 > 0$

$\therefore (2m+1)\tan^{-1}(2m) < (2m+2)\tan^{-1}(2m+2)$
$(m+1)\tan^{-1}(m) < (m+1)\tan^{-1}(m+1)$

so, $S_m < (2m+1)\tan^{-1}(2m) - (m+1)\tan^{-1}(m) + \frac{1}{2}\ln\left(\frac{\ldots}{\ldots}\right)$

$\therefore S_m < (2m+2)\tan^{-1}(2m+2) - (m+1)\tan^{-1}(m-1) + \frac{1}{2}\ln\left(\ldots\right)$

ii) Lower bound to S_{1000} is:

$2000\tan^{-1}(2000) - 1000\tan^{-1}(1000) + \frac{1}{2}\ln\left(\frac{1+10^6}{1+4\times10^6}\right)$
$= 1570.103179\ldots\ldots$

upper bound is:
$2002\tan^{-1}(2002) - 1001\tan^{-1}(1001) + \frac{1}{2}\ln\left(\frac{1+10^6}{1+4\times10^6}\right)$
$= 1571.6739$

$\therefore S_{1000} \doteqdot 1570$, (3 sig. figures)

* Full marks were awarded if the both bounds were calculated correctly.

* Some students ($\doteqdot 50$), forgot to put their calculator into RADIANS MODE, so they got:
upper limit $= 90\,089.30684$
lower limit $= 89\,999.30684$
$\therefore S_{1000} = 90044$
$= 90\,000$ (3 sig figures)
they were awarded one mark maximum.

$\frac{5}{6}$

(c) (i) $P(z) = z^n + z + p$ ——— ①

Supposing $P(z) = Q(z)R(z)$ ——— ②

then by ①, $P(0) = p$

& so by ②

$$P(0) = Q(0)R(0) = p$$

Compulsory

As p is prime & Q, R are integer polynomials (i.e. will return integers for integer inputs), then either $Q(0) = \pm 1 \Rightarrow R(0) = \pm p$ or $Q(0) = \pm p$ while $R(0) = \pm 1$.

(ii) If we take, wlog, the case where $Q(0) = \pm 1$, then ± 1 is the constant term of Q.

Hence, for $\alpha_1, \ldots, \alpha_k$ the roots of $Q(z) = 0$, we have

$$|\alpha_1 \alpha_2 \cdots \alpha_k| = |\pm 1| = 1.$$

*Explanation needed to be clear and convincing!

Now, if all the roots were > 1, then $|\alpha_1 \cdots \alpha_k| > 1$ *

So at least one root must be less than or equal to 1.

*A lot of students just wrote the question out again, and hence scored 0 mks

So there is at least one α_j such that

$$|\alpha_j| \leq 1.$$

(iii) Take $z = \alpha_j$ in $P(z) = Q(z)R(z)$

and in $P(z) = z^n + z + p.$ — ①

Then:

$$P(\alpha_j) = Q(\alpha_j)R(\alpha_j)$$

$$= 0 \quad — ②$$

So $P(\alpha_j) = \alpha_j^n + \alpha_j + p = 0$ by ① & ②

But then $\alpha_j^n + \alpha_j = -p$

so $|\alpha_j^n + \alpha_j| = |p|$

But $|\alpha_j^n + \alpha_j| \leq |\alpha_j^n| + |\alpha_j|$.

i.e. $|p| \leq |\alpha_j^n| + |\alpha_j|$

Since $|\alpha_j| \leq 1 \rightarrow |\alpha_j^n| \leq 1$

& so $|p| \leq |\alpha_j^n| + |\alpha_j| \leq 1+1 = 2$

i.e. $|p| \leq 2$

But $p > 2$ & prime, so $p \geq 3$. #

Hence such polynomials cannot be factored as a product of integer polynomials. Therefore, $P(z)$ is irreducible.