Name: ______________________

Class: ______________________

2018

Term 4 Assessment (HSC)

Mathematics Extension 2

General Instructions

- Working time – 90 minutes + 5minutes reading
- NESA approved calculators may be used
- Write using black pen
- All necessary working should be shown in all questions
- Write your student number at the top of every page of your answer sheets.

4 questions – 15 marks each

Total marks – 60

Attempt all questions

Start each question on a new sheet of paper.

Question 1 (15 Marks)

Marks

(a) Consider the complex numbers $\omega = -1 + \sqrt{3}i$ and $z = \sqrt{3} + 2i$

(i) Evaluate $\omega\bar{z}$ 2

(ii) Evaluate $|\omega|$ 1

(iii) Find the value of $\arg\omega$ 1

(iv) Evaluate $\frac{\omega}{z}$ 2

(v) Find w^5 in mod-arg form 1

(b) Find the square roots of $8 + 4\sqrt{5}i$ 3

(c) On an Argand diagram, show the region where the inequalities $1 \le |Z| \le 3$ and $\frac{\pi}{4} \le \arg Z \le \frac{\pi}{2}$ hold simultaneously. 3

(d) On an Argand diagram sketch $|Z + 1| = |Z - i|$ 2

Marks

Question 2 (15 Marks) Start a New Sheet of paper

(a) The complex numbers z and w are such that $z = \frac{3a-5i}{1+2i}$ and w = 1-13bi, 2

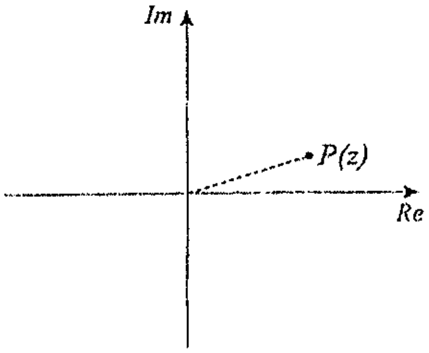
where a and b are real numbers. Given that $\bar{z} = w$, find the values of a and b.

(b) Sketch the locus of

(i) $\arg\frac{(z+1)}{(z-1)} = \frac{\pi}{2}$ 2

(ii) $2|z| = z + \bar{z} + 4$ 2

(c)



In the diagram above, the point P represents the complex number z, where

$|z| = 2$ and $0 < \arg z < \frac{\pi}{4}$. Copy the diagram and indicate clearly the mod and arg of

(i) The point Q representing iz 1

(ii) The point R representing $\bar{z}$ 1

(iii) The point S representing $\frac{-3z}{2}$ 1

(iv) The point T representing z^2 2

(d)

(i) Express $(1 - i)(5 + i)^2$ in the form a+ib where a and b are real 2

(ii) Hence prove that $\tan^{-1}\left(\frac{7}{17}\right) + 2\tan^{-1}\left(\frac{1}{5}\right) = \frac{\pi}{4}$ 2

Question 3 (15 Marks) Start a New Sheet of paper

(a)

(i) Find the roots of $z^4 - i = 0$, giving your answers in mod-Arg form. 2

(ii) Plot the roots on an Argand diagram 1

(iii) Find the exact area of the quadrilateral they form 1

(i) If λ is a root of the equation then show that λ^5 is also a root. 2

(b) The complex number z represents a variable point P in the Argand plane.

If $|z| = |z - \alpha|$ where α is real evaluate $\arg(z^2 - z\alpha)$ 3

(c) The complex number is represented by z such that $|z - 3\sqrt{3} - 3i| = 3$

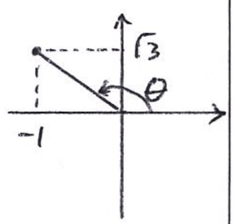
(i) Find the maximum value of $|z|$ 2

(ii) Find the range of values of arg z 2

(d) If $z = (\cos 2\theta - 1) - i\sin 2\theta$ show that $\arg z = \frac{\pi}{2} - \theta$ 2

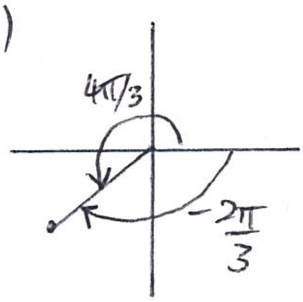
JRAHS : Task 1 Term 4 : 2018

MATHEMATICS Extension 2: Question 1

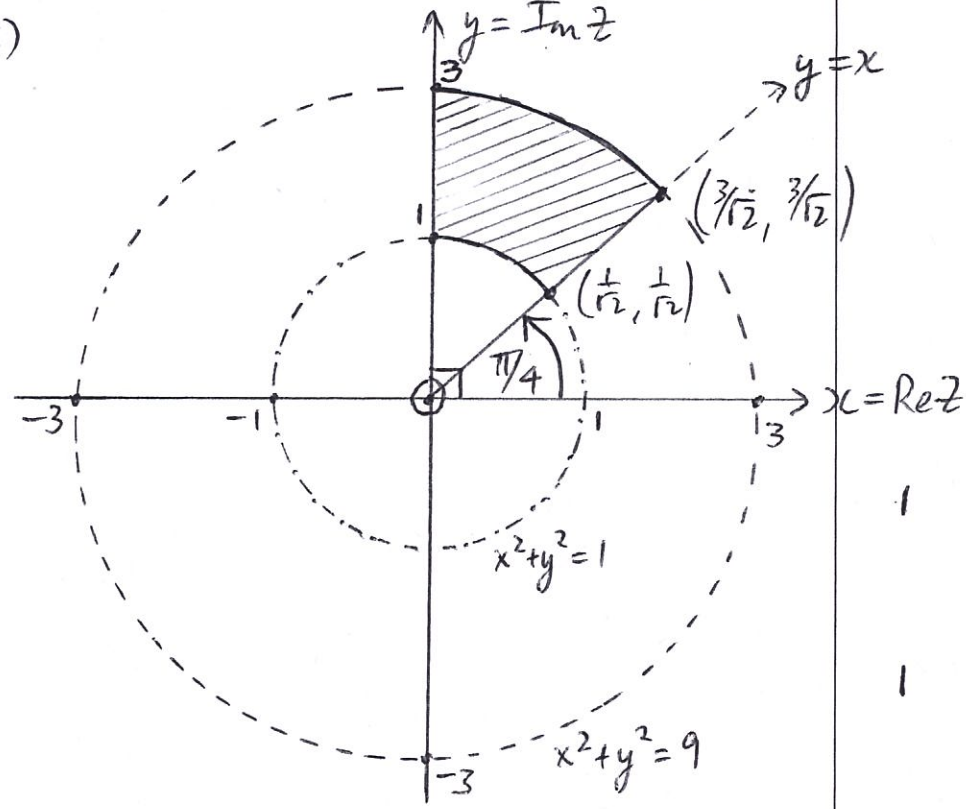
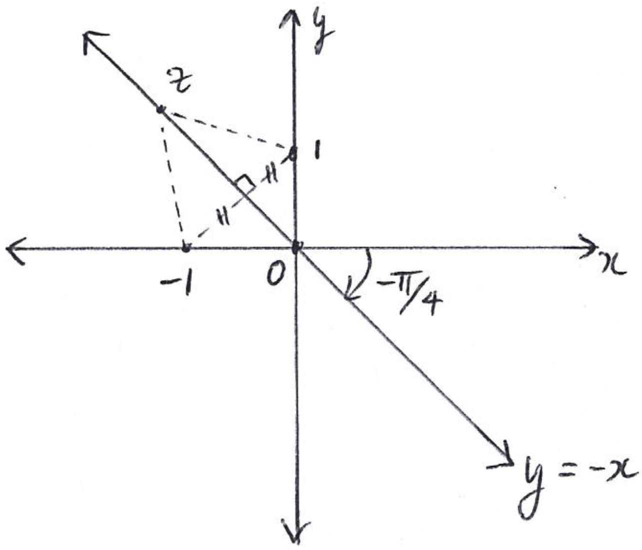
Suggested Solutions	Marks Awarded	Marker's Comments
a) (i) $w\bar{z} = (-1+\sqrt{3}i)(\sqrt{3}-2i)$	1	For conjugate of z
$= -\sqrt{3} + 5i + 2\sqrt{3}$		
$= \sqrt{3} + 5i$	1	For correct product of w and $\bar{z}$
(ii) $\lvert w\rvert = \sqrt{1+3} = 2$	1	
(iii) $\arg w = \tan^{-1}(-\sqrt{3})$		
i.e. $\theta = \frac{2\pi}{3}$ 	1	For correct argument
		Note: principle argument: $-\pi < \arg z \leq \pi$
(iv) $\frac{w}{z} = \frac{w\bar{z}}{z\bar{z}}$	1	For $z\bar{z} = \lvert z\rvert^2$
$= \frac{w\bar{z}}{\lvert z\rvert^2}$		
$= \frac{\sqrt{3}+5i}{7}$	1	Correct Answer
OR $\frac{w}{z} = \frac{-1+\sqrt{3}i}{\sqrt{3}+2i} \times \frac{\sqrt{3}-2i}{\sqrt{3}-2i}$ etc. etc.	1	For multiplying by conjugate to realise
	1	For simplification to final answer, correctly

JRAHS TASK 1 TERM 4 2018

MATHEMATICS Extension 2: Question 1

a) Suggested Solutions	Marks Awarded	Marker's Comments
(v) $w = -1+\sqrt{3}i$ $= \lvert w\rvert[\cos\theta + i\sin\theta]$ $= 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$		$\theta = \tan^{-1}(-\sqrt{3})$ (from iii)
$\therefore w^5 = 2^5\left(\cos\frac{10\pi}{3} + i\sin\frac{10\pi}{3}\right)$ … by De Moivre's		
$= 32\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$		$\left(2\pi + \frac{4\pi}{3} = \frac{10\pi}{3}\right)$
$= 32\left[\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right]$	1	For final mod-arg form (with principle argument.)
		<u>Note</u> (i)  (ii) cannot leave answer in $cis\theta$ form

MATHEMATICS Extension 2: Question 1		
Suggested Solutions	**Marks Awarded**	**Marker's Comments**
(b) Let $z = x + iy$ and $z^2 = 8 + 4\sqrt{5}\, i$ $\Rightarrow (x+iy)^2 = 8 + 4\sqrt{5}\, i$ $\Rightarrow x^2 - y^2 + 2xyi = 8 + 4\sqrt{5}\, i$ Equating reals and imaginary parts, $x^2 - y^2 = 8$ --- (I) $xy = 2\sqrt{5}$ --- (II)	1	For equating reals and imaginaries and arriving at two equations in two unknowns.
Solving simultaneously, one gets $x = \pm\sqrt{10}$ and $y = \pm\sqrt{2}$	1	For x and y, correct
Hence the square root of $8 + 4\sqrt{5}\, i$ is $z = \pm(\sqrt{10} + \sqrt{2}\, i)$	1	For correct root (Must Answer Q)
Method II: $8 + 4\sqrt{5}\, i = 2(4 + 2\sqrt{5}i)$ $= 2[(\sqrt{5})^2 + i^2 + 2\sqrt{5}\, i]$ $= 2[(\sqrt{5})^2 + 2\sqrt{5}\, i + i^2]$ $= 2(\sqrt{5} + i)^2$	2	Sweet!
$\therefore \sqrt{8 + 4\sqrt{5}\, i} = \pm\sqrt{2}(\sqrt{5} + i)$ $= \pm(\sqrt{10} + \sqrt{2}\, i)$	1	

MATHEMATICS Extension 2: Question 1		
Suggested Solutions	**Marks Awarded**	**Marker's Comments**
c) 	1	For region between the circles
	1	For region between $\pi/2$ and $\pi/4$
	1	Correct intercepts
d) 	1	St. line, correct orientation
	1	for equation of locus (either $y=-x$ or slope (-) 45°

MATHEMATICS: Question 2

Suggested Solutions	Marks	Marker's Comments

a) $Z = \frac{3a-5i}{1+2i} \times \frac{1-2i}{1-2i}$

$= \frac{(3a-10)}{5} - \frac{(6a+5)i}{5}$ ①

$\overline{Z} = \frac{(3a-10)}{5} + \frac{(6a+5)}{5} i$

$\overline{Z} = w$, equating real and imaginary parts

$\frac{3a-10}{5} = 1$

$3a - 10 = 5$

$3a = 15$

$a = 5$

$\frac{6a+5}{5} = -13b$

$\frac{35}{5} = -13b$

$b = \frac{-7}{13}$ ①

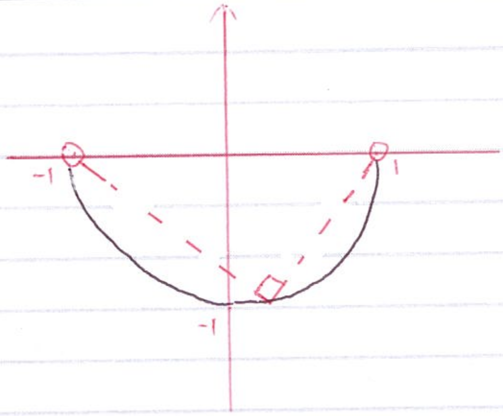
$\boxed{\text{Note } \overline{Z} \neq \frac{3a+5i}{1+2i}}$

The denominator must be real before you can just change the sign in the middle!

MATHEMATICS: Question

Suggested Solutions	Marks	Marker's Comments

b) i)



1 for recognising that it's a semi-circle

1 All other details:
* Bottom half
* open dots at $(\pm 1, 0)$
* marking the right angle

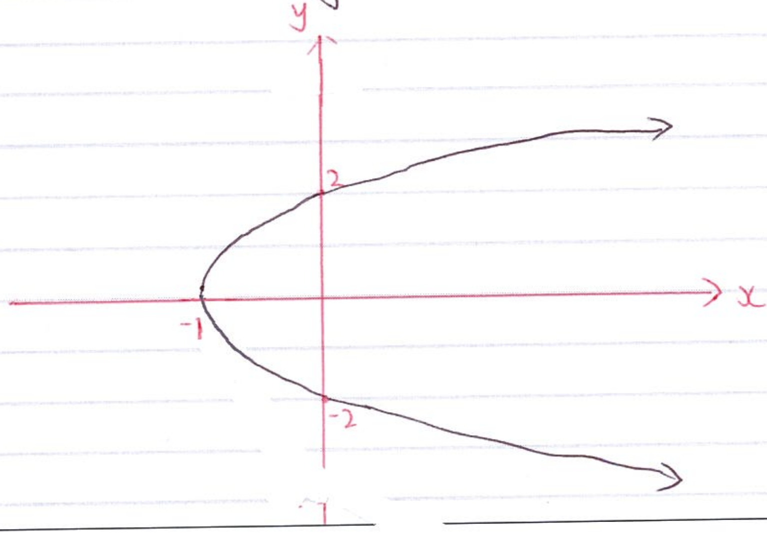
ii) let $z = x + iy$

$$\therefore 2\sqrt{x^2 + y^2} = (x + iy) + (x - iy) + 4$$

$$\therefore 4(x^2 + y^2) = (2x + 4)^2$$

$$4x^2 + 4y^2 = 4x^2 + 16x + 16$$

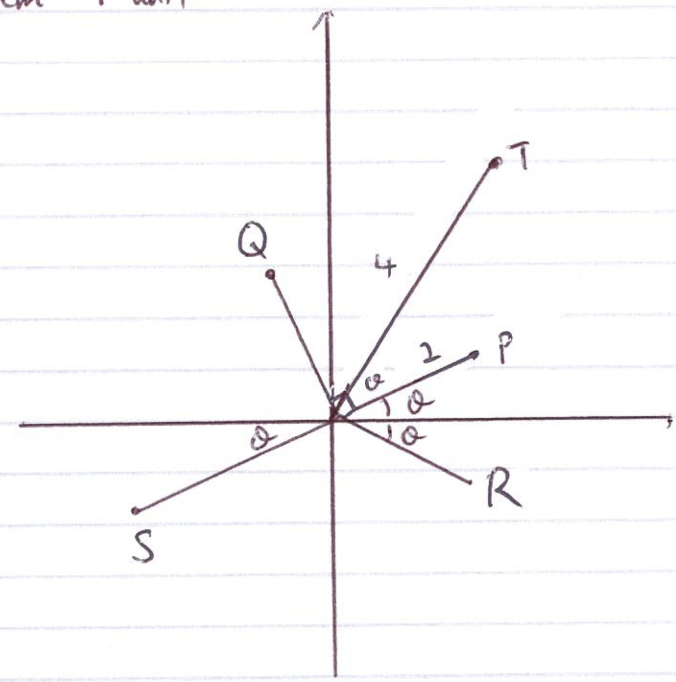
$$y^2 = 4(x+1)$$ 1



1 labelling all intercepts!

MATHEMATICS: Question		
Suggested Solutions	**Marks**	**Marker's Comments**

c) 1cm : 1 unit



Let $P = 2\operatorname{cis}\theta$, where $\operatorname{cis}\theta = \cos\theta + i\sin\theta$

i) $Q = 2\operatorname{cis}\left(\theta + \frac{\pi}{2}\right)$

ii) $R = 2\operatorname{cis}(-\theta)$

iii) $S = 3\operatorname{cis}(\theta - \pi)$

iv) $T = 4\operatorname{cis}2\theta$ (De Moivre's)

Note * 1 mark is automatically deducted if $\arg z = \frac{\pi}{4}$ from T.

* If you do not mark information on your diagram, one mark is deducted unless you wrote the working as shown above.

MATHEMATICS: Question		
Suggested Solutions	**Marks**	**Marker's Comments**
d)		
i) $(1-i)(5+i)^2$		
$= (1-i)(24+10i)$		1
$= 24+10i-24i+10$		
$= 34-14i$		1
ii) $(1-i)(5+i)^2 = 34-14i$ (part i)		
$\therefore \arg[(1-i)(5+i)^2] = \arg(34-14i)$		1
$\arg(1-i) + \arg(5+i)^2 = \arg(34-14i)$		
$\arg(1-i) + 2\arg(5+i) = \arg(34-14i)$		
$-\frac{\pi}{4} + 2\tan^{-1}\left(\frac{1}{5}\right) = \tan^{-1}\left(\frac{-7}{17}\right)$		
$-\frac{\pi}{4} + 2\tan^{-1}\left(\frac{1}{5}\right) = -\tan^{-1}\left(\frac{7}{17}\right)$		
$\therefore \tan^{-1}\left(\frac{7}{17}\right) + 2\tan^{-1}\left(\frac{1}{5}\right) = \frac{\pi}{4}$		1
* Because the answer is given, it must be clear that you have taken the arguments of expressions, if not you only receive 1/2.		
* If you did the question using inverse trig methods you will receive 1/2.		
* If you got part i) wrong, you would only get 1/2 maximum for part ii).		

<table>
<tr><th colspan="3">MATHEMATICS Extension 2: Question 3</th></tr>
<tr><th>Suggested Solutions</th><th>Marks</th><th>Marker's Comments</th></tr>
<tr><td>
(a) (i) For the roots of $z^4 - i = 0$:

Let $z = r(\cos\theta + i\sin\theta) \equiv r\text{cis}\,\theta$ where $r \geq 0$ and $\theta = \text{Arg}\,(z)$ for $-\pi < \text{Arg}(z) \leq \pi$.

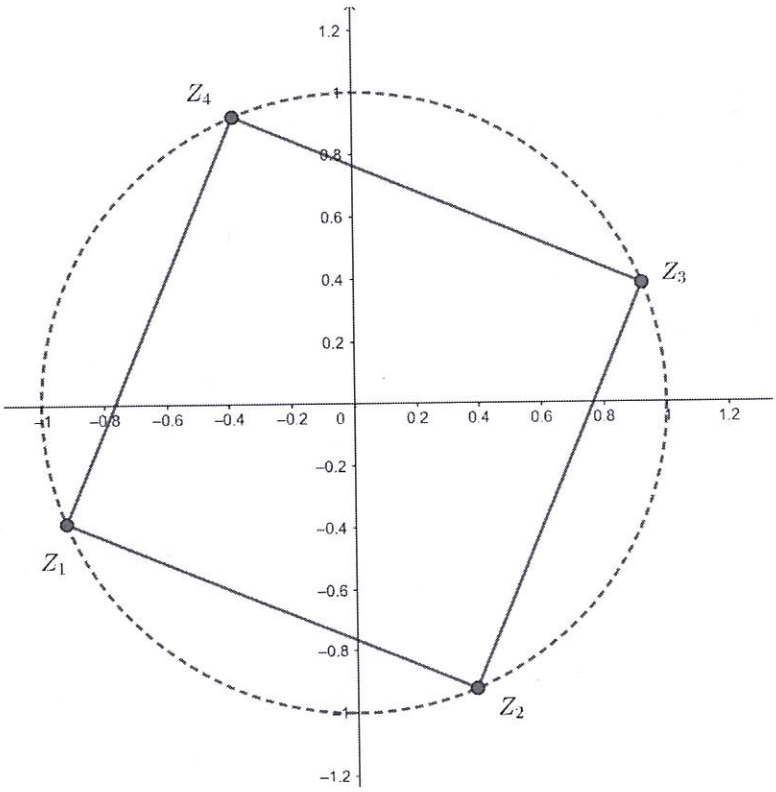
Then
$$|z^4| = |i|$$
$$|z|^4 = 1$$
$$|r(\cos\theta + i\sin\theta)|^4 = 1$$
$$r^4 \cdot |(\cos\theta + i\sin\theta)|^4 = 1$$
$$r^4 \cdot 1^4 = 1$$
so
$$r = 1$$
and
$$z^4 = (\text{cis}\,\theta)^4$$
$$= \text{cis}(4\theta) \quad \text{by de Moivre's theorem}$$
$$= \text{cis}\frac{\pi}{2}$$
$$= i$$
So,
$$4\theta = \frac{\pi}{2} + 2n\pi \rightarrow \theta = \frac{\pi}{8} + \frac{n\pi}{2}$$
for $n \in \mathbb{Z}$ such that $-\pi < \theta \leq \pi$; i.e. for integers n where
$$-\pi < \frac{\pi}{8} + \frac{n\pi}{2} \leq \pi$$
$$-2\frac{1}{4} < n \leq 1\frac{3}{4}$$
Hence, required n are $-2, -1, 0, 1$. Anything else would break the restriction on the argument.

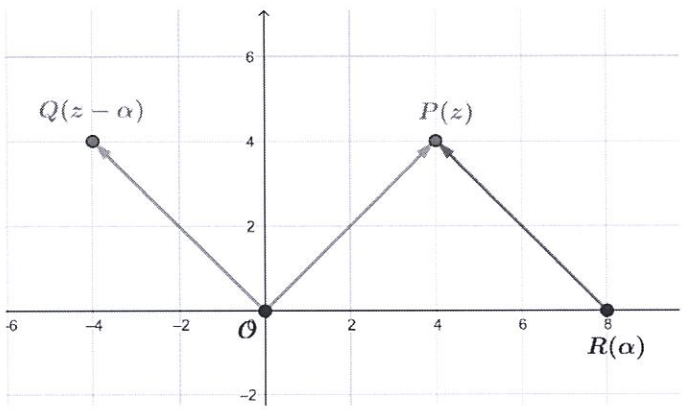
So
$$z = \text{cis}\left(\frac{\pi}{8} + \frac{n\pi}{2}\right)$$
for $n = -2, -1, 0, 1$; that is, the roots are (in **mod-Arg** form):
</td><td>1</td><td>
Question handled well on the whole. The primary error was that of incorrect arguments within the cis function.

You should, unless otherwise stated, assume the principal branch for the argument θ:
$$-\pi < \theta \leq \pi$$
Technically, there **is** a difference between lower-cased $\arg(*)$ and upper-cased $\text{Arg}(*)$. They are related as such:
$$\arg(z) = \text{Arg}(z) + 2n\pi$$
for all $n \in \mathbb{Z}$ where
$$\phi < \text{Arg}(z) \leq \phi + 2\pi$$
or
$$\phi \leq \text{Arg}(z) < \phi + 2\pi$$
The value of ϕ is chosen at the outset, something convenient given the context of the problem.

The **default** setting is
$$-\pi < \text{Arg}(z) \leq \pi$$
In the HSC, the syllabus fails to make explicit mention of this difference, and tacitly assumes
$$-\pi < \arg(z) \leq \pi$$
You should always work with a principal argument θ where $-\pi < \theta \leq \pi$ unless otherwise stated.

One mark deducted if arguments of complex roots weren't within the principal range.
</td></tr>
</table>

$z_1 = \cos\left(-\frac{7\pi}{8}\right) + i\sin\left(-\frac{7\pi}{8}\right)$ $z_2 = \cos\left(-\frac{3\pi}{8}\right) + i\sin\left(-\frac{3\pi}{8}\right)$ $z_3 = \cos\left(\frac{\pi}{8}\right) + i\sin\left(\frac{\pi}{8}\right)$ $z_4 = \cos\left(\frac{5\pi}{8}\right) + i\sin\left(\frac{5\pi}{9}\right)$	1	
(ii) Plot: 	1	Too many students did not label the plot sufficiently, but were awarded the mark if the orientation of position vectors were correct. If it were to have been a two-mark question, a mark would have been deducted. **Label everything** and be as neat as possible.
(iii) The points mark out the vertices of a square with diagonal lengths of 2. Hence the area is $\frac{1}{2} \times (\text{diagonal lengths}) = \frac{1}{2} \times 2 \times 2 = 2$ units squared.	1	Most candidates determined the correct area. Too many, though, did not provide adequate reasoning (some no reasoning at all) but were awarded the mark since the numerical result was correct. Similar to the last question, if this were to have been a two-mark question, a mark would have been deducted for insufficient reasoning. It doesn't take much: explain from where you get your numbers.

(iv) Let $\lambda \in \mathbf{C}$ be a root of the equation $z^4 - i = 0$. Then $\lambda^4 = i$. Now, substituting λ^5 into the equation gives, $(\lambda^5)^4 - i = (\lambda^4)^5 - i$ $= i^5 - i$ $= i - i$ $= 0$ Hence λ^5 is a solution to $z^4 - i = 0$ if λ is a solution.	1 1	Most candidates handled this well, though there was one primary and persistent error: **'specialising' the problem**: students assumed λ to be **one** of the roots established in (i), took that to the power of five and then showed it equivalent to i. Handling the problem this way is insufficient because λ is meant to be **any** of the roots. You therefore need to show that λ^5 works for all roots. Others took each of the four roots and then raised them each to the power of five: this is correct, but inefficient.
(b) Let $z \in \mathbf{C}$ be represented by the point P in the complex plane such that $\|z\| = \|z - \alpha\|$ for $\alpha \in \mathbb{R}$. In the example construction below, $\alpha > 0$, $P(z)$ is the point representing z, $Q(z-\alpha)$ is the point representing $z - \alpha$. The vector $\overrightarrow{RP}$ is equivalent to the position vector of $z - \alpha$ and will be used throughout.  <u>Algebraic Solution</u> Let $z = x + iy$. Then $\|x + iy\| = \|(x - \alpha) + iy\|$ $x^2 + y^2 = x^2 - 2\alpha x + \alpha^2 + y^2$ $\alpha(\alpha - 2x) = 0$ So $\alpha = 0$ or $\alpha = 2x$	1	First mark for a demonstrated correct appraisal of situation. Anyone working through an algebraic solution correctly handled the problem well and there was no real confusion.

If $\alpha \neq 0$, then

$$\begin{aligned}
\arg(z^2 - \alpha z) &= \arg(x^2 - y^2 + 2ixy - \alpha x - i\alpha y) \\
&= \arg\big((x^2 - y^2 - \alpha x) + i(2x - \alpha)y\big) \\
&= \arg(x^2 - y^2 - 2x^2 + i(2x - 2x)y) \\
&= \arg\big(-(x^2 + y^2)\big)
\end{aligned}$$

1

where $2x$ has been substituted for α.

For all $x, y \in \mathbb{R}$ such that $(x, y) \neq (0, 0)$ (else argument undefined), it follows that $-(x^2 + y^2)$ is a negative real number always. Hence

$$\arg(-(x^2 + y^2)) = \pi + 2n\pi$$

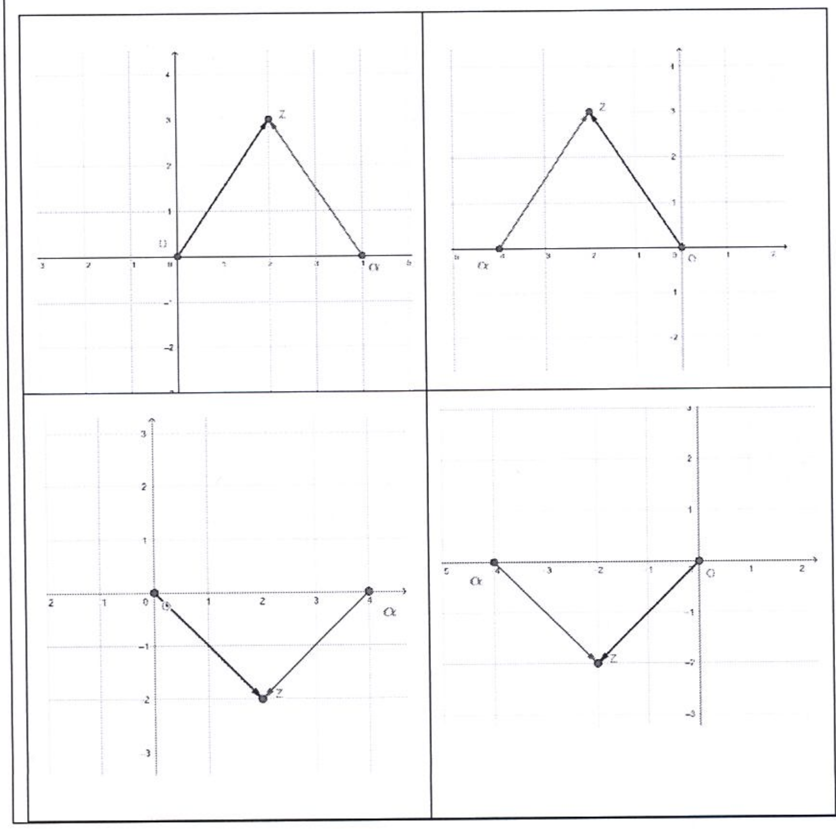
1

for all $n \in \mathbb{Z}$.

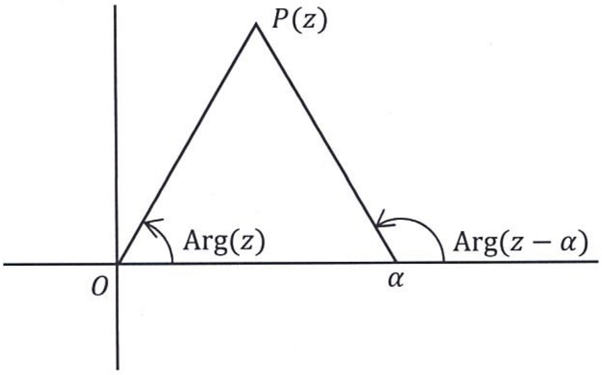
If $\alpha = 0$, then $\arg(z^2 - \alpha z) = 2\arg z$, and nothing more could be said

<u>Geometrical Solution</u>

Cases for $\alpha \neq 0$:



Note that in all situations, since we have $|z| = |z - \alpha|$, any triangles formed will be isosceles. Hence equal angles will lie opposite equal sides. We also have that $\arg(z^2 - \alpha z) = \arg\big(z(z-\alpha)\big) = \arg(z) + \arg(z-\alpha)$.



Given the geometry above, we have

$$\operatorname{Arg}(z-a) = \operatorname{Arg}(z) + \pi - 2\operatorname{Arg}(z)$$

given that the exterior angle equals the sum of the opposite two interior angles, and that the angle sum of triangle is π.

Hence

$$\operatorname{Arg}(z) + \operatorname{Arg}(z-\alpha) = \pi$$

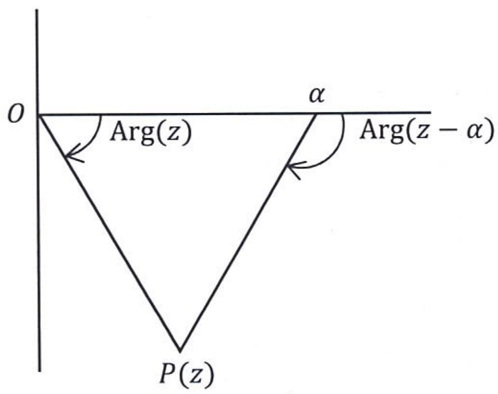
A similar construction could be made for the second situation where $\operatorname{Im}(z) > 0$ and $\alpha < 0$.

Hence generalising,

$$\arg(z^2 - \alpha z) = \pi + 2n\pi$$

for all $n \in \mathbb{Z}$ when $\operatorname{Im}(z) > 0$ and $\alpha \neq 0$.

When $\operatorname{Im}(z) < 0$ (i.e. triangles 'below' the real axis), we have **negatively-valued** arguments whose sum is $-\pi$ (assuming principal arguments) by following similar geometrical arguments as laid out above.



Candidates who arrived at π as the correct result by considering only one geometrical configuration weren't penalised.

Candidates who did not consider the situation for $\alpha = 0$ were not penalised.

When considering the geometrical solution, a mark was awarded if candidates demonstrated their knowledge that

$$\arg(z^2 - \alpha z) = \arg(z) + \arg(z-\alpha)$$

There was no penalty for **Arg** vs **arg** (even though we are dealing with a geometrical construction (i.e. using properties of triangles), where a **single** angle is required).

So

$$\mathrm{Arg}(z) + \mathrm{Arg}(z - \alpha) = -\pi \sim \pi$$

since arguments of $-\pi$ are mapped to π when dealing with principal branch.

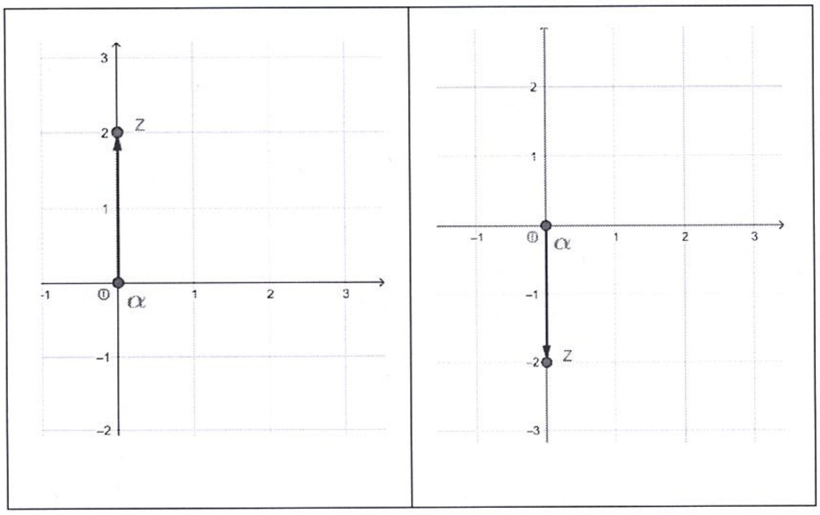
This result occurs also for $\alpha < 0$.

Generalising,

$$\begin{aligned} \arg(z^2 - \alpha z) &= -\pi + 2n\pi \\ &\equiv \pi + 2n\pi \end{aligned}$$

for all $n \in \mathbb{Z}$ (the second line is equivalent to first in the above equation since running though all integers n generates **all** arguments...hence we can choose any particular (correct) argument and start generating from there).

<u>Cases for $\lim_{\alpha \to 0} \alpha$</u>



If $\mathrm{Im}(z) > 0$ and, after forming a triangle, we make α approach zero, we arrive at the situation where $\mathrm{Arg}(z)$ and $\mathrm{Arg}(z - \alpha)$ both approach $\frac{\pi}{2}$.

Hence, in the limit, $\mathrm{Arg}(z) + \mathrm{Arg}(z - \alpha) = \pi$.

Similarly, if $\mathrm{Im}(z) < 0$ and, after forming a triangle, we take the limit as α approaches zero, we have $\mathrm{Arg}(z)$ and $\mathrm{Arg}(z - \alpha)$ each approach $-\frac{\pi}{2}$.

Hence, in the limit, $\mathrm{Arg}(z) + \mathrm{Arg}(z - \alpha) = -\pi \sim \pi$ (since $-\pi$ is equivalent to π if principal argument).

Students who listed $-\pi$ as one of the results were not penalised, but if we are dealing with the principal argument, $-\pi$ should be mapped to π.

Generalising in both cases,

$$\arg(z^2 - \alpha z) = \arg(z) + \arg(z - \alpha)$$
$$= \pi + 2n\pi$$

for all $n \in \mathbb{Z}$.

If $\alpha = 0$, we once again have $\arg(z^2)$ which would be twice the value of the argument of z; that is,

$$\arg(z^2) = 2\arg(z)$$

when $\alpha = 0$.

In all cases, then,

$$\arg(z^2 - \alpha z) = \pi + 2n\pi$$

for all $n \in \mathbb{Z}$ and $\alpha \neq 0$, and

$$\arg(z^2 - \alpha z) = 2\arg(z)$$

when $\alpha = 0$.

HSC

In all cases, then,

$$\arg(z^2 - \alpha z) = \pi$$

for all $n \in \mathbb{Z}$ and $\alpha \neq 0$, and

$$\arg(z^2 - \alpha z) = 2\arg(z)$$

when $\alpha = 0$.

(c)

(i) To find the maximum value of $z \in \mathbf{C}$ such that

$$\left|z - 3\sqrt{3} - 3i\right| = \left|z - (3\sqrt{3} + 3i)\right| = 3$$

we have, by triangle inequality,

$$|z| - \left|3\sqrt{3} + 3i\right| \leq \left|z - \left(3\sqrt{3} + 3i\right)\right|$$

$$|z| \leq 3 + 3\left|\sqrt{3} + i\right| \quad \text{(since } \left|z - \left(3\sqrt{3} + 3i\right)\right| = 3\text{)}$$
$$= 3 + 3\sqrt{3 + 1}$$
$$= 9$$

1

Many students received full award for this problem, although there was a **fatal flaw** in nearly all papers.

The flaw in reasoning depended upon the approach taken: purely geometrical versus purely algebraic.

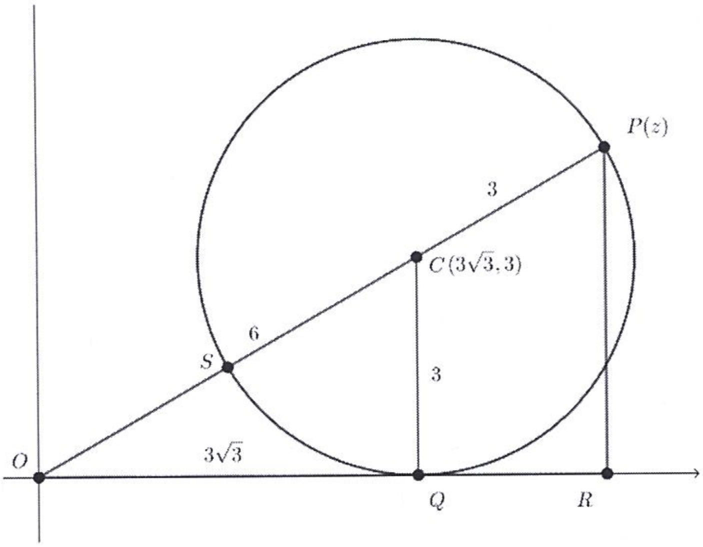
Flaw in algebraic approach:

Showing that the modulus of z cannot exceed 9 is only one part of the battle. If you fail to show that there actually **is** a z whose modulus is 9, then you haven't found the maximum value of $|z|$ (who knows, the actual maximum

Hence, the modulus of z may **not** be greater than 9. Nothing has been proven to state that 9 **is** the maximum value. All that's been shown is that $|z| \leq 9$, but the actual maximum could be 8, and the truth of the inequality would still hold.

So, we have to show that such a z exists. Any (logically sound) method will do. We may do this quickly using geometry.

The locus of z such that $|z - 3\sqrt{3} - 3i| = 3$ is a circle with centre $(3\sqrt{3}, 3)$ and radius 3.



Pythagoras' theorem gives $\mathrm{OC} = 6$, and $\mathrm{CP} = 3$ given the radial length.

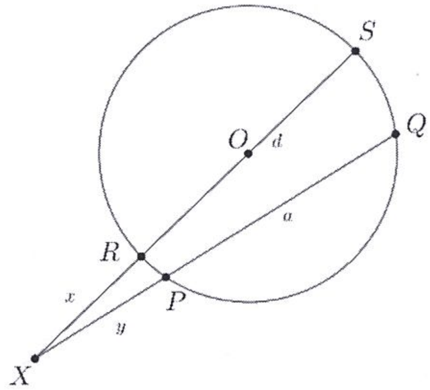
Hence there is a z on the locus with distance 9 units from the origin, and we are done.

1

on the circle may be some number **less than** 9?!). Inequalities erase information.

Flaw in geometrical approach:

Constructing the circle and taking a position vector that passes through the centre of the circle to land at the point $P(z)$, say, where $|OP| = 9$ is not (strictly speaking) sufficient to demonstrate that the value found is the maximum. There should be a reason stated as to why this construction yields the greatest value of $|z|$.



Note that by product of secants on a circle from an external point, we have

$$SX \cdot XR = QX \cdot XP$$

where secant SX runs through the centre of the circle (and hence contains diameter RS); that is,

$$(d + x)x = (a + y)y \ldots (1)$$

Also, the segment XR must be perpendicular to the tangent at R since the tangent at R is perpendicular to the diameter RS and angle XRS is a straight angle.

Hence the distance x is less than the distance y; i.e. $x < y$.

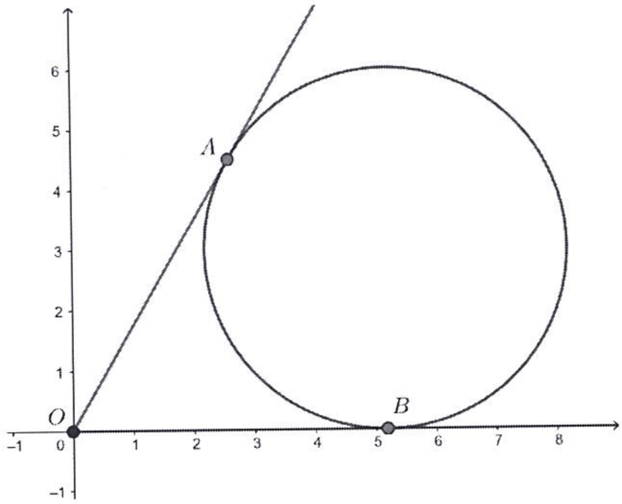
We want to show that

$$x + d < y + a$$

		so consider the difference $(x + d) - (y + a)$ $$= (x + a) - \frac{(x + d)x}{y}$$ by equation (1) $$= (x + d)\left(1 - \frac{x}{y}\right)$$ Since $0 < x < y$, we have $0 < \frac{x}{y} < 1$, i.e. $\frac{x}{y}$ is a positive number smaller than 1. So $1 - \frac{x}{y}$ is positive. Hence $$(x + d)\left(1 - \frac{x}{y}\right) > 0$$ and so $$(x + d) - (y + a) > 0$$ i.e. $$x + d > y + a$$ So, *the secant running from a point external to a circle and passing through the circle's centre is greater in length than any other secant to the circle from the same point*. There were no significant issues in regard to finding the range of allowable arguments of z for the locus.

(ii) To find all possible $\arg(z)$:

Let $y = mx$ and find all m such that the line is tangent to the circle.



Substituting mx for y in locus of z:

1

First mark awarded for a logical approach.

$$\left(x - 3\sqrt{3}\right)^2 + (mx - 3)^2 = 9$$

$$(1 + m^2)x^2 - 6\left(m + \sqrt{3}\right)x + 27 = 0$$

The lines are tangent when the discriminant is zero:

$$36\left(m + \sqrt{3}\right)^2 - 108(1 + m^2) = 0$$

implies that

$$m = 0 \text{ or } m = \sqrt{3}$$

so,

$$0 \leq \mathrm{Arg}(z) \leq \frac{\pi}{3}$$

giving,

$$2n\pi \leq \arg(z) \leq \frac{\pi}{3} + 2n\pi$$

for all $n \in \mathbb{Z}$, and so

$$0 \leq \mathrm{Arg}(z) \leq \frac{\pi}{3}$$

1

Second mark awarded for correct answer.

(d) Let $z = (\cos 2\theta - 1) - i\sin 2\theta$. Then $z = 1 - 2\sin^2\theta - 1 - 2i\sin\theta\cos\theta$ $= -2\sin\theta\,(\sin\theta + i\cos\theta)$ $= -2\sin\theta\left(\cos\left(\frac{\pi}{2} - \theta\right) + i\sin\left(\frac{\pi}{2} - \theta\right)\right)$ Hence the argument may be read as: $\mathrm{Arg}(z) = \frac{\pi}{2} - \theta$	 1 1	This question was mishandled by a good proportion of the candidature. Missteps included: • Ignoring negative signs and therefore not obtaining a result that would lead naturally to the conclusion, but still attempting to draw that conclusion. • Obtaining $\tan^{-1}(\cot\theta)$ and moving directly to $-\frac{\pi}{2}$. It's a 'show that' question, so you must demonstrate every step, including that of $\tan^{-1}(\cot\theta) = \tan^{-1}\left(\tan\left(\theta - \frac{\pi}{2}\right)\right) = \theta - \frac{\pi}{2}$ It's not sufficient to operate an inverse function on a function that is not *immediately* related to the inverse when the target result is known because you know what the answer is meant to be, so an examiner must be sure that **you** know what comes next; i.e. that you're not just taking a 'leap of faith' in the hope of scoring points. This type of question was handled poorly in the 2018 HSC for this very reason.

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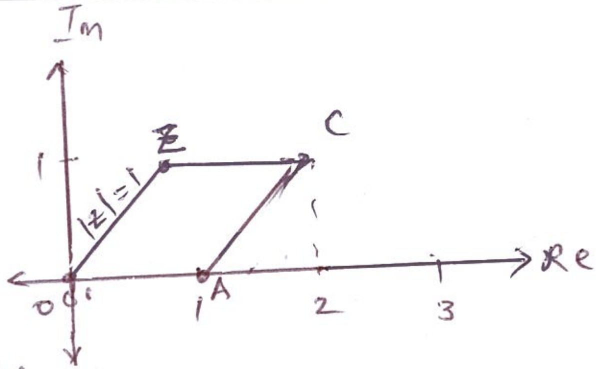
Suggested Solutions	**Marks**	**Marker's Comments**

a)



Method 1

i) $|\vec{OA}| = 1$ } given
$|z| = 1$

$\vec{AC} = z+1-1$
$= z$

$\vec{ZC} = z+1-z$
$= 1$ — (1)

$\therefore |\vec{AC}| = 1$ $\qquad |\vec{ZC}| = 1$

$\therefore$ OACZ is a rhombus as all sides have a modulus of one — (1)

Method 2:

midpoint $AZ = \left(\frac{x+1}{2} + i\left(\frac{y+0}{2}\right)\right)$; $m_{AZ} = \frac{x+1}{-y}$

$OC = \left(\frac{0+x+1}{2}, i\left(\frac{y+0}{2}\right)\right)$; $m_{OC} = \frac{y}{x+1}$ — (1)

$\therefore AZ \perp OC$

midpt $AZ = OC$

OACZ is a rhombus (diagonals bisect at $90°$) — (1)

MATHEMATICS Extension 2: Question 4		2/8
Suggested Solutions	**Marks**	**Marker's Comments**

Method 3

$|z| = 1$

$OZ = OA$

$\vec{OC} = z + 1$

$= \vec{OZ} + \vec{OA}$

but $\vec{OC} = \vec{OA} + \vec{AC}$

$\vec{AC} = \vec{OZ}$

$ZC = OA = 1$

$ZC \parallel OA$ — 1

$\vec{AC} = \vec{OZ}$ and $\vec{AC} \parallel \vec{OZ}$

OACZ is a parallelogram

OACZ is rhombus

(parallelogram with one pair adjacent sides equal)

or

1 pair of opposite sides equal and parallel) — 1

Method 1

ii) $z - 1$ is $\vec{ZA}$

$z + 1$ is $\vec{OC}$

$\therefore z - 1$ and $z + 1$ are the diagonals of the rhombus.

Diagonals of rhombus bisect at right angle

$\therefore \arg(z-1) - \arg(z+1) = \frac{\pi}{2}$

$\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{2}$

$\therefore \frac{z-1}{z+1}$ is purely imaginary.

MATHEMATICS Extension 2: Question 4		3/8
cont. 4 aii) Suggested Solutions	Marks	Marker's Comments
Method 2 $\vec{OC} = z+1 \qquad \vec{AZ} = z-1$ OC, AZ are diagonals of rhombus OACZ OC ⊥ AZ (diagonals of rhombus OACZ are perpendicular bisectors of each other) $\vec{OC} \times ki = \vec{AZ}$ $(k \in \mathbb{R})$ since $\lvert\vec{OC}\rvert \neq \lvert\vec{AZ}\rvert$ $\dfrac{\vec{OC}}{\vec{AZ}} = ki$ $\therefore \dfrac{z-1}{z+1} = ki$ $\dfrac{z-1}{z+1}$ is purely imaginary. iii) Method 1 $\angle ZOC = \frac{1}{2}\theta$ (diagonals of a rhombus bisect the vertex angles) $\angle ZCO = \frac{1}{2}\theta$ (equal angles opposite equal sides) $\angle OZC = 180 - \frac{1}{2}\theta - \frac{1}{2}\theta$ (angle sum of ΔOZC) $= 180 - \theta$ $\lvert OC\rvert^2 = \lvert OZ\rvert^2 + \lvert ZC\rvert^2 - 2\lvert OZ\rvert\lvert ZC\rvert\cos(180-\theta)$ $= 1 + 1 + 2\cos\theta$ $\lvert OC\rvert = \sqrt{2+2\cos\theta}$ —— 1 $\arg(OC) = \frac{1}{2}\theta$ —— 1		some understanding to show and follows through to show $\dfrac{z-1}{z+1}$ is purely imaginary

MATHEMATICS Extension 2: Question 4		4/8
cont 4 a iii) **Suggested Solutions**	**Marks**	**Marker's Comments**

Method 2

$|z+1| = 2OB$ (diagonals bisect angles through where they pass)

$\cos \angle ZOB = \frac{OB}{1}$

$OB = \cos\frac{\theta}{2}$

$|z+1| = 2\cos\frac{\theta}{2}$

Hence mod $= 2\cos\frac{\theta}{2}$

arg $= \frac{\theta}{2}$.

* You can use cos rule

$|z+1| = |\cos\theta + i\sin\theta + 1|$

$= \sqrt{(\cos\theta+1)^2 + \sin^2\theta}$

$= \sqrt{\cos^2\theta + 2\cos\theta + 1 + \sin^2\theta}$

$= \sqrt{2 + 2\cos\theta}$

Method 1

b) i) $z^4 + 1 = 0$

ω is a root.

$\omega^4 + 1 = 0$

$(\omega+1)(\omega^3 + \omega^2 + \omega + 1) = 0$

$\omega^3 + \omega^2 + \omega + 1 = 0$

LHS

$1 + \omega^2 + \omega^4 + \omega^6$

$= 1 + 3\omega^2 + 3(\omega^2)^2 + (\omega^2)^3 - 2\omega^2 - 2(\omega^2)^2$

$= (1+\omega^2)^3 - 2\omega^2(1+\omega^2)$

$= (1+\omega^2)((1+\omega^2)^2 - 2\omega^2)$

$= (1+\omega^2)(1 + 2\omega^2 + \omega^4 - 2\omega^2)$

$= (1+\omega^2)(1+\omega^4)$

$= (1+\omega^2)(1+(-1))$ as $\omega^4 = -1$

$= (1+\omega^2)(0)$

$= 0$

$= RHS$

$1 + \omega^2 + \omega^4 + \omega^6 = 0$

Ext 2

MATHEMATICS: Question 4		
Suggested Solutions	**Marks**	**Marker's Comments**

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method 2

Given z is a root of $z^4+1=0$

$\omega^4 = -1$ —— i

RTP $(1+\omega^2+\omega^4+\omega^6=0)$ $\omega^4=-1$

LHS $1+\omega^2+\omega^4+(\omega^4)\omega^2$

$= 1+\omega^2+(-1)+(-1)\omega^2$

$= 1+\omega^2 - 1 - \omega^2$

$= 0$

$=$ RHS.

b ii) $(1+\omega)(1+\omega^2)(1+\omega^4)$

$= (1+\omega)(1+\omega^2)(1+(-1))$ from (i)

$= 0$.

method 1

c) i) $|z_1+z_2|^2+|z_1-z_2|^2 = 2(|z_1|^2+|z_2|^2)$

Let $z_1 = x+iy$ $z_2 = a+ib$.

$|z_1+z_2|^2 = |x+iy+a+ib|^2$

$= (x+a)^2+(y+b)^2$

$|z_1-z_2|^2 = |x+iy-a-ib|^2$

$= (x-a)^2+(y-b)^2$

$|z_1+z_2|^2+|z_1-z_2|^2 = (x+a)^2+(y+b)^2+(x-a)^2+(y-b)^2$

$= x^2+2ax+a^2+y^2+2by+b^2+x^2-2ax$
$+a^2+y^2-2by+b^2$

$= 2x^2+2a^2+2y^2+2b^2$

$= 2(x^2+y^2+a^2+b^2)$

$2(|z_1|^2+|z_2|^2) = 2(|x+iy|^2+|a+ib|^2)$

$= 2(x^2+y^2+a^2+b^2)$

$= |z_1+z_2|^2+|z_1-z_2|^2$

Ext 2

MATHEMATICS: Question 4 6/8

Suggested Solutions	Marks	Marker's Comments

<u>method 2</u>

$$|z_1+z_2|^2+|z_1-z_2|^2$$

$$=(z_1+z_2)(\overline{z_1+z_2})+(z_1-z_2)(\overline{z_1-z_2})$$ — 1

$$=z_1\bar{z}_1+z_1\bar{z}_2+z_2\bar{z}_1+z_2\bar{z}_2+z_1\bar{z}_1-z_1\bar{z}_2-\bar{z}_1z_2+z_2\bar{z}_2$$

$$=2|z_1|^2+2|z_2|^2$$

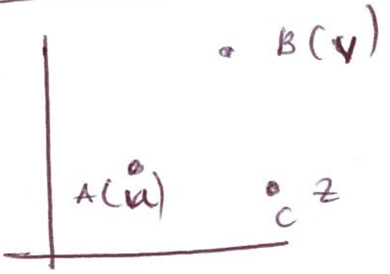
$$=2(|z_1|^2+|z_2|^2)$$

c(ii) The sum of the squares of the diagonals of a parallelogram is equal to the sum of the square of the sides — 1

OR.

the sum of the squares of the diagonals in a parallelogram is equal to twice 2 the sum of the squares of the adjacent side

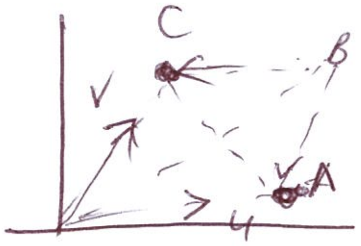
d) <u>method 1</u>



$\vec{AB}=v-u$ — 1

$\vec{BC}=\dfrac{u-ikv}{1-ik}-v$

Ext 2

MATHEMATICS Ext 2 Question 4		7/8
Suggested Solutions	**Marks**	**Marker's Comments**
$= \dfrac{u - ikv - v + ikv}{1 - ik}$ $= \dfrac{u - v}{1 - ik}$ — 1 $\vec{AC} = \dfrac{u - ikv - u}{1 - ik}$ $= \dfrac{u - ikv - u + iku}{1 - ik}$ $= \dfrac{ik(u - v)}{1 - ik}$ — 1 $\therefore \vec{AC} = ik\,\vec{BC}$ $\therefore \vec{AC}$ is rotated 90° from BC — 1 So ABC is a right angle triangle. Method 2  Let $\vec{OA} = u$ $\vec{OC} = v$ $\vec{OB} = \dfrac{u - ikv}{1 - ik}$ $\vec{AB} = v - u$ — 1		1 mark for each vectors. 1 mark for conclusion.

Ext 2.

MATHEMATICS Question 4		8/8
Suggested Solutions	**Marks**	**Marker's Comments**
$\vec{BC} = v - \dfrac{u - ikv}{1-ik}$ $= \dfrac{v - ikv - u + ikv}{1-ik}$ — 1 $= \dfrac{v-u}{1+ik}$ $\vec{BA} = u - \dfrac{u-ikv}{1-ik}$ $= \dfrac{u - uik - u + ikv}{1-ik}$ $= \dfrac{-uik + ikv}{1-ik}$ $= \dfrac{ik(v-u)}{1-ik}$ — 1 $\dfrac{\vec{BA}}{\vec{BC}} = \dfrac{ik(v-u)}{1-ik} \times \dfrac{1-ik}{(v-u)}$ $= ik$ $\vec{BA} \perp \vec{BC}$ $\therefore \angle ABC = 90^\circ$ $\therefore \triangle ABC$ is a right angle triangle - the point from the vertices of a right angled triangle.	1	Let $\dfrac{u-ikv}{1-ik} = *$. must be consistent its either $* - u$ $* - v$ or $u - *$ $v - *$. if not in this order max mark awarded was 2

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