Name: ____________________

Class: ____________________

# YEAR 12
# ASSESSMENT TEST
# TERM 1, 2018

# Mathematics Extension 2

*General Instructions*

- Working time – 120 minutes + 5 minutes reading
- Board approved calculators may be used
- Write using black or blue pen
- All necessary working should be shown in all questions
- Write your student number at the top of every page of your answer sheets.

**5 Multiple Choice Questions-1 mark each**

**5 Long Response Questions–15 marks each**

**Total marks – 80**

**Attempt all questions**

**Start each question on a new sheet of paper.**

## Multiple Choice Questions

## Answer the following questions in the answer sheet provided

1) What is the multiplicity of the root $x = -1$ of the equation $3x^5 - 5x^4 - 35x - 27 = 0$ ?

(A) one

(B) two

(C) three

(D) four

2) Using a suitable substitution, the definite integral $\int_0^{\frac{\pi}{24}} tan2xsec^2 2x\,dx$ is equivalent to

(A) $\int_0^{\frac{\pi}{24}} \frac{u}{2}\,du$

(B) $\int_0^{2-\sqrt{3}} 2u\,du$

(C) $\int_0^{\frac{\pi}{24}} 2u\,du$

(D) $\int_0^{2-\sqrt{3}} \frac{u}{2}\,du$

3) Given the eccentricity of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $e$, then the eccentricity of the ellipse $\frac{x^2}{a^2+b^2} + \frac{y^2}{b^2} = 1$ is

(A) $\sqrt{e}$

(B) $\frac{1}{e}$

(C) $e$

(D) $e^2$

4) For a certain function $y = f(x)$, the function $= \ f(|x|)$ is represented by

(A) A reflection of $y = f(x)$ in the $y$ axis

(B) A reflection of $y = f(x)$ in the $x$ axis

(C) A reflection of $y = f(x)$ in the $x$ axis for $y \geq 0$

(D) A reflection of $y = f(x)$ in the $y$ axis for $x \geq 0$

5) Let $w$ be the complex root of unity such that $w^n = 1, w \neq 1$. Find the value of $\sum_{k=0}^{n}\left(w^k + \frac{1}{w^k}\right)$ where $n \geq 3$

(A) 0

(B) 1

(C) 2

(D) 3

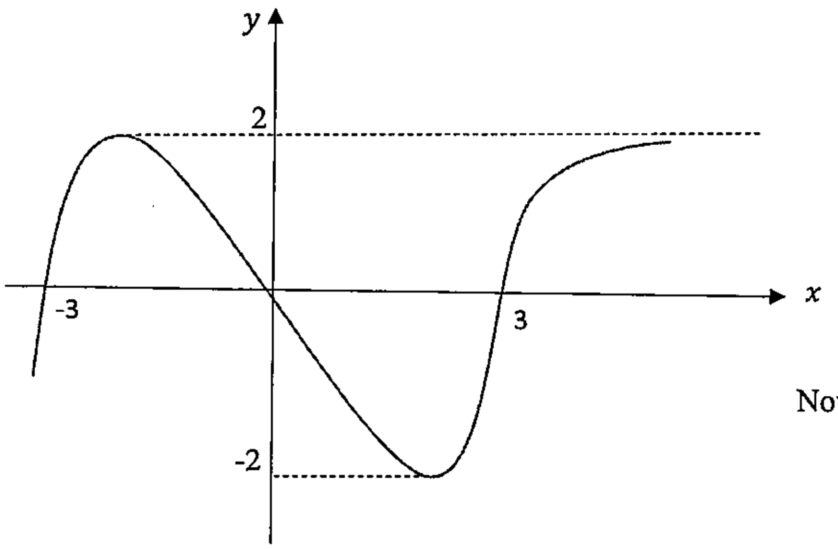
**QUESTION 6** **(Start a new sheet)** **15 Marks**

a) i) Find constants $a, b$ and $c$ such that $\frac{x^2+2x}{(x^2+4)(x-2)} = \frac{ax+b}{x^2+4} + \frac{c}{x-2}$ 2

ii) Hence, find $\int \frac{x^2+2x}{(x^2+4)(x-2)}dx$ 2

b) Find $\int \frac{9x^3+9x^2+5x+4}{3x+1}dx$ 3

c) The diagram shows $y = f(x)$




Not to scale

Draw separate one third page sketch of the following graph

i) $y = f(x) + |f(x)|$ 2

ii) $y = \frac{1}{f(x)}$ 2

iii) $y^2 = f(x)$ 2

iv) $y = e^{f(x)}$ 2

## QUESTION 7 (Start a new sheet) 15 Marks

a) i) Show that the equation of the tangent to the curve $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $P(x_1, y_1)$ on it is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$ 2

ii) Hence, show that the equation of the chord of contact to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ from the point $R(x_0, y_0)$ is $\frac{xx_0}{a^2} - \frac{yy_0}{b^2} = 1$ 2

iii) Hence, show that the chord of contact to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ from a point on the directrix is a focal chord 1

b) $\int \sin(lnx)\, dx$ using $u = ln\, x$ or otherwise 3

c) Evaluate $\int_0^{ln2} \frac{e^{2x}}{1+e^x} dx$ 3

d) Let $z = cos\theta + isin\theta$

i) Show that $z^n + z^{-n} = 2cosn\theta$, where $n$ is a positive integer 2

ii) Let $m$ be a positive integer. Show that 2

$$(2cos\theta)^{2m} = 2\left[cos2m\theta + \binom{2m}{1}\cos(2m-2)\,\theta + \binom{2m}{2}\cos(2m-4)\,\theta + \cdots + \binom{2m}{m-1}\cos 2\theta\right] + \binom{2m}{m}$$

## QUESTION 8 (Start a new sheet) 15 Marks

a) i) By using the substitution $t = tan\frac{x}{2}$, show that $\int_0^{\frac{\pi}{2}} \frac{dx}{1+sinx} = 1$ 2

ii) Show that $\int_0^a f(x)dx = \int_0^{\frac{a}{2}} \{f(x) + f(a-x)\}dx$ 2

iii) Hence evaluate $\int_0^{\pi} \frac{x}{1+sinx} dx$ 1

b) i) Find the real values of $k$ for which the equation $\frac{x^2}{6-k} + \frac{y^2}{4-k} = 1$, defines respectively an ellipse and an hyperbola 2

ii) Sketch the curve for k=5, showing its important features 2

iii) Explain how the shape of this graph changes as $k$ increases from 5 to 6 and determine if the graph has a limiting position for this increase. 1

c) i) Show that the condition for the line $y = mx + c$ to be tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is that $c^2 = a^2m^2 + b^2$ 2

ii) Hence, or otherwise, prove that the pair of tangents from the point (3,4) to the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ are at right angles to each other. 3

**QUESTION 9** **(Start a new sheet)** **15 Marks**

a) Suppose $k$ is a constant greater than 1. Let $f(x) = \frac{1}{1+(tanx)^k}$, where $0 \leq x \leq \frac{\pi}{2}$.

You may assume $f\left(\frac{\pi}{2}\right) = 0$

i) Show that $f(x) + f\left(\frac{\pi}{2} - x\right) = 1$ for $0 \leq x \leq \frac{\pi}{2}$ 2

ii) Sketch $y = f(x)$ for $0 \leq x \leq \frac{\pi}{2}$ 3

* *(There is no need to find $f'(x)$ but assume $y = f(x)$ has a horizontal tangent at $x = 0$. Your graph should exhibit the property of a)i) and all of the information given)*

iii) Hence, or otherwise, evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{1+(tanx)^k}$ 2

b) For the curve $x^2y^2 - x^2 + y^2 = 0$

i) Demonstrate why $|y|<1$ 1

ii) Demonstrate why $|y| \leq |x|$ 1

iii) Use implicit differentiation to show $\frac{dy}{dx} = \frac{x(1-y^2)}{y(1+x^2)}$ 2

iv) State the coordinates of any critical points (where the derivatives are undefined) 1

v) Explain why the curve has no stationary points 1

vi) Sketch the curve 2

## QUESTION 10 (Start a new sheet) 15 Marks

a) Let $I_n = \int_0^{\frac{\pi}{2}} (sinx)^n dx$ where $n$ is an integer, $n \geq 0$.

i) Using integration by parts, show that for $n \geq 2$, $I_n = \frac{(n-1)}{n} I_{n-2}$ 2

ii) Evaluate $I_4$ 2

b) Suppose $\alpha, \beta, \gamma$ and $\delta$ are the four roots of the polynomial equation

$P(x) = x^4 - Ax - 1 = 0$, where $A$ is real

i) By considering $P''(x)$, or otherwise, show that at most two of the roots are real 2

ii) Prove that the polynomial equation with the four roots $\alpha^2, \beta^2, \gamma^2$ and $\delta^2$ is $x^4 - 2x^2 - A^2x + 1 = 0$ 2

iii) By considering another suitable polynomial equation, or otherwise, prove that $(\alpha^2 + 1)(\beta^2 + 1)(\gamma^2 + 1)(\delta^2 + 1) = A^2$ 2

c) Let the points $A_1, A_2, \ldots, A_n$ represent the $nth$ roots of unity, $w_1, w_2, \ldots, w_n$,

and suppose $P$ represents any complex number $z$ such that $|z| = 1$

i) Show that $(PA_i)^2 = (z - w_i)(\bar{z} - \bar{w}_i)$ for $i = 1,2, \ldots, n$ 2

ii) Prove that $\sum_{i=1}^{n} (PA_i)^2 = 2n$, given that $w_1 + w_2 + \cdots + w_n = 0$ 3

****END OF EXAMINATION****

MULTIPLE CHOICE (1) B (2) D (3) B (4) D (5) C

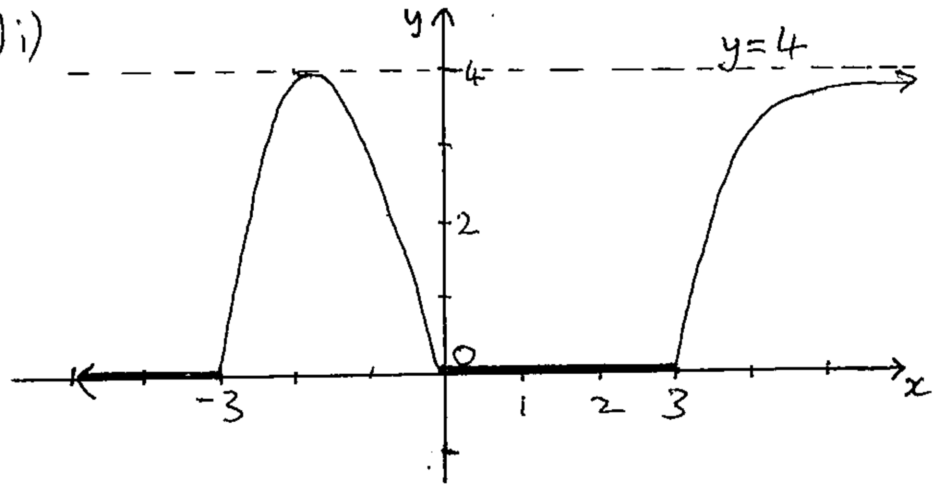
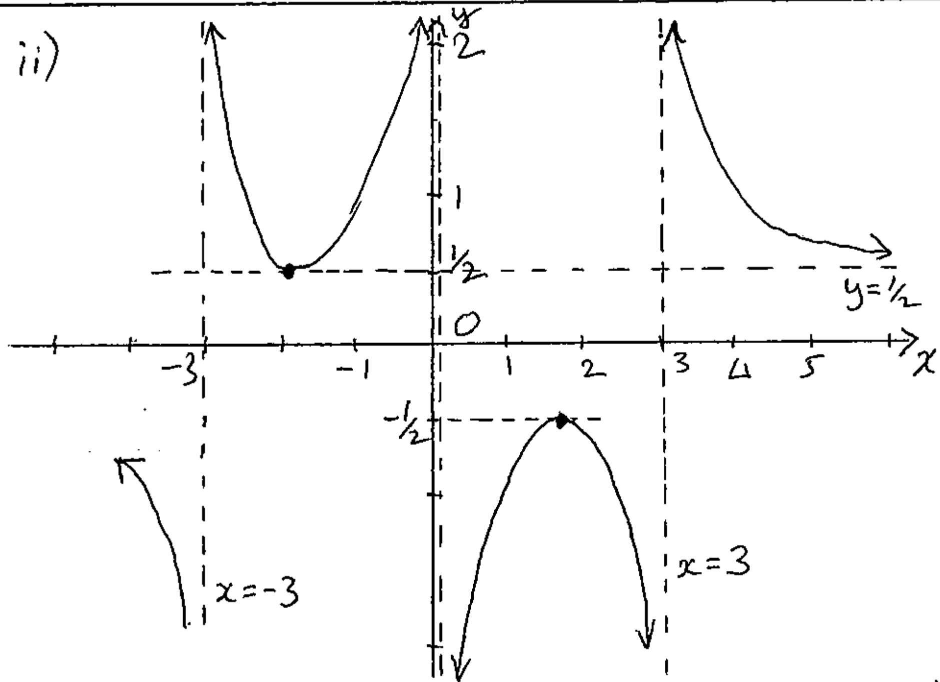
**MATHEMATICS Extension 2: Question 6**

Sheet 1 of 3

| Suggested Solutions | Marks | Marker's Comments |
|---|---|---|
| 6 a) i) $x^2+2x \equiv (x-2)(ax+b)+c(x^2+4)$<br>Substitute $x=2$:<br>$8 = 8c \quad \therefore \underline{\underline{c=1}}$<br>Substitute $x=0$:<br>$0 = -2b+4 \quad \therefore \underline{\underline{b=2}}$<br>Equate coefficients of $x^2$:<br>$1 = a+1 \quad \therefore \underline{\underline{a=0}}$<br>$\therefore \underline{\underline{(a,b,c) = (0,2,1)}}$ | 2 | One mark if only 2 values correct. (Essentially one mark off per error) Any combination of coeff. eqns and substitution would do. |
| ii) $\int \frac{x^2+2x\,dx}{(x^2+4)(x-2)} = \int \frac{2\,dx}{x^2+4} + \int \frac{dx}{x-2}$<br>$= \underline{\underline{\tan^{-1}\left(\frac{x}{2}\right) + \ln\lvert x-2\rvert + k}}$ | 2 | One mark for each integral. |
| b) $3x+1 \overline{)\,9x^3+9x^2+5x+4}$ quotient $3x^2+2x+1$ r 3<br>$9x^3+3x^2$<br>$6x^2+5x$<br>$6x^2+2x$<br>$3x+4$<br>$3x+1$<br>$3$<br>(Long Division) | * | Note the use of absolute values in log integrals. Although marks were not deducted this time, Ext 2 students should use them when appropriate. |
| $I = \int 3x^2+2x+1+\frac{3}{3x+1}\,dx$ | 1 | |
| $= \underline{\underline{x^3+x^2+x+\ln\lvert 3x+1\rvert + k}}$ | 2 | |

**MATHEMATICS Extension 2: Question** 6

sheet 2 of 3

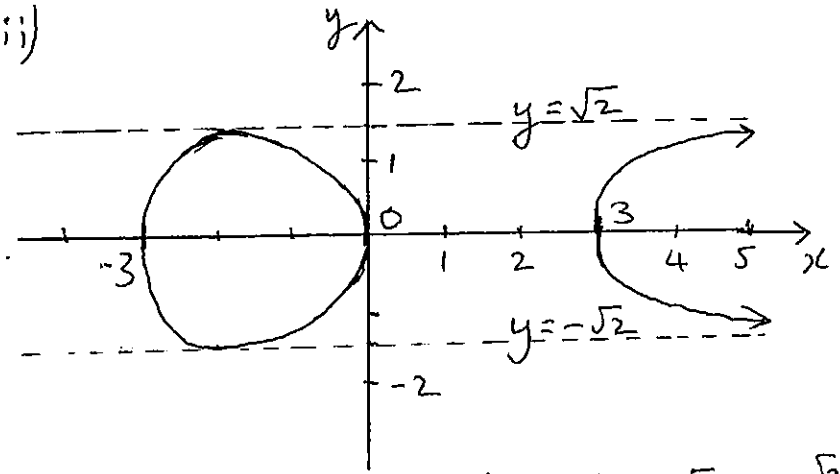
| Suggested Solutions | Marks | Marker's Comments |
|---|---|---|
| c) i)   1. Maximum of $y=4$<br>2. Labelled asymptote at $y=4$<br>3. Cut of values at $\pm 3, 0$ labelled<br>4. Some acknowledgement that $y=0$ for $x \leq -3$ and $0 \leq x \leq 3$. | 2 | Surprising number of people made no mark for $x \leq -3$ and $0 \leq x \leq 3$. The function exists there but equals 0. I need to see something different or some words. |
| c) ii)   1. Labelled asymptotes at $x=3, x=0, x=-3, y=\frac{1}{2}$<br>2. Max and min values pinned to $\pm\frac{1}{2}$<br>3. Four parts of curve in correct alignment. | 2 | In this case (and (iv)) clearer with different scales. Equal scales preferable. Whichever is used, scales should be defined. |

**MATHEMATICS Extension 2 : Question 6**



| Suggested Solutions | Marks | Marker's Comments |
|---|---|---|

c) iii)

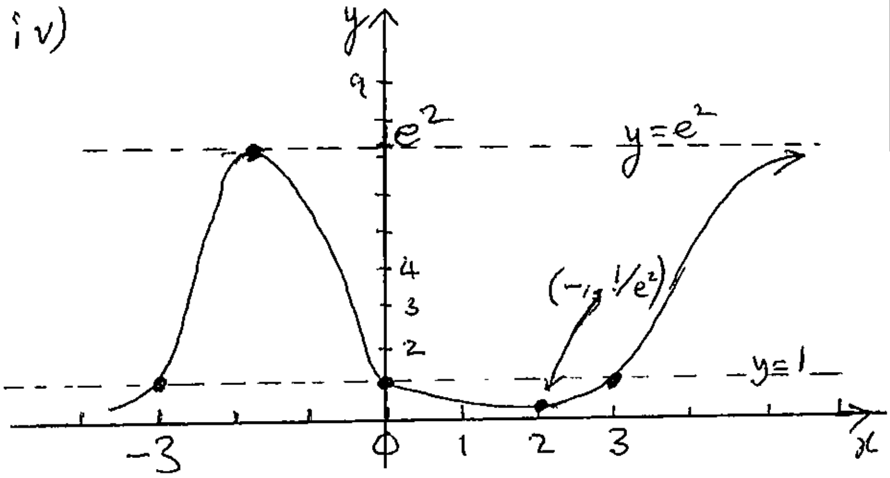



1. Labelled asymptotes at $y=\sqrt{2}$, $y=-\sqrt{2}$
2. Vertical lines (vertical tangents) at all $x$ intercepts.
3. Points pinned as appropriate at $x=-3, 0, 3$ and $y=\pm 2$
4. Symmetry in $x$ axis.

Marks: 2

Marker's Comments: Shape is always important. In this case it is important to show the verticals at the $x$ intercepts, ($\frac{dy}{dx} \to \infty$)

c) iv)




1. Labelled asymptote at $y=e^2$
2. Three crossings (marked some way) of the line $y=1$
3. Minimum value $1/e^2$ shown somehow
4. Maximum of $e^2$ shown somehow

Marks: 2

Marker's Comments: Little information for $x<-3$ but $y$ certainly does not go negative.

$e^2 \approx 7+$ It would be nice if that is shown in the scaling

| Mathematics: Extension 1, Question 7 | | |
|---|---|---|
| **Suggested Solutions** | **Marks** | **Marker's Comments** |
| (a)<br>(i) To find equation of tangent to curve at $P(x_1, y_1)$:<br>$$\frac{d}{dx}\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)=\frac{d}{dx}(1)$$<br>$$\frac{2x}{a^2}-\frac{2yy'}{b^2}=0$$<br>$$y'=\frac{b^2x}{a^2y}$$<br>At $P(x_1, y_1)$, the gradient $m$ is therefore,<br>$$m=\frac{b^2x_1}{a^2y_1}$$ | 1 for gradient. | |
| Hence the equation of the tangent to the hyperbola at $P$ is<br>$$y-y_1=\frac{b^2x_1}{a^2y_1}(x-x_1)$$<br>$$a^2y_1y-a^2y_1^2=b^2x_1x-b^2x_1^2$$<br>$$b^2x_1^2-a^2y_1^2=b^2xx_1-a^2yy_1$$<br>$$\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}=\frac{xx_1}{a^2}-\frac{yy_1}{b^2}$$ after divding through by $a^2b^2$<br>Since $P(x_1, y_1)$ lies on the hyperbola, $\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}=1$, and so<br>$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$$<br>as required. | 1 for tangent. | Should state why we obtain '1' (i.e. P satisfies the equation). |
| (ii) Let the tangents through $R(x_0, y_0)$ meet the curve at points $P(x_1, y_1)$ and $Q(x_2, y_2)$. The tangent at $P$ and $Q$ are given by the result of (i) respectively by<br>$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1 \quad \ldots(1)$$<br>and<br>$$\frac{xx_2}{a^2}-\frac{yy_2}{b^2}=1 \quad \ldots(2)$$ | 1 for acquiring tangent at Q and explaining correctly that (3) and (4) follow because R lies on both lines. | Tangent at Q needn't be derived, but at least quote 'similarly' for Q as P. |

<table>
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<td>
Since each tangent passes through $R$, we must have by (1) and (2),

$$\frac{x_0x_1}{a^2}-\frac{y_0y_1}{b^2}=1 \quad (3)$$

$$\frac{x_0x_2}{a^2}-\frac{y_0y_2}{b^2}=1 \quad (4)$$

Now, consider the line $l$:

$$l:\ \frac{xx_0}{a^2}-\frac{yy_0}{b^2}=1$$

Then by (3), $P$ lies on $l$, and by (4), $Q$ lies on $l$.

Since two points determine a line uniquely, the chord $PQ$ lies on $l$. Hence the chord (of contact) is given by

$$\frac{xx_0}{a^2}-\frac{yy_0}{b^2}=1$$
</td>
<td>1 for reasoning correctly why the equation required was that containing the chord of contact.</td>
<td>Many students constructed the two equations (3) and (4) correctly, but then failed to state (logically) how this could lead to the chord of contact equation.<br><br>The first two equations are results that we keep to the side. We then consider a line of form given by line l. THEN, since equations (3) and (4) SHOW that P, Q satisfy l, and since there is only one line passing through two points, it MUST be the case that l is the equation of a line connecting P and Q; i.e. a line containing the chord of contact (a line segment).</td>
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<td>
(iii) Let $\left(\pm\frac{a}{e},y_0\right)$ be an arbitrary point on one of the directrices. The chord of contact from this point to the hyperbola is then

$$\frac{x\left(\pm\frac{a}{e}\right)}{a^2}-\frac{yy_0}{b^2}=1$$

$$\pm\frac{x}{ae}-\frac{yy_0}{b^2}=1 \quad (5)$$

The chord passes through the corresponding focus if and only if $(\pm ae, 0)$ satisfies equation (5):

$$LHS_{of(5)}=\pm\frac{\pm ae}{ae}-\frac{y(0)}{b^2}$$

$$=1-0$$

$$=1$$

$$=RHS_{of(5)}$$

Hence, the chord of contact from a point on the directrix is a focal chord (passing through the corresponding focus).
</td>
<td>1 for showing that the chord passing through arbitrary point on directrix was focal chord.<br><br>Did not penalise for assuming $(\frac{a}{e},y_0)$ only.</td>
<td>Many students assumed only a point on the directrix of the positive arm of the hyperbola. If this were a 2-mark question, one mark would not have been awarded.<br><br>Also, too many students did not select an arbitrary point on the directrix (as the external point for chord of contact), and then use that with chord of contact equation to derive the appropriate equation.<br><br>Once you have the chord of contact for</td>
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|---|---|---|
| | | this situation, you're then tasked with proving that it passes through the focal point. That is, you're being asked to show that **every** such chord passes through the appropriate focus. Choosing $(\frac{a}{e}, 0)$ as your external point does not do this. |
| (b) In<br>$\int \sin(\log x)\, dx$<br>let $u = \log x$. Then $x = e^u$ and so $dx = e^u du$, giving<br>$\int \sin(u)\, e^u\, du$<br>Now, let $U = \sin u$ and $dV = e^u du$. Then $dU = \cos u\, du$ and $V = e^u$, so that:<br>$\int \sin(u)\, e^u\, du = e^u \sin u - \int \cos(u)\, e^u\, du$<br>For<br>$\int \cos(u)\, e^u\, du$<br>let $U = \cos u$ and $dV = e^u\, du$. Then $dU = -\sin u\ \, du$ and $V = e^u$, so,<br>$\int \sin(u)\, e^u\, du = e^u \sin u - \int \cos(u)\, e^u\, du$<br>$= e^u \sin u - \left[e^u \cos u - \int (-\sin u)\, e^u\, du\right]$<br>$= e^u \sin u - e^u \cos u - \int \sin(u)\, e^u\, du$<br>So,<br>$2\int \sin(u)\, e^u\, du = e^u \sin u - e^u \cos u$<br>giving<br>$\int \sin(\log x)\, dx = \frac{x}{2}(\sin(\log x) - \cos(\log x)) + C$ | 1 for correct integrand after substitution.<br><br>1 for a correct intermediate integral.<br><br>1 for correct final result (must have substituted back for original variable). | The integration was handled well.<br><br>The only cause for concern was that of students not keeping track of their working/brackets, and letting trouble with minus signs ensue.<br><br>Also, you must give your final answer in terms of the original variable – don't leave you integral in substituted form.<br><br>Also, do not forget to include a constant of integration. No penalisation here for such instances, but, there are situations in a mathematics problem where the constant of integration is important (e.g. you may have to derive an original function) – leaving it off leads to errors. |

| | | |
|---|---|---|
| (c) $\displaystyle\int_0^{\log 2} \frac{e^{2x}}{1+e^x}\,dx$ | | |
| Let $u = 1 + e^x$. Then $u = 2$ when $x = 0$ and $u = 3$ when $x = \log 2$. | | |
| Also, $du = e^x\,dx \Rightarrow \frac{du}{e^x} = dx \Rightarrow \frac{du}{u-1} = dx$. So, | | The integration was handled well. |
| $\displaystyle\int_0^{\log 2} \frac{e^{2x}}{1+e^x}\,dx = \int_2^3 \frac{(u-1)^2}{u}\frac{du}{(u-1)}$ <br> $\displaystyle= \int_2^3 \frac{u-1}{u}\,du$ <br> $\displaystyle= \int_2^3 1 - \frac{1}{u}\,du$ | 1 for a correct integral. | Errors crept in where people did not take care with the algebra once substitutions were made. <br><br> There were also errors with signs (again), leading to incorrect final answer. |
| $= u - \log u\vert_2^3$ <br> $= (3 - \log 3) - (2 - \log 2)$ | 1 for correct integration. | |
| $= 1 + \log 2 - \log 3$ <br> $\displaystyle= 1 + \log\frac{2}{3}$ | 1 for correct result. | |
| 'or otherwise...' <br> $\displaystyle\int_0^{\log 2} \frac{e^{2x}}{1+e^x}\,dx = \int_0^{\log 2} \frac{e^x}{1+e^x}\,de^x$ <br> $\displaystyle= \int_0^{\log 2} \frac{1+e^x-1}{1+e^x}\,de^x$ <br> $\displaystyle= \int_0^{\log 2} 1 - \frac{1}{1+e^x}\,de^x$ <br> $= e^x - \log(1+e^x)\,\vert_0^{\log 2}$ <br> $= (2 - \log 3) - (1 - \log 2)$ <br> $\displaystyle= 1 + \log\frac{2}{3}$ | | A sizeable portion of the candidature approached the problem using techniques illustrated left. <br><br> It is mathematically 'correct', but should **not be used for the HSC.** <br><br> In this integral, we choose to integrate with respect to the function $e^x$. Note that $de^x = e^x dx$, so we recover the original integrand. |

| | | |
|---|---|---|
| (d)<br>(i) Let $z = \cos\theta + i\sin\theta$.<br>Then for $n \in \mathbb{N}_0$,<br>$z^n + z^{-n} = (\cos\theta + i\sin\theta)^n + (\cos\theta + i\sin\theta)^{-n}$<br>$= (\cos n\theta + i\sin n\theta) + (\cos(-n\theta) + i\sin(-n\theta)) \quad \dots(*)$<br>$= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$<br>$= 2\cos n\theta$<br>where $(*)$ follows by De Moivre's Theorem. | 1 for correct use of De Moivre's Theorem.<br><br>1 for correct result. | No problems with using De Moivre's theorem to prove the result in (i). |
| (ii) $(2\cos\theta)^{2m} = (z + z^{-1})^{2m}$ by (i) with $n = 1$.<br>Now, by the binomial theorem,<br>$(2\cos\theta)^{2m} = (z+z^{-1})^{2m} = \sum_{k=0}^{2m} \binom{2m}{k} z^{2m-k}(z^{-1})^k$<br>$= \sum_{k=0}^{2m} \binom{2m}{k} z^{2m-2k} \quad \dots(1)$<br>Note that there are an **odd** number of terms (counting begins at zero and ends on an even number). When $k = m$, we have<br>$\binom{2m}{m} z^{2m-2m} = \binom{2m}{m} z^0 = \binom{2m}{m}$ | 1 for correct use of binomial theorem, leading to something of consequence to the solution of the problem. | Many of those who attempted part (ii) recognised the symmetry involved and demonstrated so.<br><br>It was important to show why $\binom{2m}{m}$ appeared as a lone term, and explain the consequence of symmetry here (Pascal's relation). |
| For the remaining $2m$ other terms, we can pair the $k^{\text{th}}$ term with its symmetrical partner, the $(2m-k)^{\text{th}}$ term, as follows:<br>$\binom{2m}{k} z^{2m-2k}$ \| $\binom{2m}{2m-k} z^{2m-2(2m-k)} = \binom{2m}{2m-k} z^{-(2m-2k)}$<br>$= \binom{2m}{k} z^{-(2m-2k)}$ (using $\binom{n}{r} = \binom{n}{n-r}$)<br>As there are $2m$ terms being paired, we have $m$ pairings, and hence a sum from $k = 0$ to $m - 1$. | 1 for remainder of complete solution. | Some students provided working, but not enough in the sense that, to obtain the final mark, you had to show directly how the result followed from your working. This is important because you're given the solution to the problem (if you weren't given the solution, and you produced the correct answer, the assessor would give you the mark. Without steps, you're asking the marker to essentially 'believe you' when you make the leap |

| | | |
|---|---|---|
| So, $(2\cos\theta)^{2m} = (z+z^{-1})^{2m} = \sum_{k=0}^{2m}\binom{2m}{k} z^{2m-2k}$ <br> $= \sum_{k=0}^{m-1}\binom{2m}{k}\left(z^{2m-2k} + z^{-(2m-2k)}\right) + \binom{2m}{m}$ <br> $= \sum_{k=0}^{m-1}\binom{2m}{k}(2\cos(2m-2k)\theta) + \binom{2m}{m}$ by result of part (i) <br> $= 2\sum_{k=0}^{m-1}\binom{2m}{k}\cos(2m-2k)\,\theta + \binom{2m}{m}$ <br> as required. | | from limited working to the desired conclusion). |

| MATHEMATICS Extension 2 : Question..8.... | | |
|---|---|---|
| **Suggested Solutions** | **Marks** | **Marker's Comments** |
| (i) $t = \tan\frac{x}{2}$ $\quad \frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$ <br> $\frac{dx}{dt} = \frac{2}{\sec^2 x/2}$ <br> $dx = \frac{2\,dt}{1+t^2}$ <br> $x = 0 \quad t = 0$ <br> $x = \pi/2 \quad t = 1$ <br> $\boxed{\begin{matrix} 0 \sim \pi/2 \\ 0 \sim 1 \end{matrix}}$ Confusing. | | |
| $\int_0^1 \frac{\frac{2dt}{1+t^2}}{1+\frac{2t}{1+t^2}}\,dt$ | 1 | $t$-formula on reference sheet no excuse for getting incorrect. |
| $= \int_0^1 \frac{2\,dt}{(1+t)^2}$ | | Be careful of notation |
| $= 2\left[\frac{-1}{1+t}\right]_0^1$ <br> $= 2\left[-\frac{1}{2}+1\right]$ | 1 | |
| $= 1$ | | |
| (ii) Method 1 <br> $\int_0^a f(x)\,dx = \int_0^{a/2} f(x)\,dx + \int_{a/2}^{a} f(x)\,dx$ | 1 | |
| let $x = a-u$ $\quad dx = -du$ <br> $x = \frac{a}{2} \quad u = \frac{a}{2}$ <br> $x = a \quad u = 0$ <br> $= \int_0^{a/2} f(x)\,dx - \int_{a/2}^{0} f(a-u)\,du$ <br> $= \int_0^{a/2} f(x) + \int_0^{a/2} f(a-x)\,dx$ using dummy variable. | 1 | Needed to see $-\int_{a/2}^{a} f(a-u)$ step for second mark. |

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| Suggested Solutions | Marks | Marker's Comments |
|---|---|---|
| Method 2 | | |
| Let primitive of $f(x)$ be $F(x)$ | | |
| $\int_0^a f(x)\,dx = F(a) - F(0)$ & | 1 | |
| RHS = | | |
| $\int_0^{a/2} f(x)\,dx + \int_0^{a/2} f(a-x)\,dx$ since $\frac{d}{dx} f(a-x) = -f'(a-x)$ | 1 | |
| $= F(\frac{a}{2}) - F(0) - \left[F(a-\frac{a}{2}) - F(a)\right]$ | | |
| $= F(a) - F(0)$ | | |
| $=$ LHS | | |
| Method 3 was method 1 backwards. | | |
| students who used $\int_0^a f(x) = \int_0^a f(a-x)$ property used it incorrectly. | | |
| (iii) $\int_0^{\pi} \frac{x}{1+\sin x}\,dx$ from part (ii) | | |
| $= \int_0^{\pi/2} \frac{x}{1+\sin x} + \frac{\pi - x}{1+\sin(\pi - x)}\,dx$ | | students didn't look closely and used $\frac{\pi}{2} - x$. |
| $= \int_0^{\pi/2} \frac{x+\pi-x}{1+\sin x}\,dx$ since $\sin(\pi - x) = \sin x$. | | |
| $= \int_0^{\pi/2} \frac{\pi}{1+\sin x}\,dx$ from part c) | 1 | |
| $= \pi$ | | |

**MATHEMATICS Extension 2: Question....8...**

| Suggested Solutions | Marks | Marker's Comments |
|---|---|---|
| b(i) $\dfrac{x^2}{6-k}+\dfrac{y^2}{4-k}=1$ | | Students who looked at e made question much harder and didn't get correct answer |
| to be ellipse must be of form. | | |
| $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1.$ | | |
| $\therefore$ $6-k>0$ and $4-k>0$ | | |
| $k<6$ and $k<0$ | 1 | |
| so $\underline{k<0}$ | | |
| For hyperbola | | |
| $6-k>0$ and $4-k<0$ | | |
| $k<6$ and $k>4.$ | 1 | |
| $4<k<6$ | | |
| OR | | didn't lose marks for not considering this. |
| $\left(\begin{array}{l} 6-k<0 \text{ and } 4-k>0 \\ k>6 \text{ and } k<4 \\ \text{so no solution}\end{array}\right)$ | | |
| (ii) see over. | | |
| (iii) As k increases from 5 to 6 graph becomes steeper vertex approaches origin and hyperbola approaches the vertical line $x=0$ or y-axis | 1 | Needed to indicate it was becoming vertical and approaching 0. poorly done. |

**MATHEMATICS Extension 2: Question…8…**

| Suggested Solutions | Marks | Marker's Comments |
|---|---|---|
| | | |

(ii) $a^2 = b^2(e^2-1)$

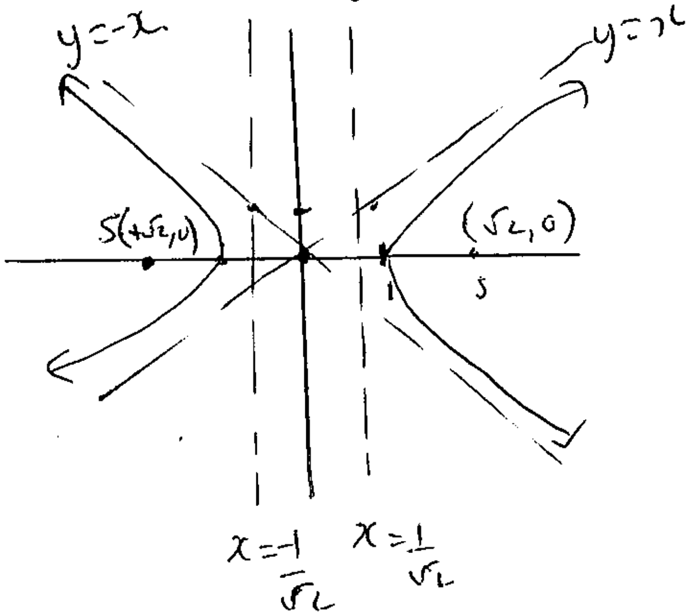
$1 = 1(e^2-1)$

$e^2 = 2$

$e = \sqrt{2}$

D : $x = \pm\dfrac{1}{\sqrt{2}}$ focus $(\pm\sqrt{2}, 0)$

asymptote $y = \pm x$.




1 for hyperbola and asymptotes

1 for foci, and directrices and $e = \sqrt{2}$

| MATHEMATICS Extension 2 : Question 8...... | | |
|---|---|---|
| **Suggested Solutions** | **Marks** | **Marker's Comments** |
| c)(i) $\dfrac{x^2}{a^2}+\dfrac{(mx+c)^2}{b^2}=1$ | | |
| $\therefore b^2x^2+a^2(m^2x^2+2mcx)+a^2c^2=a^2b^2$ | | well done |
| $\therefore (b^2+a^2m^2)x^2+2ma^2cx+a^2c^2-a^2b^2=0$ | 1 | some poor algebra. |
| for tangent $\Delta=0$ | | some people went off on wrong path. |
| $4a^4m^2c^2-4(b^2+a^2m^2)(a^2c^2-a^2b^2)=0$ | | by finding equation of tangent. |
| $a^2\cancel{m^2}c^2-b^2c^2+b^4-a^2m^2c^2+a^2m^2b=0$ | 1 | |
| $\therefore -c^2+b^2+a^2m^2=0$ | | |
| $c^2=b^2+a^2m^2$ as required | | |
| (ii) $\dfrac{x^2}{16}+\dfrac{y^2}{9}=1$ $\quad c^2=16m^2+9$ | | 8 |
| at $(3,4)$ $\quad 4=3m+c.$ | 1 | |
| $\therefore c=4-3m$ | | Some solved quadratic then multiplied. |
| $(4-3m)^2=16m^2+9$ | | |
| $7m^2-24m-7=0$. Roots are $m_1$ and $m_2$ | 1 | |
| products of roots $=\dfrac{-7}{7}$ | | |
| $=-1$ | | |
| $\therefore m_1\times m_2=-1$ | 1 | |
| tangents are perpendicular | | |



5.

Mathematics Ext 2 Question 9

| Suggested Solutions | Marks | Marker's Comments |
|---|---|---|

a) i $f(\frac{\pi}{2}-x) = \dfrac{1}{1+\tan(\frac{\pi}{2}-x)^k}$

$= \dfrac{1}{1+\dfrac{\sin^k(\frac{\pi}{2}-x)}{\cos^k(\frac{\pi}{2}-x)}}$

$= \dfrac{\cos^k(\frac{\pi}{2}-x)}{\cos^k(\frac{\pi}{2}-x)+\sin^k(\frac{\pi}{2}-x)}$

$= \dfrac{\sin^k x}{\sin^k x+\cos^k x}$ ——— 1

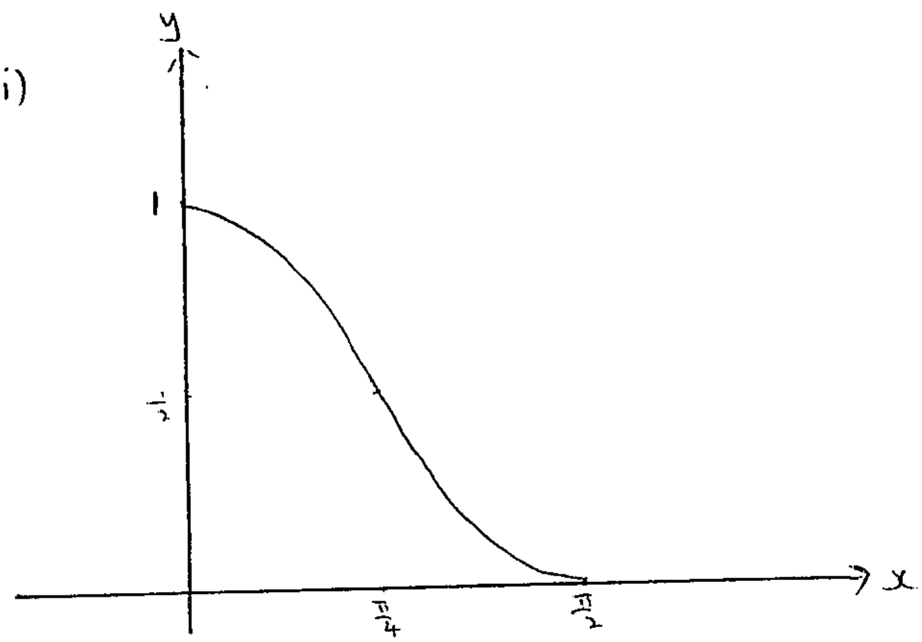
$f(x) = \dfrac{1}{1+\dfrac{\sin^k x}{\cos^k x}}$

$= \dfrac{\cos^k x}{\cos^k x+\sin^k x}$

$\therefore f(x)+f(\frac{\pi}{2}-x) = \dfrac{\cos^k x+\sin^k x}{\cos^k x+\sin^k x}$

$= 1$ ——— 1

ii)

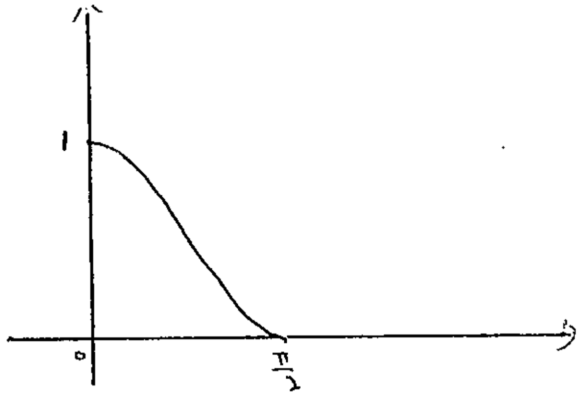



3 marks

Mathematics ________ Question ______

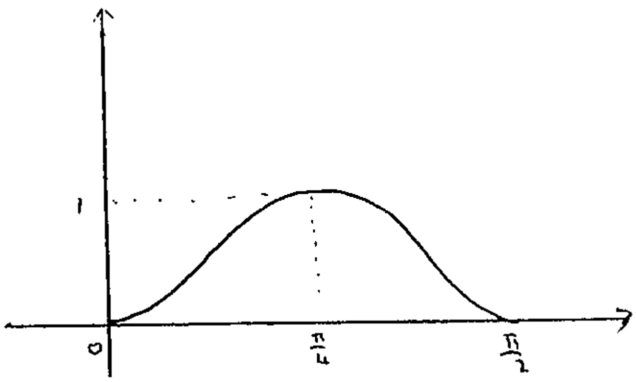
| Suggested Solutions | Marks | Marker's Comments |
|---|---|---|

Other graphs.



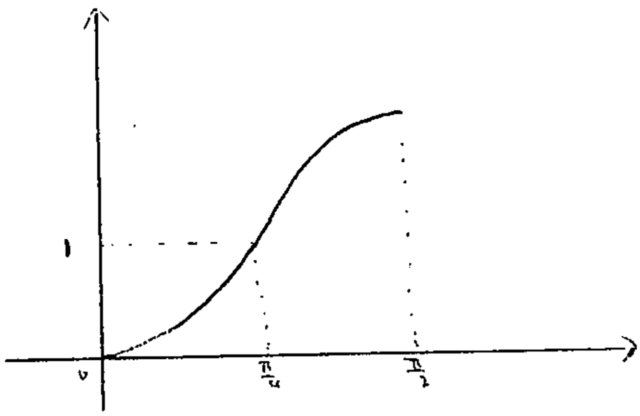

2 marks

no $\left(\frac{\pi}{4}, \frac{1}{2}\right)$



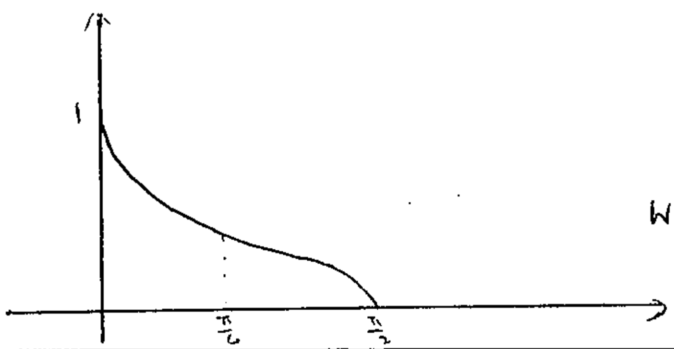

1 mark

(wrong y-intercept)




2 marks

(back to front)




2 marks

Wrong concavity

| Mathematics _______ Question _____ | Marks | Marker's Comments |
|---|---|---|
| Suggested Solutions | | |
| iii) <u>Method 1</u><br>Since Area is exactly half the area of rectangle with length $\frac{\pi}{2}$ and height 1. (By graph)<br>$\therefore I = \frac{1}{2} \times \frac{\pi}{2} \times 1$<br>$= \frac{\pi}{4}$ | | |
| iv) Let $I = \int_0^{\frac{\pi}{2}} \frac{1}{1+\tan^k x}\, dx$<br>$= \int_0^{\frac{\pi}{2}} \frac{1}{1+\tan^k(\frac{\pi}{2}-x)}\, dx$<br>$\therefore 2I = \int_0^{\frac{\pi}{2}} 1 \, dx$ (from i) | 1 | |
| $\therefore 2I = [x]_0^{\frac{\pi}{2}}$<br>$I = \frac{\pi}{2} \times \frac{1}{2}$<br>$= \frac{\pi}{4}$ | 1 | |
| Anyone who wrote $2I = 1$ and $\therefore I = \frac{1}{2}$ received 0 marks! | | |

Mathematics ______ Question ______

| Suggested Solutions | Marks | Marker's Comments |
|---|---|---|

b) i) $x^2y^2 - x^2 + y^2 = 0$

$x^2y^2 + y^2 = x^2$

$y^2(x^2+1) = x^2$

$y^2 = \dfrac{x^2}{x^2+1}$

$< 1$ (Since $x^2+1 > x^2$)

$\therefore |y| < 1$ ——— 1

ii) $x^2y^2 - x^2 + y^2 = 0$

$\therefore x^2 - y^2 = x^2y^2$

$\geq 0$

$\therefore x^2 - y^2 \geq 0$

$x^2 \geq y^2$

$y^2 \leq x^2$

$|y| \leq |x|$ ——— 1

Many assumed that $\dfrac{y^2}{1-y^2} \geq 1$ and $\therefore |x| \geq 1$

Wrong!!!

iii) $2xy^2 + 2y\frac{dy}{dx}x^2 - 2x + 2y\frac{dy}{dx} = 0$ ——— 1

$\frac{dy}{dx}(2yx^2 + 2y) - (2x - 2xy^2) = 0$

$\frac{dy}{dx}(2yx^2 + 2y) = 2x - 2xy^2$

$\frac{dy}{dx} = \dfrac{2x - 2xy^2}{2yx^2 + 2y}$

$= \dfrac{x(1-y^2)}{y(1+x^2)}$ ——— 1

Mathematics _______ Question _____

| Suggested Solutions | Marks | Marker's Comments |
|---|---|---|

iv) $\frac{dy}{dx}$ is undefined with $y(1+x^2) = 0$

$\therefore x = 0$

When $y=0$, $x=0$

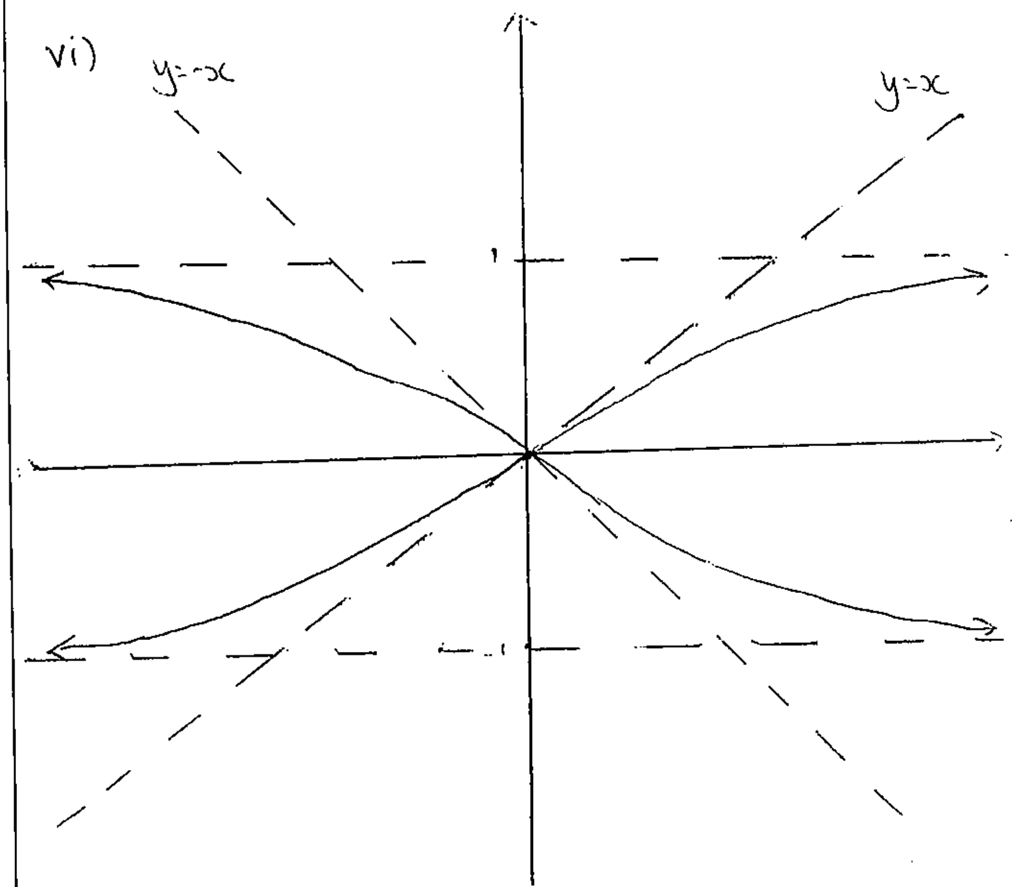
$\therefore$ critical point $(0,0)$ — ①

v) $\frac{dy}{dx} = 0 \Rightarrow x(1-y^2) = 0$

$\therefore x = 0$ or $y = \pm 1$

Since $\frac{dy}{dx}$ at $(0,0)$ is undefined (part iv) and $|y| < 1$, $\therefore y \neq \pm 1$. There are no stationary points.

vi)




1 – shape

1 – lying between $y = \pm 1$ and $y = \pm x$.

JRAHS : Maths Extension 2 : Task 2 Term 1 2018

**MATHEMATICS Extension 2: Question 10**

| Suggested Solutions | Marks | Marker's Comments |
|---|---|---|
| a). (i) $I_n = \int_0^{\pi/2} \sin^n x \, dx = \int_0^{\pi/2} \sin^{n-1} x \sin x \, dx$ | | Method I |
| $\boxed{\begin{array}{l} \text{Let } u = \sin^{n-1} x \Rightarrow du = (n-1)\sin^{n-2} x \cos x \, dx \\ \text{and } dv = \sin x \, dx \Rightarrow v = -\cos x \end{array}}$ | | |
| $\therefore I_n = \sin^{n-1} x(-\cos x)\Big\vert_0^{\pi/2} - (n-1)\int_0^{\pi/2} (-\cos x)\sin^{n-2} x \cos x \, dx$ ... (by parts) | | |
| $= 0 + (n-1)\int_0^{\pi/2} \cos^2 x \sin^{n-2} x \, dx$ | 1 | Must show what happens to $-\sin^{n-1} x \cos x \Big\vert_0^{\pi/2}$ to gain maximum marks |
| $= (n-1)\int_0^{\pi/2} (1-\sin^2 x)\sin^{n-2} x \, dx$ | | |
| $= (n-1)\int_0^{\pi/2} (\sin^{n-2} x - \sin^n x) \, dx$ | | |
| $= (n-1)\left[\int_0^{\pi/2} \sin^{n-2} x \, dx - \int_0^{\pi/2} \sin^n x \, dx\right]$ | | |
| $= (n-1)\left[I_{n-2} - I_n\right]$ | 1 | For recognising that $I_n = \int_0^{\pi/2} \sin^n x \, dx$ and $I_{n-2} = \int_0^{\pi/2} \sin^{n-2} x \, dx$ |
| $I_n + (n-1)I_n = (n-1)I_{n-2}$ | | |
| $nI_n = (n-1)I_{n-2}$ | | |
| $\therefore I_n = \frac{(n-1)}{n} I_{n-2}$ as required | | |

**MATHEMATICS Extension 2: Question 10**

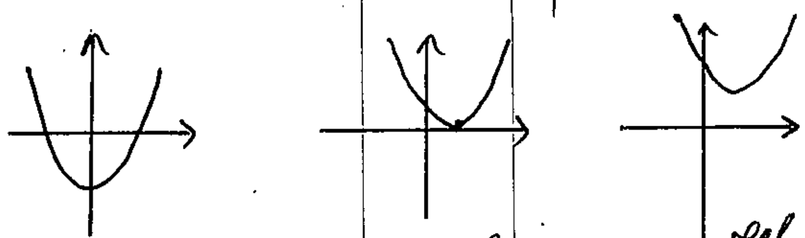
| Suggested Solutions | Marks | Marker's Comments |
| --- | --- | --- |
| Method 2 : | | |
| $I_n = \int_0^{\pi/2} \sin^{n-1}x \sin x \, dx$ | | |
| $= \int_0^{\pi/2} \sin^{n-1}x \cdot \frac{d}{dx}(-\cos x)\, dx$ | | |
| $= \sin^{n-1}x(-\cos x)\Big\vert_0^{\frac{\pi}{2}} - \int_0^{\pi/2} -\cos x \frac{d}{dx}(\sin^{n-1}x)\, dx$ ... (by parts) | | |
| $= 0 + (n-1)\int_0^{\pi/2} \cos^2 x \cdot \sin^{n-2}x \, dx$ | 1 | |
| $= (n-1)\int_0^{\pi/2} (1-\sin^2 x)\sin^{n-2}x \, dx$ | | |
| $= (n-1)\int_0^{\pi/2} (\sin^{n-2}x - \sin^n x)\, dx$ | | |
| $= (n-1)\left[I_{n-2} - I_n\right]$ | 1 | |
| $\therefore I_n + (n-1)I_n = (n-1)I_{n-2}$ | | |
| $n I_n = (n-1)I_{n-2}$ | | |
| $\therefore I_n = \frac{n-1}{n} I_{n-2}$ | | |

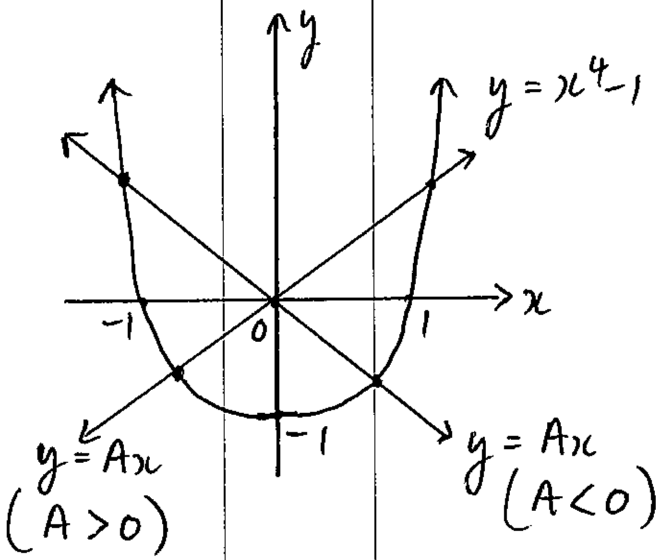
JRAHS : Maths Ext 2 : Task 2 Term 1 2018

| MATHEMATICS Extension 2: Question 10 | | |
|---|---|---|
| **Suggested Solutions** | **Marks** | **Marker's Comments** |
| a(ii) $I_n = \int_0^{\pi/2} \sin^n x\,dx$ ... given | | |
| $I_n = \frac{n-1}{n} I_{n-2}$ ... proved above | | |
| $\therefore I_4 = \frac{3}{4} I_2$ | | |
| $= \frac{3}{4}\left(\frac{1}{2} I_0\right)$ | | |
| $= \frac{3}{8}\int_0^{\pi/2} \sin^0 x\,dx$ | 1 | First mark for getting $I_2 = \frac{\pi}{4}$ |
| $= \frac{3}{8}\int_0^{\pi/2} 1 \,.\, dx$ | | |
| $= \frac{3}{8}\left(x\right)\Big\vert_0^{\pi/2}$ | | |
| $= \frac{3\pi}{16}$ | 1 | 2nd mark for getting $I_4 = \frac{3\pi}{16}$ |
| Note: $I_2 = \int_0^{\pi/2} \sin^2 x\,dx$ could have been used to get $\pi/4$ | | |

JRAHS : Maths Ext 2 : Task 2 Term 1 2018

**MATHEMATICS Extension 2: Question** 10

| Suggested Solutions | Marks | Marker's Comments |
|---|---|---|
| b) Method I :<br>(1) $P(x) = x^4 - Ax - 1$<br>$P'(x) = 4x^3 - A$<br>$P''(x) = 12x^2$<br>$\Rightarrow P''(x) > 0$ for all $x \in \mathbb{R}$<br>$\Rightarrow$ curve is concave up | 1 | For concluding curve is concave up for all $x$ |
| <br>distinct roots    equal roots    no real roots | 1 | For demonstrating that there could be 2 distinct roots, a double root, or no real roots |
| $\therefore$ At most there are 2 real roots | | |
| For 2nd mark, there is only 1 turning point at $x = \sqrt[3]{A/4}$<br><br>OR $P''(x) = 0$, when $x = 0$<br>But $P(0) \neq 0$, hence no double roots and with 2 complex roots (conjugate pairs as coefficients are real), there must be at most 2 real roots | | |

| MATHEMATICS Extension 2: Question 10 | | |
|---|---|---|
| **Suggested Solutions** | **Marks** | **Marker's Comments** |
| b) (i) Method III:<br>$P(x) = x^4 - Ax - 1$<br>$P(x) = 0$<br>$\Rightarrow$<br>$x^4 - 1 = Ax$<br>Graph $y = x^4 - 1$ and $y = Ax$ | 1 | |
|  | | |
| Clearly there are 2 points of intersection when $x^4 - 1 = Ax$ and hence $P(x) = x^4 - Ax - 1$ has at most 2 real roots | 1 | |

JRAHS : Maths Ext 2 : Task 2 Term 1 2018

| MATHEMATICS Extension 2: Question 10 | | |
|---|---|---|
| **Suggested Solutions** | **Marks** | **Marker's Comments** |
| b) Method II :<br>(1) $P(x) = x^4 - Ax - 1, \quad A \in \mathbb{R}$<br>Clearly polynomial with real coefficients<br>$\Rightarrow$ P(x) can be expressed as a product of linear and ~~OR~~ quadratic irreducible factors, by FTA | | 1st mark |
| But $P(x)$ = quartic polynomial (at most 4 roots) | | |
| • 4 complex roots : $(\text{quadratic irreducible})^2$<br>• 2 complex + double root : $(\text{linear})^2 \times$ quadratic irreducible<br>• 2 complex + 2 distinct roots : linear $\times$ linear $\times$ quadratic irreducible<br>Clearly this quartic polynomial has <u>at most</u> 2 real roots | | 2nd mark |

JRAHS : Ext 2 Task 2 Term 1 2018

**MATHEMATICS Extension 2: Question** 10

| Suggested Solutions | Marks | Marker's Comments |
|---|---|---|
| b ii) If $\alpha, \beta, \gamma, \delta$ are the roots of $x^4 - Ax - 1 = 0$, then $\alpha^2, \beta^2, \gamma^2, \delta^2$ must satisfy | | |
| $\left(x^{\frac{1}{2}}\right)^4 - A\left(x^{\frac{1}{2}}\right) - 1 = 0$ | 1 | For making Transition to the new equation |
| $\Rightarrow x^2 - A\sqrt{x} - 1 = 0$ | | |
| $x^2 - 1 = A\sqrt{x}$ | | |
| $(x^2-1)^2 = A^2 x$ | 1 | For algebraic manipulation |
| $\therefore x^4 - 2x^2 - A^2x + 1 = 0$ is the required equation | | Note: if $m = \alpha^2$ then $\alpha = \pm\sqrt{m}$ |
| (iii) Since $\alpha^2, \beta^2, \gamma^2, \delta^2$ are roots of $x^4 - 2x^2 - A^2x + 1 = 0$, then $\alpha^2+1, \beta^2+1, \gamma^2+1, \delta^2+1$ must satisfy | | |
| $(x-1)^4 - 2(x-1)^2 - A^2(x-1) + 1 = 0$ | 1 | For transition to new equation |
| But product of roots $= (\alpha^2+1)(\beta^2+1)(\gamma^2+1)(\delta^2+1) = \frac{e}{a}$ | | |
| $= (-1)^4 - 2(-1)^2 + A^2 + 1$ | 1 | Algebraic manipulation |
| $= 1 - 2 + A^2 + 1$ | | |
| $= A^2$ as required | | |



Note: If $x = \pm\sqrt{m-1}$, then $x^4 - Ax - 1 = 0$ is the equation into which $\pm\sqrt{m-1}$ needs to be substituted

## QUESTION 10

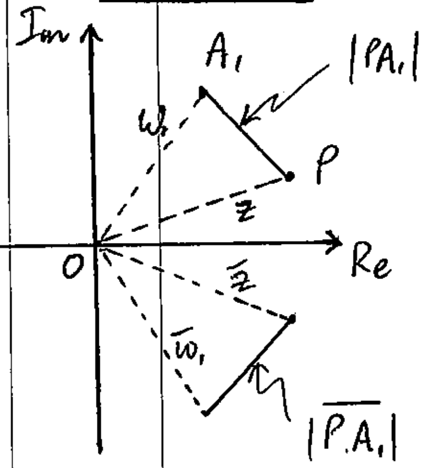
(c)

Note:

$PA_1 = PO + OA_1$

$= -z + w_1$

$= w_1 - z$




(i) $|PA_i| = |z - w_i|$

$\therefore |PA_i|^2 = |z - w_i|^2$

or $PA_i^2 = |z - w_i|^2$

$= (z - w_i)(\overline{z - w_i})$ since $|z|^2 = z\bar{z}$ — 1 + 1

$= (z - w_i)(\bar{z} - \bar{w}_i)$

for $i = 1, 2, \ldots n$

for linking the distance $PA_i$ to $|z - w_i|$ and using $|z|^2 = z \cdot \bar{z}$ and using mod of the difference = difference of the mods.

If RHS $= (z - w_i)(\bar{z} - \bar{w}_i)$ must say why!

$= (z - w_i)(\overline{z - w_i})$ must say why!

$= |z - w_i|^2$

$= (z - w_i)^2$ must say why!

$= (PA_i)^2$

to get maximum marks.

Note: must mention $|z|^2 = z \cdot \bar{z}$ and $(\overline{z - w_i}) = (\bar{z} - \bar{w}_i)$ to get maximum marks.

JRAHS : Maths Ext 2 : Term 1 Task 2 : 2018

## Question 10

| | | |
|---|---|---|
| (c) (ii) $\sum_{i=1}^{n} PA_i^2 = \sum_{i=1}^{n} (z-w_i)(\bar{z}-\bar{w}_i)$ | | from (i) |
| $= \sum_{i=1}^{n} (z\bar{z} - z\bar{w}_i - \bar{z}w_i + w_i\bar{w}_i)$ | | For expanding AND |
| $= \sum_{i=1}^{n} (1 - z\bar{w}_i - \bar{z}w_i + 1)$ | 1 | For using $z\bar{z} = \|z\|^2 = 1$ |
| $= \sum_{i=1}^{n} (2 - z\bar{w}_i - \bar{z}w_i)$ | | $w_i\bar{w}_i = \|w_i\|^2 = 1$ |
| $= 2n - z\sum_{i=1}^{n} \bar{w}_i - \bar{z}\sum_{i=1}^{n} w_i$ | 1 | for recognising that $z\bar{z}$ and $w_i\bar{w}_i$ are each taken n-times (MUST BE EXPLICIT) |
| $= 2n - z(\bar{w}_1+\bar{w}_2+\cdots+\bar{w}_n) - \bar{z}(w_1+w_2+\cdots+w_n)$ | | |
| $= 2n - z(0) - \bar{z}(0)$ | 1 | For expanding $w_i$'s correctly and using the fact that $w_1+w_2+\cdots+w_n=0$ and $\overline{w_1+w_2+\cdots+w_n}=0$ or $\bar{w}_1+\bar{w}_2+\cdots+\bar{w}_n=0$ |
| $= 2n - 0 - 0$ | | |
| $= 2n$ as required | | |