Student Number:

Maths Class: ____________

James Ruse Agricultural High School

2020 **HSC Task 1**

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 1 hour and 20 minutes
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- Show relevant mathematical reasoning and/or calculations
- Answers to the Multiple Choice questions and Question 7(a) should be completed on the separate sheet and handed in with the rest of question 7.

Total Marks: 70

Section I – 5 marks

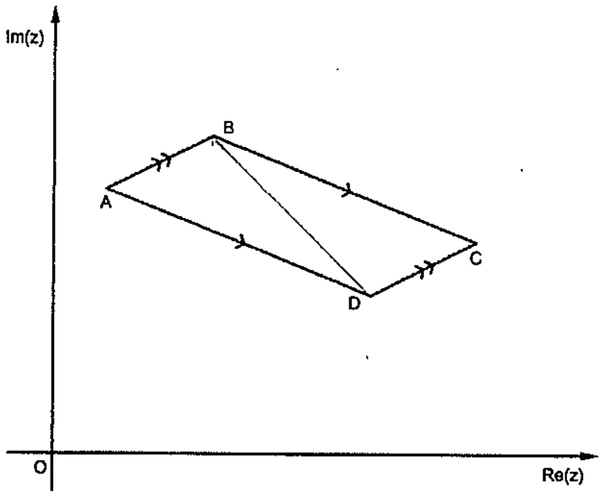
- Attempt Questions 1–5

Section II – 65 marks

- Attempt Questions 6-10

Section 1 – Multiple Choice (5 Marks)

Attempt questions 1 – 5
Record your answers on the tear-off sheet provided.



1. The above diagram shows a parallelogram *ABCD* drawn in the first quadrant of the complex plane. The points *A*, *B* and *C* represent the complex numbers z_1, z_2 *and* z_3 respectively. The vector $\overrightarrow{DB}$ represents the complex number

A) $z_1 + z_3 - 2z_2$ B) $z_2 - z_1 - z_3$

C) $2z_2 - z_1 - z_3$ D) $2z_2 - z_1 + z_3$

2. Which of the following is equal to i^i ?

A) -1 B) 1 C) $e^{-\frac{\pi}{2}}$ D) $e^{\frac{\pi}{2}}$

3. If $z_1 = -2i$ and $z_2 = 1 - i\sqrt{3}$, which one of the following statements is true?

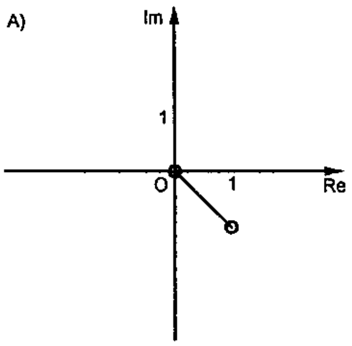
A) $\arg(z_1 + z_2) = -\frac{5\pi}{12}$ B) $|z_1 - z_2| = 2 + \sqrt{3}$

C) $\arg\left(\frac{z_1}{z_2}\right) = \frac{\pi}{6}$ D) $\arg(z_1 z_2) = -\frac{\pi}{3}$

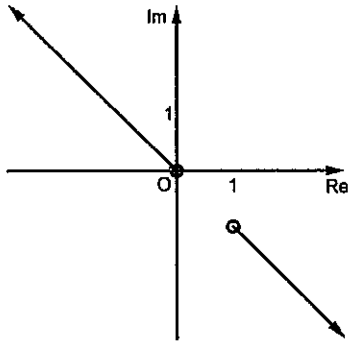
4. Which of the following diagrams best represents the solutions to the equation

$\arg(z) = \arg(z + 1 - i)$

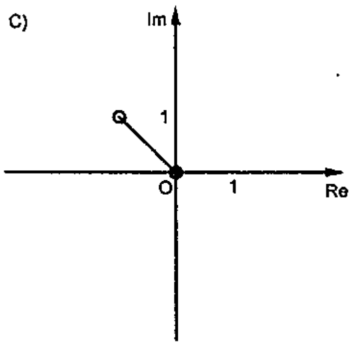
A)



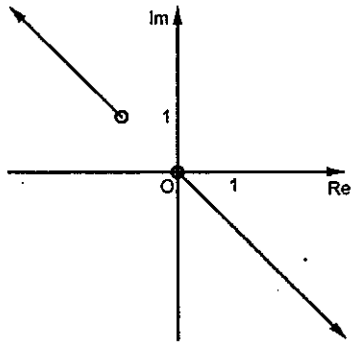
B)



C)



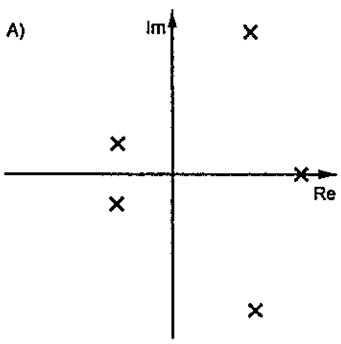
D)



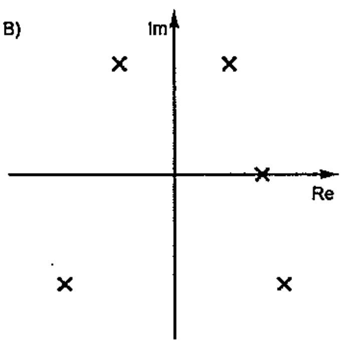
5. Which of the following Argand planes could represent the position of the roots of

$z^5 + z^2 - z + k = 0$ where k is a real number ?

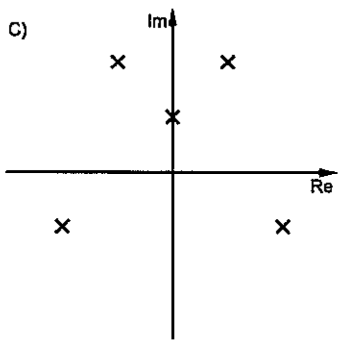
A)



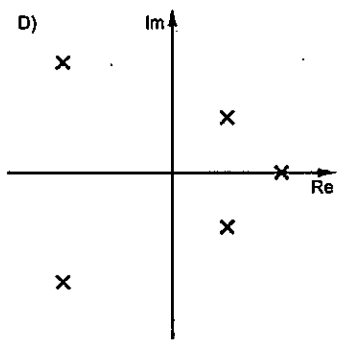
B)



C)



D)



Section 2 – (65 Marks)

Start each of questions 6 – 10 on a fresh piece of paper.
Each of these questions is worth 13 marks.

Question 6. **(13 Marks) Start on a fresh sheet of paper** **Marks**

a) Let $z = 1 + 2i$ and $w = 3 - i$.
Find, in the form $x + iy$ $(x, y$ real),

i) $z\overline{w}$ ii) $\overline{\left(\frac{10}{z}\right)}$ **2**

b) Let $\alpha = 1 - i$

i) Write α in mod-arg form and exponential form. **2**

ii) Hence show that $\alpha^4 + 4 = 0$. **1**

c) Show that the complex number $z = \frac{6-2i}{3+4i} - \frac{6}{5i}$ is real. **2**

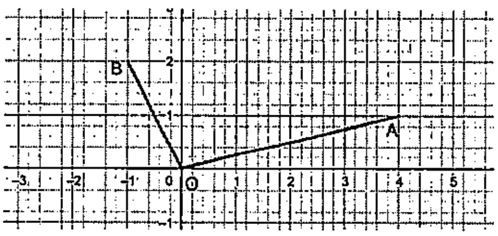
d) i) Solve the equation $z^6 + 8 = 0$, writing the roots in exponential form. **3**

ii) Illustrate these roots on a clear Argand diagram. **1**

iii) Write $z^6 + 8$ in factorised form over the field of real numbers. **2**

Question 7. **(13 Marks) Start on a fresh sheet of paper** **Marks**

a) On the Argand diagram below the points A and B represent the complex numbers α and β respectively.



On the tear-off sheet provided construct the points P, Q, R, S and T which respectively represent :

i) $\alpha + \beta$ ii) $\bar{\alpha}$ iii) $-\beta$ iv) $\beta - \alpha$ v) $2i\beta$ **5**

Ensure that the diagram sheet (including Multiple Choice answers) is attached to the other answers of question 7.

(Question 7 continues on the next page.)

Question 7 (continued) **Marks**

b) The complex number z satisfies $|z - 2 + 4i| = 3$ and w satisfies $\arg(w + 2 + 3i) = \frac{\pi}{4}$. The point P represents z on an Argand diagram and Q represents w.

i) Sketch the loci of P and Q on a clear Argand diagram. **4**

ii) If $w = x + iy$ (x, y being real numbers), write down the equation of the locus of Q in terms of x and y. **2**

iii) Hence deduce the minimum value of $|z - w|$ in surd form. **2**

Question 8. **(13 Marks) Start on a fresh sheet of paper** **Marks**

a) i) Find all real numbers x and y that satisfy $(x + iy)^2 = 20i - 21$. **3**

ii) Hence solve the equation $z^2 + 6z + 30 - 20i = 0$. **2**

b) Find all the complex numbers $z = a + ib$, where a and b are real, such that $|z|^2 + 5\bar{z} + 10i = 0$. **3**

c) You may assume for this question that, if $z = \cos\theta + i\sin\theta$ then $\left(z^n + \frac{1}{z^n}\right) = 2\cos n\theta$ for positive integers n. Let $w = z + \frac{1}{z}$.

i) Prove that $w^3 + w^2 - 2w - 2 = \left(z + \frac{1}{z}\right) + \left(z^2 + \frac{1}{z^2}\right) + \left(z^3 + \frac{1}{z^3}\right)$. **2**

ii) Hence, or otherwise, find all solutions of $\cos\theta + \cos 2\theta + \cos 3\theta = 0$ in the range $0 \leq \theta \leq 2\pi$. **3**

Question 9. **(13 Marks) Start on a fresh sheet of paper** **Marks**

a) i) If $z = \cos\theta + i\sin\theta$, use de Moivre's Theorem to prove that

$$\left(z^n - \frac{1}{z^n}\right) = 2i \sin n\theta \ . \qquad 2$$

ii) Hence, or otherwise, prove that $16\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta$ 3

b) On an Argand diagram shade and label the region specified by 3

$$\{z : \operatorname{Im}(z) \le 4\} \cap \{z : |z - 4 - 5i| \le 3\}$$

c) The points P and Q represent the complex numbers z and w respectively on an Argand diagram where $w = \frac{3z+2}{z-1}$. 3
By writing an expression for z in terms of w, or otherwise, determine and sketch the locus of Q as P traces out the circle $|z| = 1$.

d) If $\operatorname{Arg}\left(\frac{z_1+z_2}{z_1-z_2}\right) = \frac{\pi}{2}$ show that $|z_1| = |z_2|$. 2

Question 10. **(13 Marks) Start on a fresh sheet of paper** **Marks**

a) i) Show that, if $w = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$, then the equation $z^7 - 1 = 0$ has distinct roots $1, w, w^2, w^3, w^4, w^5$ and w^6. 2

ii) Explain why the equation $z^6 + z^5 + z^4 + z^3 + z^2 + z + 1 = 0$ has roots w, w^2, w^3, w^4, w^5 and w^6. 1

iii) Find the value of $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7}$. 2

iv) Find the monic quadratic equation with numerical coefficients whose roots are $w + w^2 + w^4$ and $w^3 + w^5 + w^6$. 2

b) The number c is real and non-zero. It is also known that $(1 + ic)^5$ is real.

i) Expand $(1 + ic)^5$ and deduce that $c^4 - 10c^2 + 5 = 0$. 2

ii) Hence show that $c = \sqrt{5 - 2\sqrt{5}}$, $-\sqrt{5 - 2\sqrt{5}}$, $\sqrt{5 + 2\sqrt{5}}$ or $-\sqrt{5 + 2\sqrt{5}}$. 1

iii) Let $1 + ic = re^{i\theta}$. Show that the smallest possible positive value of θ is $\frac{\pi}{5}$. 1

iv) Hence determine the exact value of $\tan\frac{\pi}{5}$, justifying your answer. 2

2020 Y-11/12 4r Q6

Q6

a) $z = 1+2i \quad w = 3-i$

i) $z\bar{w} = (1+2i)(3+i)$

$= 3+7i-2$

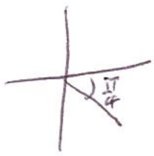
$= 1+7i$ 1m

ii) $\overline{\left(\frac{10}{z}\right)} = ?$ $\quad \frac{10}{1+2i} \times \frac{1-2i}{1-2i} = \frac{10-20i}{5}$

$\overline{\left(\frac{10}{z}\right)} = \frac{10+20i}{5} = 2+4i$ 1m

b) i) $\alpha = 1-i = \sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right]$ # 1m

$= \sqrt{2}\, e^{-\frac{\pi}{4}i}$ 1m



ii) $\alpha^4 = \left(\sqrt{2}\ \mathrm{cis}\, -\frac{\pi}{4}\right)^4 = (\sqrt{2})^4\left(\mathrm{cis}\, -\frac{\pi}{4} \times 4\right)$ (D. Moivre's Th)

$= 4 \times \mathrm{cis}(-\pi)$ 1m

$= 4 \times -1 = -4$

$\therefore \alpha^4 + 4 = -4 + 4 = 0$ 1m

c) $z = \frac{6-2i}{3+4i} - \frac{6}{5i}$

$\frac{6-2i}{(3+4i)} \frac{(3-4i)}{(3-4i)} = \frac{18-8-6i-24i}{9+16} = \frac{10-30i}{25} = \frac{2-6i}{5}$ 1m

$\frac{6}{5i} \times \frac{5i}{5i} = \frac{30i}{-25} = \frac{-6i}{5}$

$\therefore z = \frac{2-6i}{5} - \frac{-6i}{5} = \frac{2}{5} \in R$ 1m

1

6di) $z^6 = -8$

$= 2^3 e^{(\pi + 2k\pi)i}, \ k \in \mathbb{Z}$

$z = (\sqrt{2})\, e^{\left(\frac{\pi + 2k\pi}{6}\right)i}$

1m for $\sqrt{2}$

1m for $(\pi + 2k\pi)/6$

1m for final answers

Roots are $\therefore z = \sqrt{2}e^{-i(\frac{\pi}{6})}, \ \sqrt{2}e^{i\frac{\pi}{6}}, \ \sqrt{2}e^{-i(\frac{\pi}{2})}, \ \sqrt{2}e^{i\frac{\pi}{2}}, \ \sqrt{2}e^{-i\frac{5\pi}{6}}, \ \sqrt{2}e^{i\frac{5\pi}{6}}$

Alternatively $z = r\,\text{cis}\,\theta$

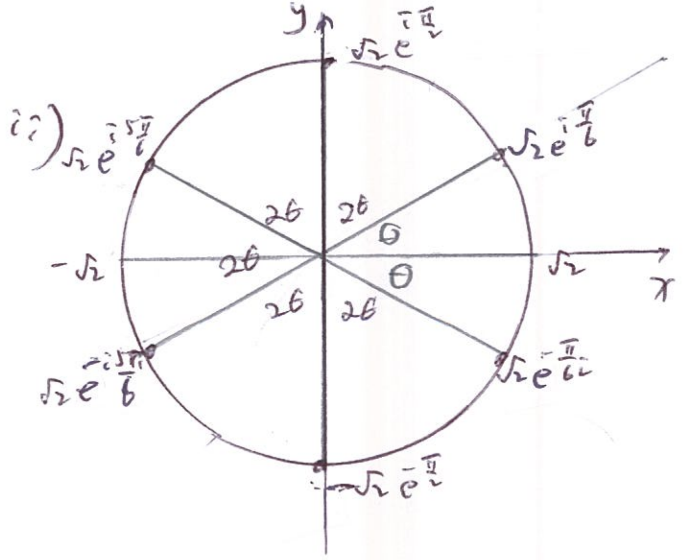
$z^6 = r^6 \,\text{cis}\, 6\theta$ (D. M. Th)

$r^6 \cdot \text{cis}(6\theta) = 2^3 \,\text{cis}(\pi + 2k\pi)$

$r = \sqrt{2}$, $\theta = \dfrac{\pi + 2k\pi}{6}$

$\therefore$ Roots are $\sqrt{2}e^{i\frac{\pi}{6}}, \ \sqrt{2}e^{-\frac{\pi}{6}i}, \ \sqrt{2}e^{\frac{i\pi}{2}}, \ \sqrt{2}e^{-\frac{i\pi}{2}}, \ \sqrt{2}e^{i\frac{5\pi}{6}}, \ \sqrt{2}e^{-\frac{5\pi}{6}i}$

ii)



$\theta = \dfrac{\pi}{6}$

must show $r = \sqrt{2}$ and $\theta = \dfrac{\pi}{6}$ 1m

- Angles need to be evenly distributed around circle

iii) $z^6 + 8 = (z^2)^3 + 2^3 = (z^2 + 2)(z^4 - 2z^2 + 4)$ 1m

$= (z^2 + 2)[(z^4 + 4z^2 + 4) - 6z^2)]$

$= (z^2 + 2)(z^2 + 2 - \sqrt{6}z)(z^2 + 2 + \sqrt{6}z)$ 1m

Alternatively

$z^6 + 8 = (z - \sqrt{2}e^{i\frac{\pi}{2}})(z - \sqrt{2}e^{-i\frac{\pi}{2}})[(z - \sqrt{2}e^{+\frac{\pi}{6}i})(z - \sqrt{2}e^{-\frac{\pi}{6}i})][(z - \sqrt{2}e^{i\frac{5\pi}{6}})(z - \sqrt{2}e^{-i\frac{5\pi}{6}})]$ 1m

$z^6 + 8 = (z^2 - \sqrt{2}i)(z^2 + \sqrt{2}i)(z^2 - 2\sqrt{2}\cos\frac{\pi}{6}z + 2)(z^2 - 2\sqrt{2}\cos\frac{5\pi}{6}z + 2)$ since $(z - z_1)(z - \bar{z}_1) = z^2 - 2(\text{Re}\, z_1)z + |z_1|^2$

$= (z^2 + 2)(z^2 - 2\sqrt{2}\frac{\sqrt{3}}{2}z + 2)(z^2 - 2\sqrt{2}\frac{-\sqrt{3}}{2}z + 2)$

$= (z^2 + 2)(z^2 - \sqrt{6}z + 2)(z^2 + \sqrt{6}z + 2)$ 1m

many students forget z with $\sqrt{6}$, $|z_1|^2 \neq 1$

Multiple Choice solutions

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1.

$\overrightarrow{DB}$

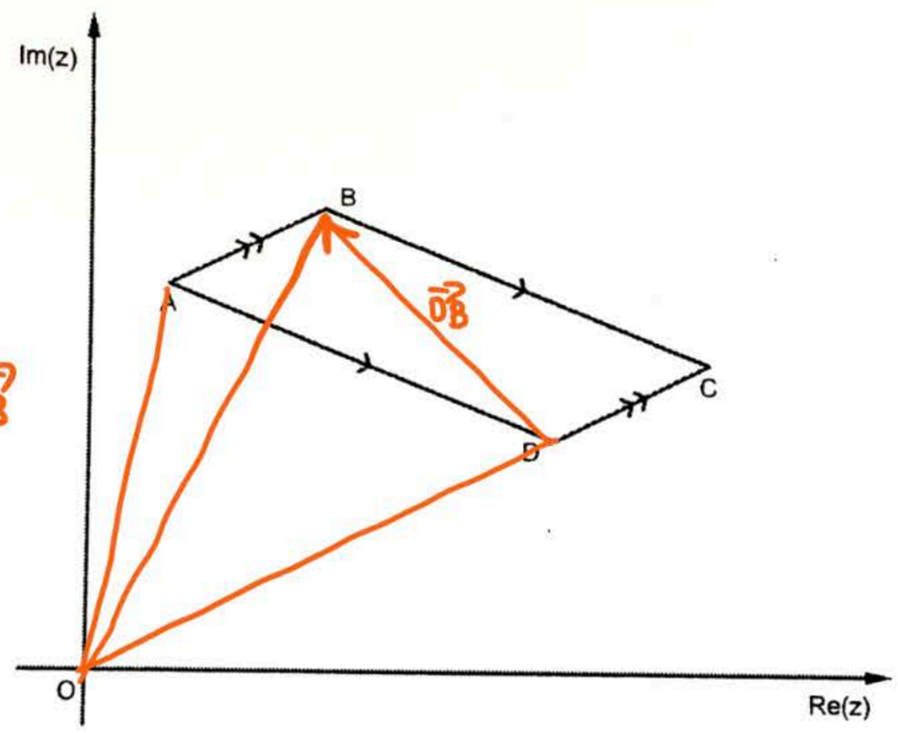
$\overrightarrow{DA} + \overrightarrow{AB}$

$= \overrightarrow{CB} + \overrightarrow{AB}$

$= \overrightarrow{CO} + \overrightarrow{OB} + \overrightarrow{AO} + \overrightarrow{OB}$

$= -z_3 + 2z_2 - z_1$

$= 2z_2 - z_1 - z_3$



1. The above diagram shows a parallelogram $ABCD$ drawn in the first quadrant of the complex plane. The points A, B and C represent the complex numbers z_1, z_2 *and* z_3 respectively. The vector $\overrightarrow{DB}$ represents the complex number

A) $z_1 + z_3 - 2z_2$ B) $z_2 - z_1 - z_3$

C) $2z_2 - z_1 - z_3$ D) $2z_2 - z_1 + z_3$

2. $i^i = \left(e^{i\pi/2}\right)^i$

$= e^{-\pi/2}$ C

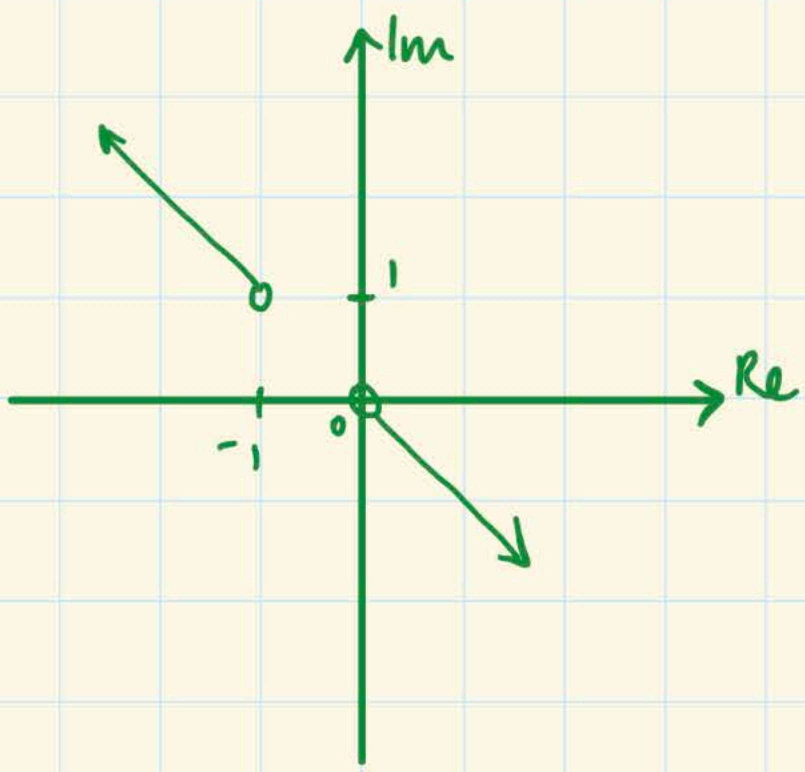
3. $z_1 + z_2 = 1 - (\sqrt{3}+2)i$

$\arg(z_1+z_2) = \tan^{-1} -(\sqrt{3}+2)$

$= -1.31$ calculator

$= -5\pi/2$ A

4.



$\arg z - \arg(z+1-i) = 0$

D

5. Real coefficients

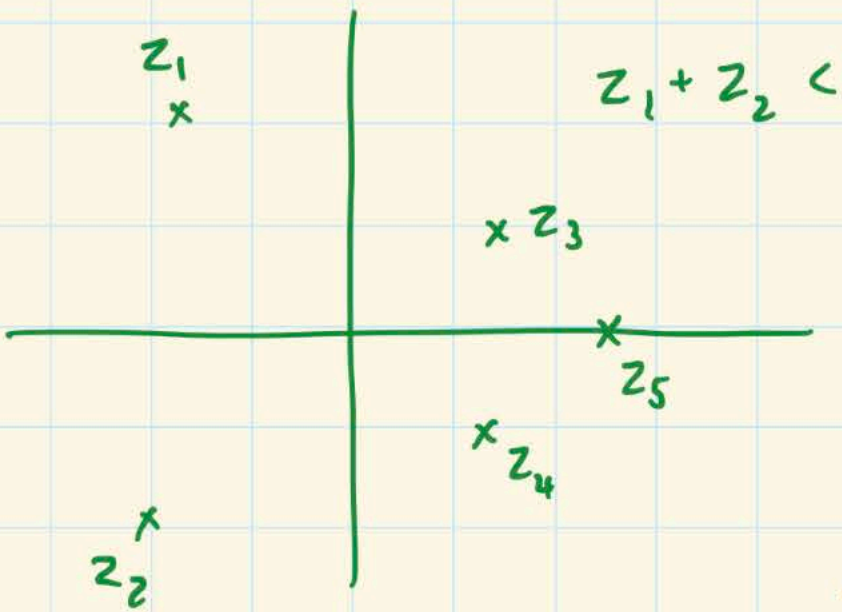
$\therefore$ Complex roots occur in conjugate pairs

$\therefore$ A or D

For z & $\bar{z}$, $z + \bar{z} = Re(z)$

$\therefore$ Sum of roots is real

& sum of roots $= 0$



$z_1 + z_2 < 0$, $z_3 + z_4 + z_5 > 0$

$\therefore z_1 + z_2 = -(z_3 + z_4 + z_5)$

only possible when $Re(z_3)$ is small.

$\therefore$ D

Student number__________________ Class______________

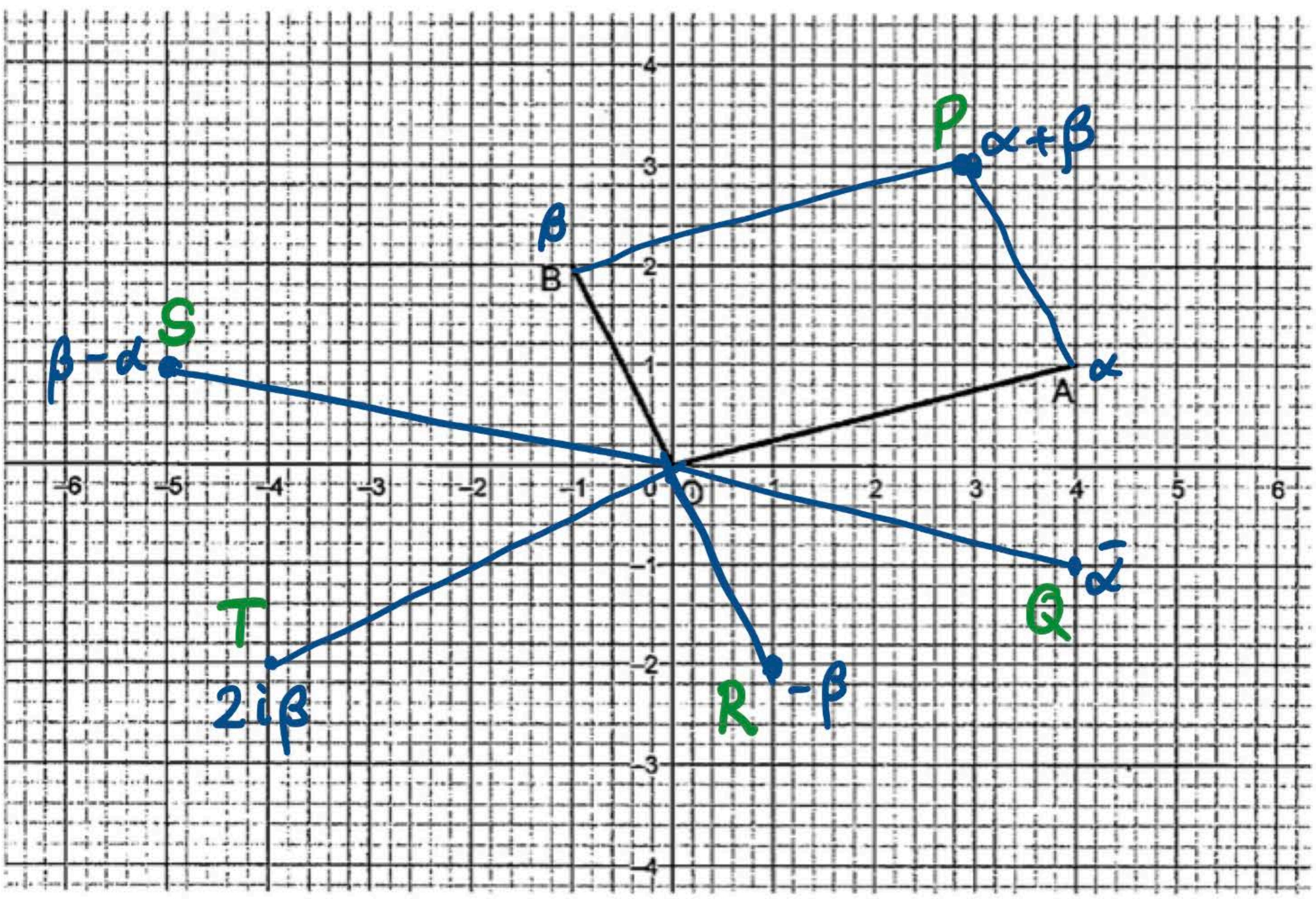
Multiple Choice Answers

Circle the letter of the most correct answer.

1.	A	B	C	D
~~2.~~	~~A~~	~~B~~	~~C~~	~~D~~
3.	A	B	C	D
4.	A	B	C	D
5.	A	B	C	D

MC Mark: /5

Question 7a) Your answers to question 7(a) should be drawn and labelled on this diagram.



This sheet should be submitted as the top sheet of question 7.

Question 7b)

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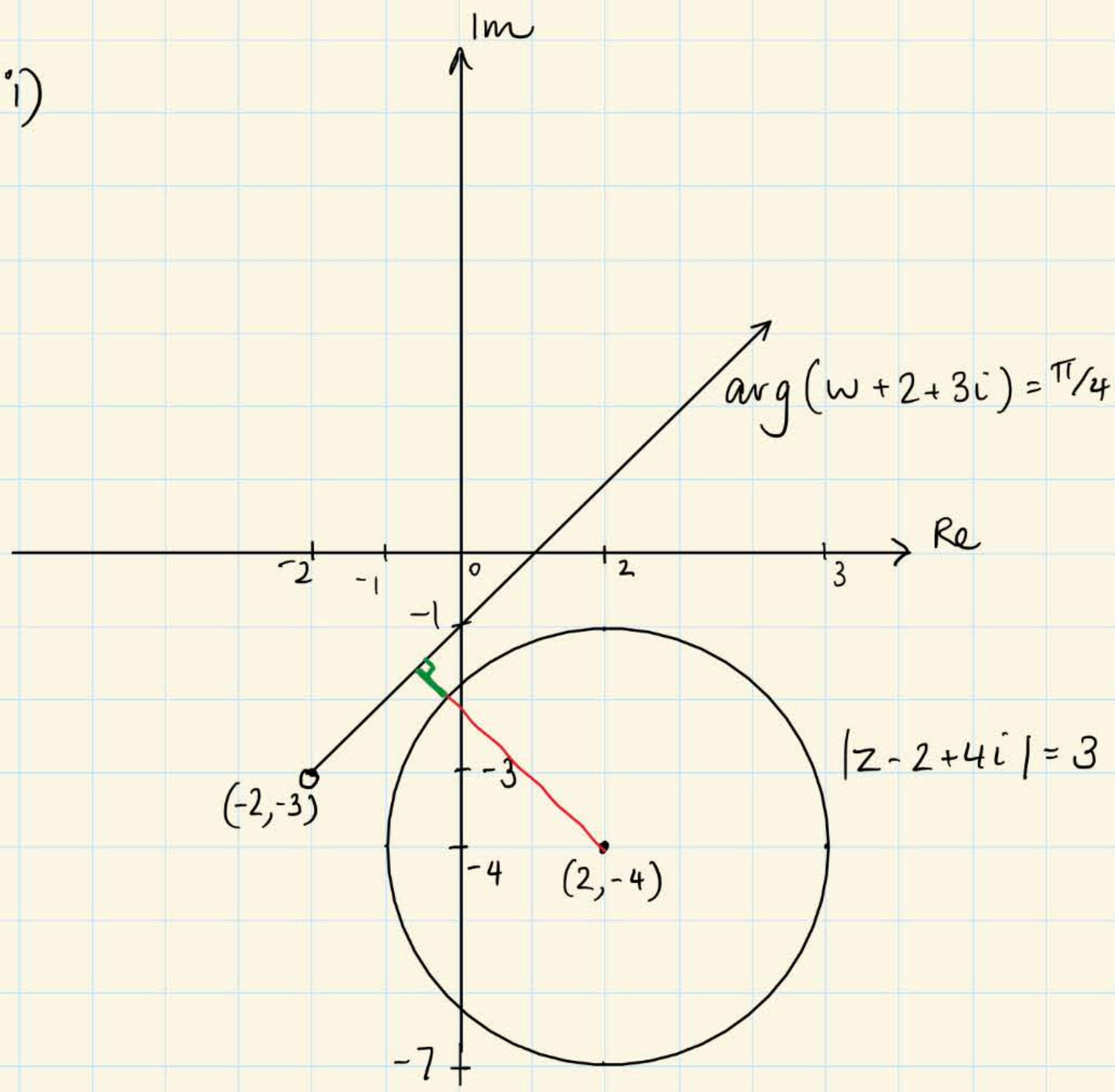
i)



$|z-2+4i|=3$

circle, centre $(2,-4)$

radius 3

- 1m circle centre $(2,-4)$
- 1m radius 3 shown

$\arg(w+2+3i)=\frac{\pi}{4}$

- 1m open circle $(-2,-3)$ straight line
- 1m indicate $\pi/4$
  - pass thru $(0,-1)$
  - $\pi/4$ labelled.

(ii) $m=1$, $b=-1$

$y=x-1 \iff x-y-1=0$

- 1m gradient of 1
- 1m equation

(iii) minimum distance, perpendicular distance from line to circle

centre to line $= \dfrac{|1\times2-1\times-4-1|}{\sqrt{1^2+1^2}}$

$= \dfrac{5}{\sqrt2}$

- 1m perp distance Centre $(2,-4)$ to line $x-y-1=0$

minimum value of $|z-w|$ $= \dfrac{5}{\sqrt2}-3$

$= \dfrac{5\sqrt2-6}{2}$

- 1m min $|z-w|$ $\dfrac{5\sqrt2-6}{2}$ or $\dfrac{5}{\sqrt2}-3$.

or $\boxed{\sqrt{\cancel{-}}3}$

OR Tangent parallel to $x-y-1=0$
is of the form $x-y+k=0$
distance from tangent to centre is 3.
$(2,-4)$, $x-y+k=0$
$$\therefore \frac{|2-1\times-4+k|}{\sqrt{2}} = 3$$
$$|6+k| = 3\sqrt{2}$$
$$k = -6+3\sqrt{2} \quad \text{or} \quad k=-6-3\sqrt{2}$$
too big

$\therefore k=-6+3\sqrt{2}$
$\therefore$ Tangent is $x-y-6+3\sqrt{2}=0$ • 1m tangent
$(0,-1)$ is a point on $x-y-1=0$
perpendicular distance from
$(0,-1)$ to $x-y-6+3\sqrt{2}=0$
$$d = \frac{|1\times0-1\times-1-6+3\sqrt{2}|}{\sqrt{2}}$$
$$= \frac{|-5+3\sqrt{2}|}{\sqrt{2}}$$
$$= \frac{5-3\sqrt{2}}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$
• 1m perp distance
$$= \frac{5\sqrt{2}-6}{2}$$

MATHEMATICS Extension 2: Question 8		
Suggested Solutions	**Marks**	**Marker's Comments**
a) i) $(x+iy)^2 = 20i - 21$ $x^2 - y^2 + 2xyi = 20i - 21$ equating real and imaginary parts $x^2 - y^2 = -21 \quad -①$ $2xy = 20$ $xy = 10$ $y = \frac{10}{x} \quad -②$ sub. ② into ① $x^2 - \left(\frac{10}{x}\right)^2 = -21$ $x^4 + 21x^2 - 100 = 0$ $(x^2-4)(x^2+25) = 0$ $x^2 = 4 \;,\; x \in \mathbb{R}$ $\therefore x = 2 \;,\; y = 5$ $x = -2 \;,\; y = -5$ $\therefore x + iy = \pm(2+5i)$	1 1 1	getting x or y and eliminating the imaginary values matching the correct x and y values together.
OR $\quad x^2 - y^2 = -21 \quad -①$ $2xy = 20$ $(x^2+y^2)^2 = (x^2-y^2)^2 + (2xy)^2$ $= (-21)^2 + (20)^2$ $= 841$ $\therefore x^2 + y^2 = 29 \quad -②$ ① + ② : $2x^2 = 8$ $x^2 = 4$ $x = \pm 2$ $\therefore x = 2,\; y = 5$ $x = -2,\; y = -5$	1 1 1	getting x or y correct and having $x^2+y^2=29$
ii) $z = \dfrac{-6 \pm \sqrt{36 - 4(30-20i)}}{2}$ $= \dfrac{-6 \pm \sqrt{80i - 84}}{2}$ $= -3 \pm \sqrt{20i - 21}$	1	many students had transcription error and wrote the equation as: $z^2 - 6z + 30 - 20i = 0$ no marks were deducted.

MATHEMATICS Extension 2: Question 8

Suggested Solutions	Marks	Marker's Comments
$= -3 + (2+5i)$ or $-3-(2+5i)$		
$= -1+5i$ or $-5-5i$	1	
b) $\|z\|^2 + 5\bar{z} + 10i = 0$		
$a^2+b^2+5(a-ib)+10i=0$		
$a^2+b^2+5a-5bi+10i=0$	1	
equating real and imaginary parts		
$a^2+b^2+5a=0$ -①		
$-5b+10=0$		
$5b=10$		
$b=2$	1	
sub. $b=2$ into ①		
$a^2+4+5a=0$		
$(a+4)(a+1)=0$		
$\therefore a=-1,-4$	1	must write as a complex number
$\therefore z=-1+2i$ or $-4+2i$		
c) i) LHS $= w^3+w^2-2w-2$		
$=(z+\frac{1}{z})^3+(z+\frac{1}{z})^2-2(z+\frac{1}{z})-2$	1	for substituting correctly
$= z^3+3z+3(\frac{1}{z})+\frac{1}{z^3}+z^2+2+\frac{1}{z^2}$	1	for correct expansion and simplifying
$-2z-2(\frac{1}{z})-2$		
$= z^3+\frac{1}{z^3}+z^2+\frac{1}{z^2}+z+\frac{1}{z}$		
$=(z+\frac{1}{z})+(z^2+\frac{1}{z^2})+(z^3+\frac{1}{z^3})$		
$=$ RHS		
ii) $\cos\theta+\cos 2\theta+\cos 3\theta=0$ multiply by 2		
$2\cos\theta+2\cos 2\theta+2\cos 3\theta=0$	1	explaining that the solution for $\cos\theta+\cos 2\theta+\cos 3\theta=0$ is the same as the solutions for $w^3+w^2-2w-2=0$
$z+\frac{1}{z}+z^2+\frac{1}{z^2}+z^3+\frac{1}{z^3}=0$		
$w^3+w^2-2w-2=0$		
$(w+1)(w^2-2)=0$		
$\therefore w=-1, \pm\sqrt{2}$	1	
$\therefore 2\cos\theta=-1$, $2\cos\theta=\pm\sqrt{2}$		
$\cos\theta=-\frac{1}{2}$ $\cos\theta=\pm\frac{\sqrt{2}}{2}$		

$\theta=\frac{2\pi}{3},\frac{4\pi}{3}$ $\theta=\frac{\pi}{4},\frac{7\pi}{4},\frac{3\pi}{4},\frac{5\pi}{4}$ 1

MATHEMATICS Extension 2: Question 8		
Suggested Solutions	**Marks**	**Marker's Comments**
OR $\cos\theta + \cos 2\theta + \cos 3\theta = 0$		
$\cos\theta + 2\cos^2\theta - 1 + \cos\theta\cos 2\theta - \sin\theta\sin 2\theta = 0$		
$\cos\theta + 2\cos^2\theta - 1 + \cos\theta(2\cos^2\theta - 1)$		
$- 2\sin^2\theta\cos\theta = 0$		
$\cos\theta + 2\cos^2\theta - 1 + 2\cos^3\theta - \cos\theta$		
$- 2(1-\cos^2\theta)\cos\theta = 0$		
$2\cos^2\theta - 1 + 2\cos^3\theta - 2\cos\theta + 2\cos^3\theta = 0$		
$4\cos^3\theta + 2\cos^2\theta - 2\cos\theta - 1 = 0$	1	
$2\cos^2\theta(2\cos\theta + 1) - (2\cos\theta + 1) = 0$		
$(2\cos^2\theta - 1)(2\cos\theta + 1) = 0$		
$\therefore \cos\theta = \pm\frac{1}{\sqrt{2}} \qquad \cos\theta = -\frac{1}{2}$	1	
$\theta = \frac{\pi}{4}, \frac{7\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4} \qquad \theta = \frac{2\pi}{3}, \frac{4\pi}{3}$	1	

MATHEMATICS Extension 2: Question 9

Suggested Solutions	Marks	Marker's Comments
a) i let $Z = \cos\theta + i\sin\theta$		
$Z^n = \cos n\theta + i\sin n\theta$ (De Moivre's)	1	Use of De Moivre's
$\frac{1}{Z^n} = Z^{-n} = \cos(-n\theta) + i\sin(-n\theta)$ (De Moivre's)		
$= \cos n\theta - i\sin n\theta$		
$\therefore Z^n - \frac{1}{Z^n} = (\cos n\theta + i\sin n\theta) - (\cos n\theta - i\sin n\theta)$		
$= 2i\sin n\theta$	1	
ii (Hence)		
$2i\sin\theta = Z - \frac{1}{Z}$		Using identity from i)
$(2i)^5 \sin^5\theta = \left(Z - \frac{1}{Z}\right)^5$	1	
$= Z^5 - 5Z^3 + 10Z - 10Z^{-1} + 5Z^{-3} - Z^{-5}$		
$= (Z^5 - Z^{-5}) - 5(Z^3 - Z^{-3}) + 10(Z - Z^{-1})$	1	
$= (2i\sin 5\theta) + 5(2i\sin 3\theta) + 10(2i\sin\theta)$ From i)		
$32i\sin^5\theta = 2i\sin 5\theta + 10i\sin 3\theta + 20i\sin\theta$		
$16\sin^5\theta = \sin 5\theta + 5\sin 3\theta + 10\sin\theta$	1	

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ii (Otherwise)

$(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$ (De Moivre's Theorem)

$(\cos\theta + i\sin\theta) = \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10i\cos^2\theta\sin^3\theta$
$+ 5\cos\theta\sin^4\theta + i\sin^5\theta$

Equating imaginary parts

$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$ — 1 Expression for $\sin 5\theta$

$= 5(1-\sin^2\theta)^2\sin\theta - 10(1-\sin^2\theta)\sin^3\theta + \sin^5\theta$

$= 5\sin\theta - 10\sin^3\theta + 5\sin^5\theta - 10\sin^3\theta + 10\sin^5\theta + \sin^5\theta$

$= 5\sin\theta - 20\sin^3\theta + 16\sin^5\theta$

$(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$ (De Moivre's Theorem)

$(\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$

Equating Imaginary parts:

$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$

$= 3(1-\sin^2\theta)\sin\theta - \sin^3\theta$

$= 3\sin\theta - 3\sin^3\theta - \sin^3\theta$

$= 3\sin\theta - 4\sin^3\theta$ — 1 Expression for $\sin 3\theta$

Sub $\sin 3\theta$ and $\sin 5\theta$ into RHS:

$RHS = (5\sin\theta - 20\sin^3\theta + 16\sin^5\theta) - 5(3\sin\theta - 4\sin^3\theta) + 10\sin\theta$

$= 5\sin\theta - 20\sin^3\theta + 20\sin^5\theta - 15\sin\theta + 20\sin^3\theta + 10\sin\theta$

$= 16\sin^5\theta$ — 1

$= LHS.$

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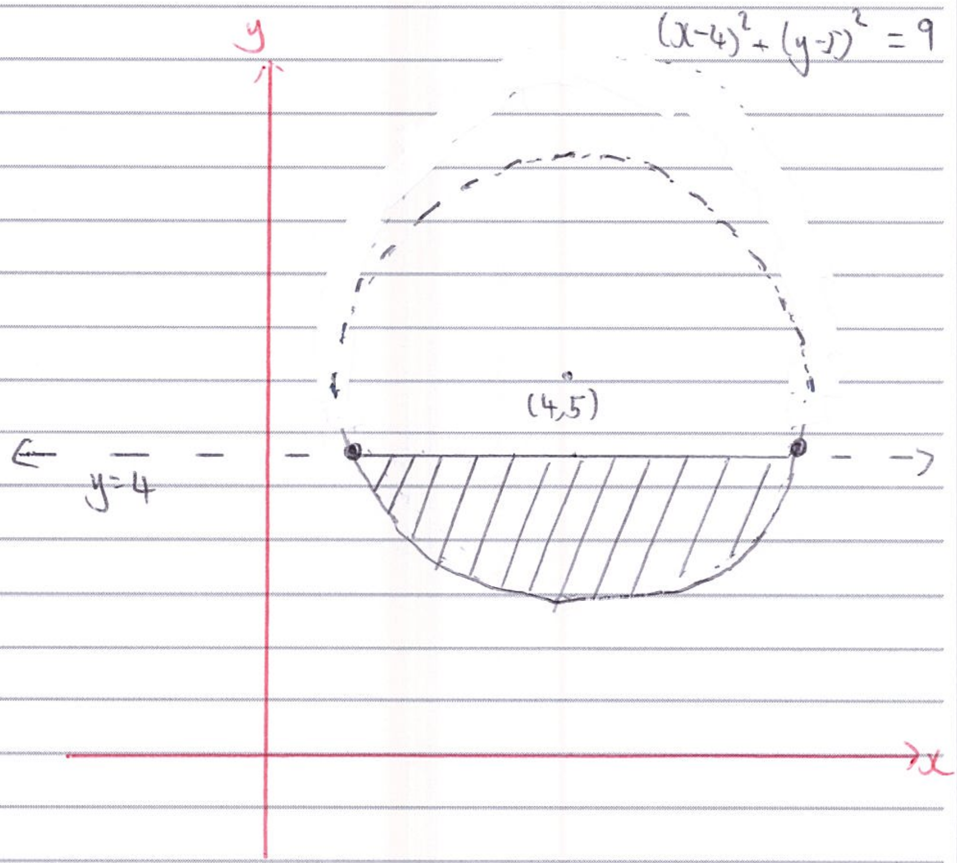
b) let $z = x + iy$		
$Im(z) \le 4 \Rightarrow y \le 4$		
$\lvert z - 4 - 5i \rvert \le 3$		
$\downarrow$		
$\lvert (x-4) + (y-5)i \rvert \le 3$		
$\sqrt{(x-4)^2 + (y-5)^2} \le 3$		
$(x-4)^2 + (y-5)^2 \le 9$		
		

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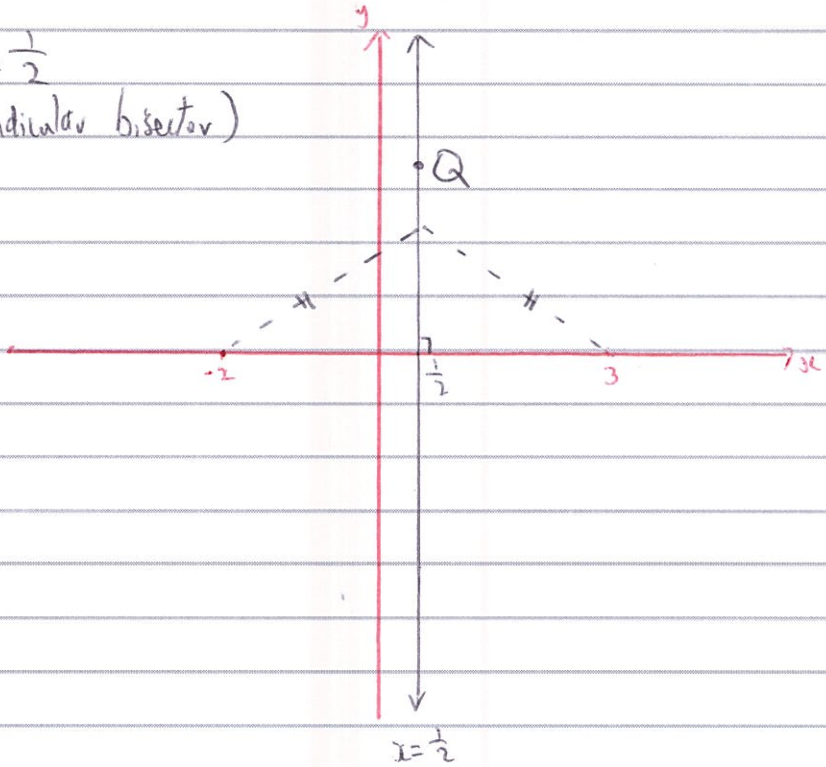
c) $w = \dfrac{3z+2}{z-1}$		
$wz - w = 3z + 2$		
$wz - 3z = w + 2$		
$z(w-3) = w+2$		
$z = \dfrac{w+2}{w-3}$		
$\lvert z\rvert = 1 \Rightarrow \left\lvert \dfrac{w+2}{w-3} \right\rvert = 1$		
$\dfrac{\lvert w+2\rvert}{\lvert w-3\rvert} = 1$		
$\lvert w+2\rvert = \lvert w-3\rvert$		
$\therefore x = \frac{1}{2}$ (perpendicular bisector) 		

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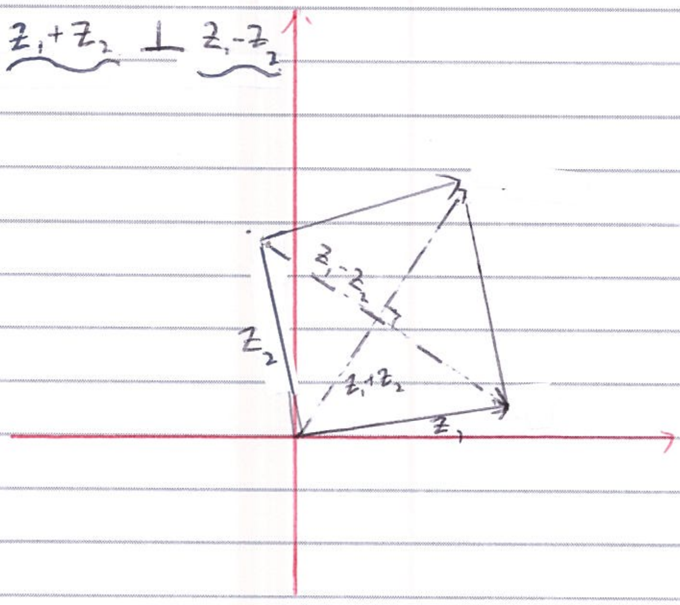
d) $\text{Arg}\left(\dfrac{z_1+z_2}{z_1-z_2}\right) = \dfrac{\pi}{2}$

$\text{Arg}(z_1+z_2) - \text{Arg}(z_1-z_2) = \dfrac{\pi}{2}$

$\text{Arg}(z_1+z_2) = \text{Arg}(z_1-z_2) + \dfrac{\pi}{2}$ ——— 1

$\therefore z_1+z_2 \perp z_1-z_2$



Since z_1+z_2 and z_1-z_2 are diagonals of the parallelogram formed by the origin, and points representing z_1, z_2, and z_1+z_2 respectively, if $z_1+z_2 \perp z_1-z_2$, then the parallelogram is a rhombus (diagonals are perpendicular).

$\therefore |z_1| = |z_2|$, (adjacent sides of a rhombus).

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d) Alternate Method.		
$\mathrm{Arg}\left(\dfrac{z_1+z_2}{z_1-z_2}\right) = \dfrac{\pi}{2}$		
$\downarrow$		
$\dfrac{z_1+z_2}{z_1-z_2} = ki \quad (k \in Z)$		
$z_1+z_2 = ki(z_1-z_2)$		
$z_1+z_2 = kiz_1 - kiz_2$		
$z_2 + kiz_2 = kiz_1 - z_1$	1	
$z_2(1+ki) = z_1(-1+ki)$		
$\lvert z_2(1+ki)\rvert = \lvert z_1(-1+ki)\rvert$		
$\lvert z_2\rvert\,\lvert 1+ki\rvert = \lvert z_1\rvert\,\lvert -1+ki\rvert$		
Since $\lvert 1+ki\rvert = \lvert -1+ki\rvert = \sqrt{1+k^2}$		
$\therefore \lvert z_1\rvert = \lvert z_2\rvert$	1	

MATHEMATICS Extension 2: Question 10

Suggested Solutions	Marks	Marker's Comments

10. (a)

(i) By subs $z=1$ is a solution

Sub $z = \cos\frac{2n\pi}{7} + i\sin\frac{2n\pi}{7}$, $n\in\mathbb{Z}$ into equation

$LHS = \left(\operatorname{cis}\frac{2n\pi}{7}\right)^7 - 1$

$= \operatorname{cis} 2n\pi - 1$ (de Moivre)

$= \cos 2n\pi + i\sin 2n\pi - 1$

$= 1 - 1$ — 1

$= 0$

$= RHS$

$\therefore\ \omega^n$ is a solution for integer n

$1, \omega, \omega^2, \ldots, \omega^6$ have distinct arguments, so the 7 roots are distinct, — 1

Other values of n will be duplicates of one of these.

(ii) $z^7 - 1 = 0 \therefore (z-1)(1+z+z^2+z^3+z^4+z^5+z^6) = 0$

$\therefore z = 1$ or $1+z+z^2+z^3+z^4+z^5+z^6 = 0$

Since the roots of $z^7-1=0$ are $1, \omega, \omega^2, \omega^3, \omega^4, \omega^5$ and ω^6

$\therefore$ the roots of $1+z+z^2+z^3+z^4+z^5+z^6 = 0$ — 1

are $\omega, \omega^2, \omega^3, \omega^4, \omega^5$ and ω^6.

(iii) $1+\omega+\omega^2+\omega^3+\omega^4+\omega^5+\omega^6 = 0$

$\omega^3\left(\frac{1}{\omega^3}+\frac{\omega}{\omega^3}+\frac{\omega^2}{\omega^3}+\frac{\omega^3}{\omega^3}+\frac{\omega^4}{\omega^3}+\frac{\omega^5}{\omega^3}+\frac{\omega^6}{\omega^3}\right) = 0,\ \omega\neq 0$

$\therefore\ \omega^3 + \frac{1}{\omega^3} + \omega^2 + \frac{1}{\omega^2} + \omega + \frac{1}{\omega} + 1 = 0$ — 1

$\omega^n + \omega^{-n} = 2\cos n\theta$

$\therefore\ 2\cos 3\theta + 2\cos 2\theta + 2\cos\theta + 1 = 0$

$2\cos\frac{6\pi}{7} + 2\cos\frac{4\pi}{7} + 2\cos\frac{2\pi}{7} = -1$

$\therefore\ \cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} = -\frac{1}{2}$ — 1

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(iv) let $\alpha = \omega + \omega^2 + \omega^4$ and $\beta = \omega^3 + \omega^5 + \omega^6$		
$\alpha + \beta = \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 = -1$		
$\alpha\beta = (\omega + \omega^2 + \omega^4)(\omega^3 + \omega^5 + \omega^6)$		
$= \omega^4(1 + \omega + \omega^3)(1 + \omega^2 + \omega^3)$		
$= \omega^4(1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 + 2\omega^3)$	1	
$= \omega^4(0 + 2\omega^3) = 2\omega^7 = 2$		
The monic quadratic equation is		
$z^2 + z + 2 = 0$.	1	
(b)		
(i) $(1+ic)^5 = 1 + 5(ic) + 10(ic)^2 + 10(ic)^3 + 5(ic)^4 + (ic)^5$		
$= 1 + 5ic - 10c^2 - 10ic^3 + 5c^4 + ic^5$		
$= 1 - 10c^2 + 5c^4 + i(5c - 10c^3 + c^5)$	1	
$(1+ic)^5$ is real $\therefore 5c - 10c^3 + c^5 = 0$		
since $c \neq 0$ $\therefore c^4 - 10c^2 + 5 = 0$	1	
(ii) $c^4 - 10c^2 + 5 = 0$		
$c^4 - 10c^2 + 25 = 20$		
$(c^2 - 5)^2 = 20$		
$c^2 - 5 = \pm 2\sqrt{5}$		
$c^2 = 5 \pm 2\sqrt{5}$		
$c = \sqrt{5 - 2\sqrt{5}}, \sqrt{5 + 2\sqrt{5}}, -\sqrt{5 - 2\sqrt{5}}$, or $-\sqrt{5 + 2\sqrt{5}}$	1	
(iii) $1 + ic = re^{i\theta}$ $\therefore (1+ic)^5 = r^5 e^{i5\theta}$ is real		
$\therefore 5\theta = 0, \pi, \ldots$		
The smallest positive value is π $\therefore \theta = \frac{\pi}{5}$	1	
(iv) $1 + ic = re^{i\pi/5} = r(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5})$		
$r\cos\pi/5 = 1$ and $r\sin\frac{\pi}{5} = c$ $\therefore \tan\frac{\pi}{5} = c$	1	
$0 \le \frac{\pi}{5} \le \pi/2$ in the first quadrant		
$\tan\pi/5 > 0$ $\therefore \tan\pi/5 = \sqrt{5 - 2\sqrt{5}}$.	1	