Student Name: ______________________

Maths class: ______________________

James Ruse Agricultural High School

2020 **YEAR 12 Task 3**

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 1 hour and 30 minutes
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided

Total Marks: 60

Attempt all questions (1 – 5). Each question is worth 12 marks.

Start your answer to each question on a separate piece of paper.

Question 1 (12 marks)

Start a new page.

a) Find $\int \frac{1}{x^2+2x+3} dx$ **2**

b) Find $\int \frac{sin^3x}{cos^2x} dx$ **2**

c) Evaluate $\int \frac{1+x}{9-x^2} dx$ by first decomposing into partial fractions. **4**

d) Find the error(s) in the following false induction proof, and state why it is an error. **1**

Claim: For all $n \in \mathbb{N}$, $(*) \sum_{i=1}^{n} i = \frac{1}{2}\left(n+\frac{1}{2}\right)^2$

Proof:

When $n = 1$, $(*)$ holds.

Let $k \in \mathbb{N}$ and suppose $(*)$ holds for $n = k$. Then

$$\sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k+1)$$

$$= \frac{1}{2}\left(k+\frac{1}{2}\right)^2 + (k+1) \quad \text{(by induction hypothesis)}$$

$$= \frac{1}{2}\left(k^2 + k + \frac{1}{4} + 2k + 2\right)$$

$$= \frac{1}{2}\left(\left(k+1+\frac{1}{2}\right)^2 - 3k - \frac{9}{4} + k + \frac{1}{4} + 2k + 2\right)$$

$$= \frac{1}{2}\left((k+1)+\frac{1}{2}\right)^2$$

Thus, $(*)$ holds for $n = k+1$ if $(*)$ holds for $n = k$.

Therefore, by the principle of mathematical induction, $(*)$ holds for all $n \in \mathbb{N}$.

e) Prove by mathematical induction that for $n \geq 3$, the sum of the interior angles of a convex polygon with n vertices is $180(n-2)°$. **3**

Question 2 (12 marks)

Start a new page.

a) Relative to a fixed origin O, point A has position vector $\left(8\underset{\sim}{i} + 13\underset{\sim}{j} - 2\underset{\sim}{k}\right)$, point B has position vector $\left(10\underset{\sim}{i} + 14\underset{\sim}{j} - 4\underset{\sim}{k}\right)$, and the point C has position vector $\left(9\underset{\sim}{i} + 9\underset{\sim}{j} + 6\underset{\sim}{k}\right)$.

The line l passes through the points A and B.

(i) Find a vector equation for the line l. **2**

(ii) Find $\left|\overrightarrow{CB}\right|$. **1**

(iii) Find the size of the acute angle between the line segment CB and the line l, giving your answer to the nearest minute. **4**

(iv) Find the shortest distance from the point C to the line l, correct to 2 decimal places. **2**

(v) Find the area of ΔCXB, where X lies on l such that $\overrightarrow{CX} \perp \overrightarrow{BX}$. **1**

b) Find $\int x cos(x)^2 dx$ **2**

Question 3 begins on the next page.

Question 3 (12 marks)

Start a new page.

a) With respect to a fixed origin O the lines l_1 and l_2 are given by the equations:

$$l_1: \underset{\sim}{r} = \begin{pmatrix} 11 \\ 2 \\ 17 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ -4 \end{pmatrix} \qquad l_2: \underset{\sim}{r} = \begin{pmatrix} -5 \\ 11 \\ p \end{pmatrix} + \mu \begin{pmatrix} q \\ 2 \\ 2 \end{pmatrix}$$

where λ and μ are parameters and p and q are constants. Given that l_1 and l_2 intersect at right angles,

(i) Show that $q = -3$. **2**

(ii) Find the value of p. **3**

(iii) Point A lies on l_1 and has position vector $\begin{pmatrix} 9 \\ 3 \\ 13 \end{pmatrix}$. Point C lies on l_2.

Given that a circle with centre C, cuts the line l_1 at the points A and B,

Find the position vector of B. **3**

b) The inequality $x > \ln(1 + x)$ holds for all real $x > 0$. (Do NOT prove this.) **4**

Use this result and the method of mathematical induction to prove that for all positive integers n,

$$\sum_{r=1}^{n} \left(\frac{1}{r}\right) > \ln(n + 1)$$

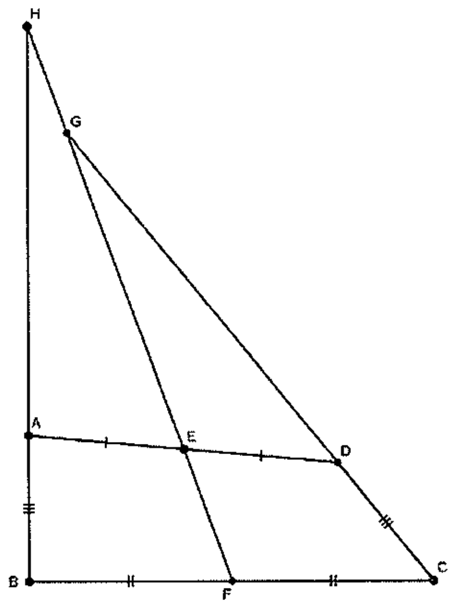
Question 4 begins on the next page.

Question 4 (12 marks)

Start a new page.

a) Prove that if $|\underset{\sim}{a}| = |\underset{\sim}{b}|$, then $(\underset{\sim}{a} + \underset{\sim}{b}) \cdot (\underset{\sim}{a} - \underset{\sim}{b}) = 0$. **1**

b) Consider the diagram below:



(i) By rewriting $\overrightarrow{EF}$ as a sum of other vectors, prove that $\overrightarrow{EF} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{DC})$. **2**

(ii) Hence, using part a) and b) i., prove that $\angle BHF = \angle CGF$. (You must approach the proof using vectors). **2**

c) (i) Find the remainder, when $x^2 + 6$ is divided by $x^2 + x - 6$. **1**

(ii) Hence, find $\int \frac{x^2+6}{x^2+x-6} dx$ **2**

d) A sequence is defined by the formula: $U_0 = 0$ and $U_n = \sqrt{U_{n-1} + 2}$ for $n = 1, 2, 3 \ldots$ **4**

Prove by mathematical induction that $U_n = 2\cos\left[\frac{\pi}{2^{n+1}}\right]$ for $n = 0, 1, 2, 3 \ldots$

Question 5 (12 marks)

Start a new page.

a) Let $I_n = \int_0^{\frac{\pi}{2}} (\sin x)^n dx$ where n is an integer, $n \geq 0$.

Using integration by parts, show that, for $n \geq 2$. **3**

$$I_n = \frac{n-1}{n} I_{n-2}.$$

b) Deduce that **3**

$$I_{2n} = \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdots \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$I_{2n+1} = \frac{2n}{2n+1} \cdot \frac{2n-2}{2n-1} \cdots \cdots \frac{4}{5} \cdot \frac{2}{3} \cdot 1.$$

c) Explain why $I_k > I_{k+1}$ *where* $k \geq 0$. **2**

d) Hence, using the fact that $I_{2n-1} > I_{2n}$ and $I_{2n} > I_{2n+1}$, show that **4**

$$\frac{\pi}{2}\left(\frac{2n}{2n+1}\right) < \frac{2^2.4^2 \ldots \ldots \ldots (2n)^2}{1.3^2.5^2 \ldots \ldots (2n-1)^2(2n+1)} < \frac{\pi}{2}$$

END OF EXAMINATION

Ext 2 Task 3 (Term 2 - 2020) 1/

Q 1.

a) $\int \frac{1}{x^2+2x+3}\,dx$ — completing the square.

$= \int \frac{1}{(x+1)^2+2}\,dx$

$= \int \frac{1}{(x+1)^2+(\sqrt{2})^2}\,dx$ — 1

$= \frac{1}{\sqrt{2}}\tan^{-1}\frac{x+1}{\sqrt{2}} + c$ — 1

b) $\int \frac{\sin^3 x}{\cos^2 x}\,dx$ METHOD 1

$= \int \frac{\sin^2 x \sin x}{\cos^2 x}\,dx$

$= \cancel{\int \frac{\sin^2 x}{\tan}}\ \int \tan^2 x \sin x\,dx$

$\int (\sec^2 x - 1)\cdot \sin x\,dx$

$= \int \frac{1}{\cos^2 x}\times \sin x\,dx - \int \sin x\,dx$ — 1

$= \int \tan x \sec x\,dx - \int \sin x\,dx$

$= \sec x + \cos x + c$ — 1

Method 2

1b) $\int \frac{(1-\cos^2 x)\sin x}{\cos^2 x}\,dx$

$= \int \frac{\sin x}{\cos^2 x}\,dx - \int \sin x\,dx$ —— (1)

$= \frac{1}{\cos x} + \cos x + C$ —— (1)

Method 3

Let $u = \cos x$.

$\frac{du}{dx} = -\sin x$

$\int \frac{\sin^3 x}{\cos^2 x}\,dx = \int \frac{\sin^2 x \cdot \sin x}{\cos^2 x}\,dx$

$= \int \frac{(1-u^2)\cdot(-1)}{u^2}\,du$

$= \int \frac{u^2-1}{u^2}\,du$ —— (1)

$= \int 1 - \frac{1}{u^2}\,du$

$= u + \frac{1}{u} + C$

$= \cos x + \sec x + C.$ —— (1)

3/

c) $\int \frac{1+x}{9-x^2}\,dx = \frac{A}{3-x} + \frac{B}{3+x}$

$1+x = A(3+x) + B(3-x)$

when $x = -3$, $B = -\frac{1}{3}$.
when $x = 3$, $A = \frac{2}{3}$ —— (1)

$\int \frac{1+x}{9-x^2}\,dx = \frac{2}{3(3-x)} - \frac{1}{3(3+x)}$

$= -\frac{2}{3}\ln|3-x| - \frac{1}{3}\ln|3+x| + C$ (2)

(1) mark for absolute value sign

* if used ~~(3+x)~~
$(x+3)(x-3)$
(values of A & B are both positive)
and final answer will be * $-\left[\frac{2}{3}\ln|x-3| + \frac{1}{3}\ln|x+3|\right] +$

d) when $n = 1$

LHS $= 1$ RHS $= \frac{1}{2}\left(1+\frac{1}{2}\right)^2$
$= \frac{9}{8}$ —— (1)

LHS $\neq$ RHS
$\therefore$ the base case does not hold true for $n = 1$

Student Number

Question 1

e) Prove true for $n=3$

A convex polygon with three vertices is a triangle, which has an angle sum of $180°$ 1

$180°$ is equal to $180(3-2)°$

$\therefore$ The statement is true for $n=3$

Assume true for $n=k$, $k \in \mathbb{Z}$, $k \geq 3$

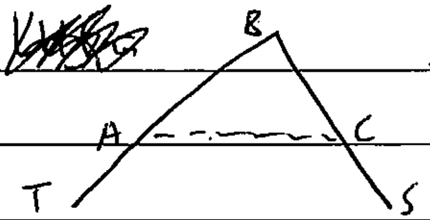
i.e. The sum of interior angles of a convex polygon with k vertices is $180(k-2)$

Prove true for $n=k+1$

RTP: The sum of interior angles of a convex polygon with $(k+1)$ vertices is $180((k+1)-2)°$

Let this sum be S_n, where n is the number of vertices

$\therefore$ RTP: $S_{k+1} = 180((k+1)-2)°$



Suppose B is the corner of a convex polygon with k vertices. If the sides AB and BC are removed and replaced with side AC, the polygon now has $(k+1)$ vertices.

$\therefore$ LHS $= S_{k+1} = S_k - \angle ABC + \angle TAC + \angle ACS$

$= S_k - \angle ABC + (\angle ABC + \angle BCA) + \angle ACS$ (exterior angle of $\triangle ABC$ equals sum of interior opposite angles) 1

$= S_k + \angle BCA + \angle ACS$

$= S_k + 180°$ (angle sum of straight angle $\angle BCS$ is $180°$)

$= 180(k-2)° + 180°$ (by assumption)

$= 180((k-2)+1)°$

$= 180((k+1)-2)°$ 1

$=$ RHS

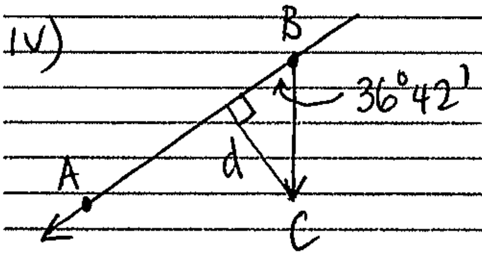
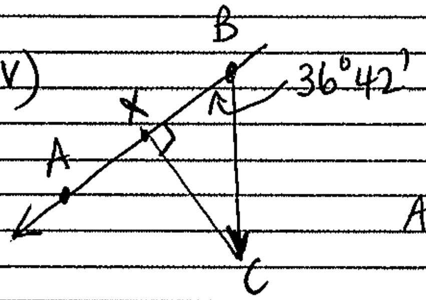
$\therefore$ If the statement is true for $n=k$, it is true for $n=k+1$

$\therefore$ The statement is true for $n \in \mathbb{Z}$, $n \geq 3$ by the principle of mathematical induction

MATHEMATICS Extension 2: Question 2

Suggested Solutions	Marks	Marker's Comments
a) i) $\overrightarrow{AB} = \begin{pmatrix}10\\14\\-4\end{pmatrix} - \begin{pmatrix}8\\13\\-2\end{pmatrix} = \begin{pmatrix}2\\1\\-2\end{pmatrix}$ $\quad \therefore \overrightarrow{BA} = \begin{pmatrix}-2\\-1\\2\end{pmatrix}$	1	Well done by most students
$\therefore l = \begin{pmatrix}8\\13\\-2\end{pmatrix} + \lambda\begin{pmatrix}2\\1\\-2\end{pmatrix}$ or $\begin{pmatrix}8\\13\\-2\end{pmatrix} + \lambda\begin{pmatrix}-2\\-1\\2\end{pmatrix}$	1	Errors came in dealing with negatives
or $\begin{pmatrix}10\\14\\-4\end{pmatrix} + \lambda\begin{pmatrix}2\\1\\-2\end{pmatrix}$ or $\begin{pmatrix}10\\14\\-4\end{pmatrix} + \lambda\begin{pmatrix}-2\\-1\\2\end{pmatrix}$		
ii) $\lvert\overrightarrow{CB}\rvert = \sqrt{(10-9)^2+(14-9)^2+(-4-6)^2}$		Well done
$= \sqrt{1+25+100}$		
$= \sqrt{126}$	1	
$= 3\sqrt{14}$		
iii) $\overrightarrow{BC} = \begin{pmatrix}9\\9\\6\end{pmatrix} - \begin{pmatrix}10\\14\\-4\end{pmatrix} = \begin{pmatrix}-1\\-5\\10\end{pmatrix}$	1	→ finding $\overrightarrow{BC}, \lvert\overrightarrow{BA}\rvert$
		Most students attained at least 3 marks
$\lvert\overrightarrow{BA}\rvert = \sqrt{(-2)^2+(-1)^2+2^2}$		
$= \sqrt{9}$		
$= 3$		
$\therefore \overrightarrow{BC}\cdot\overrightarrow{BA} = (-1)(-2) + (-5)(-1) + (10)(2)$	1	→ $\overrightarrow{BC}\cdot\overrightarrow{BA}$
$= 27$		
$27 = \lvert\overrightarrow{BC}\rvert\lvert\overrightarrow{BA}\rvert\cos\theta$	1	Unfortunately many errors were made in dealing with negatives which affected finding $\overrightarrow{BC}$, $\overrightarrow{BC}\cdot\overrightarrow{BA} = 27$
$= 9\sqrt{14}\cos\theta$		
$\therefore \cos\theta = \dfrac{3}{\sqrt{14}}$		
$\theta = 36°42'$	1	

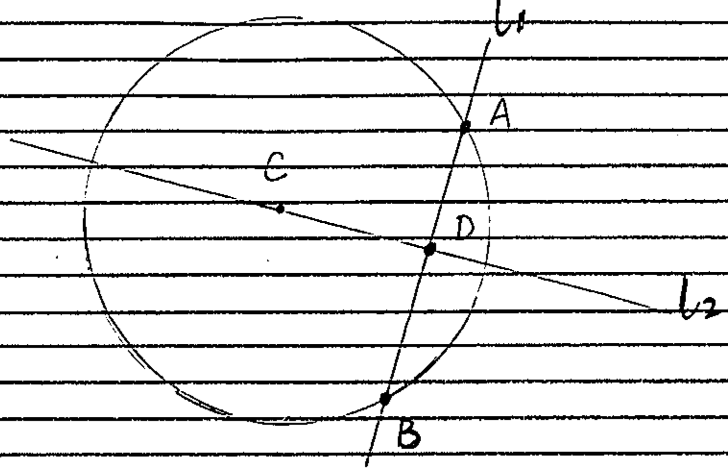
MATHEMATICS Extension 2: Question 2

Suggested Solutions	Marks	Marker's Comments
iv)  $d = \lvert\overrightarrow{BC}\rvert \sin 36°42'$	1	right-angled triangle and correct notation
$= 6.71$	1	length d being shortest distance.
v)  $\lvert\overrightarrow{XB}\rvert = \dfrac{6.71}{\tan 36°42'}$		Some students did not realise they were dealing with a right-angled triangle
Area $\Delta CXB = \dfrac{1}{2} \times \dfrac{6.71}{\tan 36°42'} \times 6.71$		
$= 30.2$	1	
b) $\int x \cos(x^2)\,dx$		
$= \frac{1}{2}\sin(x^2) + C$	1	→ for $\frac{1}{2}$ and $+C$
	1	→ for $\sin^2 x$

MATHEMATICS Extension 2: Question……..

Suggested Solutions	Marks	Marker's Comments
Q3 a) i) $l_1 \perp l_2 \Rightarrow$ direction vectors perpendicular.		Some students multiply the expression of l_1 & $l_2 = 0$
$\begin{pmatrix}-2\\1\\-4\end{pmatrix} \cdot \begin{pmatrix}b\\2\\2\end{pmatrix} = 0$	1/2	
$-2b+2-8=0$		
$-2b=6$	1/2	
$b=-3$		
ii) l_1 and l_2 intersect		some students do not consider the intersection feature of two lines.
$\begin{cases} 11-2\lambda = -5-3\mu \\ 2+\lambda = 11+2\mu \\ 17-4\lambda = p+2\mu \end{cases}$	1/3	
$\begin{cases} -2\lambda+3\mu=-16 & (1) \\ \lambda-2\mu=9 & (2) \\ -4\lambda-2\mu=p-17 & (3) \end{cases}$		
$(1)+(2)\times 2 \Rightarrow -\mu=2$, $\mu=-2$		
sub $\mu=-2$ into (2) $\Rightarrow \lambda = 2\times(-2)+9=5$	1/3	a lot of miscalculation of the value of λ.
sub $\mu=-2$, $\lambda=5$ into (3) $\Rightarrow -4(5)-2\times(-2)=p-17$		
$-20+4=p-17$		
$p=1$	1/3	

MATHEMATICS Extension 2: Question........

Suggested Solutions	Marks	Marker's Comments
Q3 a) iii) 		
Method 1 let D be the intersection of l_1 and l_2. $\because CD \perp AB$, C is the circle centre $\therefore$ CD bisects AB $\therefore \overrightarrow{AB} = 2\overrightarrow{AD}$		Need to introduce a new point
For point D $\lambda = 5$ for l_1 (or $u = -2$ for l_2) $\overrightarrow{OD} = \begin{pmatrix} 11 \\ 2 \\ 17 \end{pmatrix} + 5\begin{pmatrix} -2 \\ 1 \\ -4 \end{pmatrix} = \begin{pmatrix} 1 \\ 7 \\ -3 \end{pmatrix}$	1/3	a lot of students write $D = \begin{pmatrix} 1 \\ 7 \\ -3 \end{pmatrix}$ or $\vec{D} = \begin{pmatrix} 1 \\ 7 \\ -3 \end{pmatrix}$
$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA}$ $= \begin{pmatrix} 1 \\ 7 \\ -3 \end{pmatrix} - \begin{pmatrix} 9 \\ 3 \\ 13 \end{pmatrix} = \begin{pmatrix} -8 \\ 4 \\ -16 \end{pmatrix}$		
$\overrightarrow{AB} = 2\overrightarrow{AD} = \begin{pmatrix} -16 \\ 8 \\ -32 \end{pmatrix}$	1/3	→ significant progress towards the final answer, that is → show $\overrightarrow{AB} = 2\overrightarrow{AD}$ & $\overrightarrow{AB}$ vector value
$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$ $= \begin{pmatrix} 9 \\ 3 \\ 13 \end{pmatrix} + \begin{pmatrix} -16 \\ 8 \\ -32 \end{pmatrix} = \begin{pmatrix} -7 \\ 11 \\ -19 \end{pmatrix}$	1/3	
Method 2 pick a point from l_2, which can be either point D or $\begin{pmatrix} -5 \\ 11 \\ 1 \end{pmatrix}$ to calculate $\lvert\overrightarrow{AD}\rvert = \lvert\overrightarrow{DB}\rvert$		

MATHEMATICS Extension 2: Question……..

Suggested Solutions	Marks	Marker's Comments
Q3 b) ① When $n=1$ $\sum_{r=1}^{1}\left(\frac{1}{r}\right) = \frac{1}{1} = 1,$ $\ln(1+1) = \ln 2$ $1 > \ln 2 \doteqdot 0.69$ (2dp) or $1 = \ln e > \ln 2$ $(e > 2)$	1/4	
② Assume the statement is true for $n=k$, that is, $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{k} > \ln(k+1)$ ③ for $n=k+1$, $\sum_{r=1}^{k+1}\left(\frac{1}{r}\right) = \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{k}+\frac{1}{k+1}$ $> \ln(k+1)+\frac{1}{k+1}$	1/4	
Given $x > \ln(1+x)$ (let $x = \frac{1}{k+1}$) $\frac{1}{k+1} > \ln\left(1+\frac{1}{k+1}\right) = \ln\frac{k+2}{k+1}$	1/4	→ Many students use $k+1 > \ln(k+2)$ $\frac{1}{k+1} < \frac{1}{\ln(k+2)}$ which doesn't help with the proof. In order to prove $A > C$, need to prove $A > B$ & $B > C$ thus $A > B > C$.
$\ln(k+1)+\frac{1}{k+1} > \ln(k+1)+\ln\frac{k+2}{k+1} = \ln\left((k+1)\cdot\frac{k+2}{k+1}\right)$ $= \ln(k+2)$ $\therefore \sum_{r=1}^{k+1}\left(\frac{1}{r}\right) > \ln[(k+1)+1]$	1/4	
$\therefore$ the statement is true for $n=k+1$ By the Principle of Mathematics Induction, the statement is true for all positive integer n		

MATHEMATICS Extension 2, Question 4		
Suggested Solutions	**Marks Awarded**	**Marker's Comments**
(a) $(\underline{a}+\underline{b})\cdot(\underline{a}-\underline{b}) = \underline{a}\cdot\underline{a}+\underline{b}\cdot\underline{a}-\underline{a}\cdot\underline{b}-\underline{b}\cdot\underline{b}$ $= \lvert\underline{a}\rvert^2-\lvert\underline{b}\rvert^2$ $= 0 \quad \text{if } \lvert\underline{a}\rvert=\lvert\underline{b}\rvert$	1 mark.	Many students started proof from $\lvert\underline{a}\rvert = \lvert\underline{b}\rvert$ and moved from there to try to show $(\underline{a}+\underline{b})\cdot(\underline{a}-\underline{b}) = 0$. 'If-then' (implication statements) don't have to work like this; you can perform part of a proof, then introduce what's needed (i.e. 'Now if I have it being that *** is the case, then the conclusion will follow').
(b) (i) 'Crack' the vector in a manner that introduces what you need **efficiently**, then deal with any 'mess' that arises. Here, we need $\overrightarrow{AB}$, $\overrightarrow{DC}$ from $\overrightarrow{EF}$, so can work as: $\overrightarrow{EF} = \overrightarrow{EA}+\overrightarrow{AB}+\overrightarrow{BF}$ $\overrightarrow{EF} = \overrightarrow{ED}+\overrightarrow{DC}+\overrightarrow{CF}$ Adding, $2\overrightarrow{EF} = \left(\overrightarrow{EA}+\overrightarrow{AB}+\overrightarrow{BF}\right)+\left(\overrightarrow{ED}+\overrightarrow{DC}+\overrightarrow{CF}\right)$ $= \left(\overrightarrow{AB}+\overrightarrow{DC}\right)+\left(\overrightarrow{EA}+\overrightarrow{ED}+\overrightarrow{BF}+\overrightarrow{CF}\right)$ $= \left(\overrightarrow{AB}+\overrightarrow{DC}\right)$ since $\overrightarrow{EA} = -\overrightarrow{ED}$, $\overrightarrow{BF} = -\overrightarrow{CF}$ (see diagram on question paper). So $\overrightarrow{EF} = \frac{1}{2}\left(\overrightarrow{AB}+\overrightarrow{DC}\right)$ as required.	1 mark awarded for reasonable start. 1 mark awarded for successful complete-on.	Most students handled this well. Some solutions were 'circuitous' in the sense that much more work was put down on paper than necessary. Should attempt to strategise a little more before putting pen to paper. It makes the argument clearer as well.

(ii) We must show that $\alpha \coloneqq \angle BHF = \angle CGF =: \beta$. Now, $$\overrightarrow{HB} \cdot \overrightarrow{HF} = \lvert\overrightarrow{HB}\rvert\lvert\overrightarrow{HF}\rvert \cos\alpha$$ $$\overrightarrow{HF} \cdot \overrightarrow{GC} = \lvert\overrightarrow{HF}\rvert\lvert\overrightarrow{GC}\rvert \cos\beta$$ But $$\overrightarrow{HB} = \lambda\overrightarrow{AB}, \qquad \overrightarrow{HF} = \mu\overrightarrow{EF}, \qquad \overrightarrow{GC} = \nu\overrightarrow{DC}$$ for $\lambda, \mu, \nu \in \mathbb{R}$ scalar constants. Accordingly, $$\overrightarrow{HB} \cdot \overrightarrow{HF} = \lvert\overrightarrow{HB}\rvert\lvert\overrightarrow{HF}\rvert \cos\alpha$$ $$\to \lambda\overrightarrow{AB} \cdot \mu\overrightarrow{EF} = \lvert\lambda\overrightarrow{AB}\rvert\lvert\mu\overrightarrow{EF}\rvert \cos\alpha$$ $$\lambda\mu\overrightarrow{AB} \cdot \overrightarrow{EF} = \lambda\mu\lvert\overrightarrow{AB}\rvert\lvert\overrightarrow{EF}\rvert \cos\alpha$$ $$\overrightarrow{AB} \cdot \overrightarrow{EF} = \lvert\overrightarrow{AB}\rvert\lvert\overrightarrow{EF}\rvert \cos\alpha \quad \ldots (1)$$ and similarly, $$\overrightarrow{HF} \cdot \overrightarrow{GC} = \lvert\overrightarrow{HF}\rvert\lvert\overrightarrow{GC}\rvert \cos\beta$$ $$\to \mu\overrightarrow{EF} \cdot \nu\overrightarrow{DC} = \lvert\mu\overrightarrow{EF}\rvert\lvert\nu\overrightarrow{DC}\rvert \cos\beta$$ $$\overrightarrow{EF} \cdot \overrightarrow{DC} = \lvert\overrightarrow{EF}\rvert\lvert\overrightarrow{DC}\rvert \cos\beta \quad \ldots (2)$$ Noting $\lvert\overrightarrow{AB}\rvert = \lvert\overrightarrow{DC}\rvert$, then by part (a), $$0 = \left(\overrightarrow{AB} + \overrightarrow{DC}\right) \cdot \left(\overrightarrow{AB} - \overrightarrow{DC}\right)$$ $$= 2\overrightarrow{EF} \cdot \left(\overrightarrow{AB} - \overrightarrow{DC}\right)$$ So, $$\overrightarrow{EF} \cdot \overrightarrow{AB} = \overrightarrow{EF} \cdot \overrightarrow{DC}$$	1 mark for correct setup and reasonab-le attempt.	Most students identified correctly what needed to be found and expressed it correctly using the dot product. The level of rigour here was not typically observed, and candidates were not penalised for demonstrating through action that they understood the problem of finding angles $\angle BHF$ and $\angle CGF$ was, in vector terms, equivalent to working with the dot product of $\overrightarrow{AB}, \overrightarrow{EF}$, $\overrightarrow{DC}$ (i.e. jump to using these vectors without first dealing with $\overrightarrow{HB}, \overrightarrow{HF}, \overrightarrow{GC}$).

Hence by (1) and (2), $$\left\|\overrightarrow{AB}\right\|\left\|\overrightarrow{EF}\right\|\cos\alpha = \left\|\overrightarrow{EF}\right\|\left\|\overrightarrow{DC}\right\|\cos\beta$$ $$\therefore \cos\alpha = \cos\beta$$ so $\alpha = \beta$ (both angles are acute); that is, $\angle BHF = \angle CGF$.	1 mark for correctly proving the angles to be equal.	
(c) (i) If polynomial P (dividend) is divided by polynomial D (divisor) where $\deg D \leq \deg P$, then there is always a polynomial Q (quotient) and polynomial R (remainder) such that $$P(x) = Q(x)D(x) + R(x)$$ where $0 \leq \deg R < \deg D$. So, if we are working with $\frac{P(x)}{D(x)}$, then $$\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}$$ Note that the remainder must have degree strictly less than that of the divisor AND note that the remainder is **not** the fraction $\frac{R(x)}{D(x)}$.		This prelude is not necessary commentary within an exam paper; it is here because many students either did not correctly find the remainder, or identified the remainder as a fraction, or did not state what the remainder was.
We can find the remainder using the division algorithm or just by manipulating: $$\frac{x^2+6}{x^2+x-6} = \frac{x^2+x-6-x+12}{x^2+x-6} = \frac{x^2+x-6}{x^2+x-6} + \frac{12-x}{x^2+x-6} = 1 + \frac{12-x}{x^2+x-6}$$ The remainder is therefore $(12-x)$.	1 mark for obtaining the correct remaind-er and stating it as such.	

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(ii)

$$\int \frac{x^2+6}{x^2+x-6}\,dx = \int 1 + \frac{12-x}{x^2+x-6}\,dx$$

$$= \int 1 + \frac{12-x}{(x+3)(x-2)}\,dx$$

$$= \int 1 + \frac{15/(-5)}{x+3} + \frac{10/5}{x-2}\,dx$$

$$= \int 1 - \frac{3}{x+3} + \frac{2}{x-2}\,dx$$

$$= x - 3\log|x+3| + 2\log|x-2| + C$$

</td>
<td>1 mark awarded for reasonab-le working (i.e. getting at least some of the integral correct).

1 mark for correct integral.</td>
<td>This question was not handled well. If the remainder from (i) was incorrect, the question was marked with the assumed integrand. If the integrand made the integral easier to find, full marks could not be awarded. If the integral carried with it similar complexity to the intended integral, full marks were possible.

Simplify your results, too. Marks were not deducted here since, technically, an unsimplified integral may be correct. In the HSC, though, if it comes down to deciding between you and your competition on who should be ‘king/queen’, you lose if their working is immaculate.</td>
</tr>
<tr>
<td>

(d)

Sequence U_n defined by $U_n = \sqrt{U_{n-1}+2}$ for $n \in \mathbb{N} \cup \{0\}$ with $U_0 = 0$.

We seek to prove, by mathematical induction, that

$U_n = 2\cos\left(\frac{\pi}{2^{n+1}}\right)$ for all $n \in \mathbb{N} \cup \{0\}$

given the sequence above.

Base case:

$U_0 = 0$ (sequence seed), and $U_0 = 2\cos\left(\frac{\pi}{2^{0+1}}\right) = 0$. Hence true for $n = 0$.

Assume statement $S(n)$: $U_n = \cos\left(\frac{\pi}{2^{n+1}}\right)$ is true with $U_n = \sqrt{U_{n-1}+2}$ and $U_0 = 0$.

</td>
<td>1 mark for testing base case $n = 0$.

1 mark for correct setup and reasonab-le attempt.</td>
<td>Most students were able to establish the base case without trouble.</td>
</tr>
</table>

We seek to show that statement $S(n+1)$ is true if the hypothesis holds. We have, $$U_{n+1} = \sqrt{U_n + 2}$$ $$= \sqrt{\cos\left(\frac{\pi}{2^{n+1}}\right) + 2}$$ $$= \sqrt{2\left[2\cos^2\left(\frac{\pi}{2^{n+2}}\right) - 1\right] + 2}$$ $$= \sqrt{4\cos^2\left(\frac{\pi}{2^{n+2}}\right)}$$ $$= 2\left\lvert\cos\left(\frac{\pi}{2^{n+2}}\right)\right\rvert$$ Now, $0 < \frac{\pi}{2^{n+1}} \leq \frac{\pi}{2}$ for all $n \in \mathbb{N} \cup \{0\}$, so, $\cos\left(\frac{\pi}{2^{n+2}}\right) \geq 0$, hence the absolute value may be removed, giving $$U_{n+1} = 2\cos\left(\frac{\pi}{2^{(n+1)+1}}\right)$$ So, by the principle of mathematical induction, if $S(n)$ is true, then $S(n+1)$ is true. Since $S(n)$ is true for $n = 0$, it follows that $S(n)$ is true for all $n \in \mathbb{N} \cup \{0\}$.	1 mark for correctly making the substitut-ion required/ using correct identity. 1 mark for moving correctly from positive square root to justificati on as to why cosine could be expressed as required for the induction to work.	Most students were able to use the recursion relation correctly and move to next phase of using trig. identity of double angle for cosine. The great majority of students did not identify that $$\sqrt{f(x)^2} = \lvert f(x)\rvert$$ always, and that establishing $$\lvert f(x)\rvert = f(x)$$ was necessary (at least internally) before **justifying** why one may write $$\sqrt{4\cos^2\left(\frac{\pi}{2^{n+2}}\right)} = 2\cos\left(\frac{\pi}{2^{n+1}}\right)$$

MATHEMATICS Extension 2: Question 5

Suggested Solutions	Marks	Marker's Comments
a) $I_n = \int_0^{\frac{\pi}{2}} (\sin x)^n \, dx$		
$= \int_0^{\frac{\pi}{2}} \sin x . \sin^{n-1} x \, dx$ $u = \sin^{n-1} x$, $u' = (n-1)\sin^{n-2} x . \cos x$		
$= \left[-\cos x \sin^{n-1} x \right]_0^{\frac{\pi}{2}}$ $v' = \sin x$, $v = -\cos x$		
$- \int_0^{\frac{\pi}{2}} (n-1)\sin^{n-2} x \cos x (-\cos x) \, dx$		1 IBP correctly
$= 0 + (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x \cos^2 x \, dx$		
$= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x (1 - \sin^2 x) \, dx$		1 Applying Pythagorean Identity.
$= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x \, dx - (n-1)\int_0^{\frac{\pi}{2}} \sin^n x \, dx$		
$I_n = (n-1) I_{n-2} - (n-1) I_n$ *		
$I_n + (n-1) I_n = (n-1) I_{n-2}$		
$n I_n = (n-1) I_{n-2}$		
$I_n = \frac{(n-1)}{n} I_{n-2}$ Answer		1
* Note: Students mustn't skip from * straight to the answer!		

MATHEMATICS Extension 2: Question........		
Suggested Solutions	**Marks**	**Marker's Comments**

b)

$I_{2n} = \frac{2n-1}{2n} I_{2n-2}$ (By part a)

$= \frac{2n-1}{2n} \times \frac{(2n-2)-1}{2n-2} I_{2n-4}$

$= \frac{2n-1}{2n} \times \frac{2n-3}{2n-2} \times \frac{(2n-4)-1}{2n-4} I_{2n-6}$

$= \frac{2n-1}{2n} \times \frac{2n-3}{2n-2} \times \frac{2n-5}{2n-4} I_{2n-6}$

$\vdots$

$= \frac{2n-1}{2n} \times \frac{2n-3}{2n-2} \times \frac{2n-5}{2n-4} \times \cdots \frac{2n-(2n-1)}{2n-(2n-2)} I_{2n-2n}$

$= \frac{2n-1}{2n} \times \frac{2n-3}{2n-2} \times \frac{2n-5}{2n-4} \times \cdots \frac{1}{2} I_0$ — 1

$I_0 = \int_0^{\frac{\pi}{2}} 1 \, dx = \left[x\right]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$

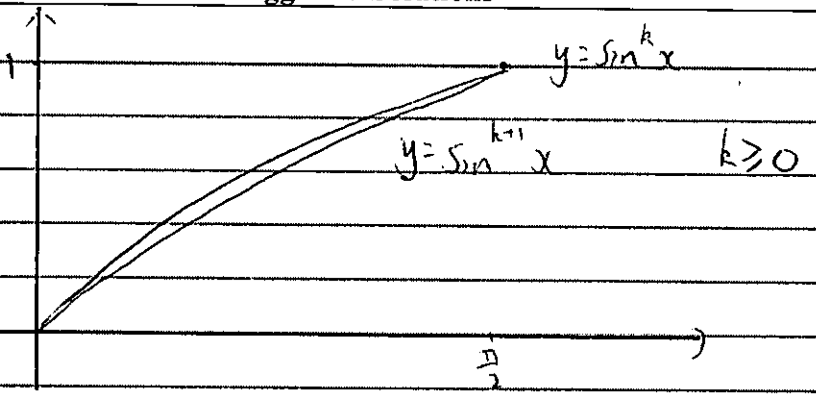
$\therefore I_{2n} = \frac{2n-1}{2n} \times \frac{2n-3}{2n-2} \times \cdots \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2}$ — 1

MATHEMATICS Extension 2: Question……..

Suggested Solutions	Marks	Marker's Comments
$I_{2n+1} = \frac{(2n+1)-1}{2n+1} I_{2n-1}$		
$= \frac{2n}{2n+1} I_{2n-1}$		
$= \frac{2n}{2n+1} \times \frac{(2n-1)-1}{2n-1} I_{2n-3}$		
$= \frac{2n}{2n+1} \times \frac{2n-2}{2n-1} I_{2n-3}$		
$\vdots$		
$= \frac{2n}{2n+1} \times \frac{2n-2}{2n-1} \times \cdots \frac{2n-(2n-2)}{2n-(2n-3)} I_{2n-(2n-1)}$		
$= \frac{2n}{2n+1} \times \frac{2n-2}{2n-1} \times \cdots \frac{2}{3} \times I_1$		
$I_1 = \int_0^{\frac{\pi}{2}} \sin x \, dx = \left[-\cos x\right]_0^{\frac{\pi}{2}}$		
$= ((-0) - (-1))$		
$= 1$		
$\therefore I_{2n+1} = \frac{2n}{2n+1} \times \frac{2n-2}{2n-1} \times \cdots \frac{2}{3} \times 1$		1
Or Vice Versa.		
* First identity worth 2 marks		
* Second identity worth 1 mark		
* Evaluating I_0 and I_1 will result in 1 mark		

MATHEMATICS Extension 2: Question……..		
Suggested Solutions	**Marks**	**Marker's Comments**

c)



Since $0 \leq \sin x \leq 1$ for $0 \leq x \leq \frac{\pi}{2}$ — 1

$\sin^k x > \sin^{k+1} x$ for $0 \leq x \leq \frac{\pi}{2}$ — 1

$\therefore$ Area under the curve of $y = \sin^k x$ will be greater than area under the curve of $y = \sin^{k+1} x$

$\therefore I_k > I_{k+1}$ for $k \geq 0$

MATHEMATICS Extension 2: Question……..

Suggested Solutions	Marks	Marker's Comments
d) $I_{2n} > I_{2n+1}$		
$\downarrow$		
$\frac{2n-1}{2n} \times \frac{2n-3}{2n-2} \times \ldots \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2} > \frac{2n}{2n+1} \times \frac{2n-2}{2n-1} \times \ldots \frac{4}{5} \times \frac{2}{3} \times 1$		
Multiply both sides by $2n \times (2n-2) \times \ldots 4 \times 2$		
$(2n-1)(2n-3) \times \ldots 3 \times 1 \times \frac{\pi}{2} > \frac{(2n)^2(2n-2)^2 \times \ldots 4^2 \times 2^2}{(2n+1)(2n-1) \times \ldots 5 \times 3 \times 1}$		
Divide both sides by $(2n-1)(2n-3) \times \ldots 3 \times 1$		
$\frac{\pi}{2} > \frac{(2n)^2(2n-2)^2 \times \ldots 4^2 \times 2^2}{(2n+1)(2n-1)^2(2n-3)^2 \times 5^2 \times 3^2}$		
$\therefore \frac{2^2 \times 4^2 \times \ldots (2n)^2}{1 \times 3^2 \times 5^2 \times \ldots (2n-1)^2(2n+1)} < \frac{\pi}{2}$		

MATHEMATICS Extension 2: Question……..

Suggested Solutions	Marks	Marker's Comments

$I_{2n} < I_{2n-1}$

$\downarrow$

$$\frac{(2n-1)(2n-3)\times 3\times 1}{2n(2n-2)\times\ldots 4\times 2}\times\frac{\pi}{2} < \frac{(2n-2)(2n-4)\ldots 4\times 2}{(2n-1)(2n-3)\times\ldots 5\times 3}\times 1$$

$\downarrow$ Multiply both sides by $2n(2n-2)(2n-4)\times\ldots 4\times 2$

$$\frac{(2n-1)(2n-3)\times\ldots 3\times 1}{1}\times\frac{\pi}{2} < \frac{2n(2n-2)^2(2n-4)^2\times\ldots 4^2\times 2^2}{(2n-1)(2n-3)\times\ldots 5\times 3}$$

$\downarrow$ Divide both sides by $(2n-1)(2n-3)\times\ldots\times 3\times 1$

$$\frac{\pi}{2} < \frac{2n(2n-2)^2(2n-4)^2\times\ldots 4^2\times 2^2}{(2n-1)^2(2n-3)^2\times\ldots 5^2\times 3^2}$$

$\downarrow$ Multiply both sides by $\frac{2n}{2n+1}$

$$\frac{\pi}{2}\left(\frac{2n}{2n+1}\right) < \frac{2^2\times 4^2\times 6^2\times\ldots\times(2n)^2}{1\times 3^2\times 5^2\times\ldots(2n-1)^2\times(2n+1)}$$

$$\therefore \frac{\pi}{2}\left(\frac{2n}{2n+1}\right) < \frac{2^2\times 4^2\times\ldots(2n)^2}{1\times 3^2\times 5^2\times\ldots(2n-1)^2(2n+1)} < \frac{\pi}{2}$$

(2) Marks per inequality proven

(1) mark for writing expansion of I_{2n-1}