Multiple Choice (5 marks)

1. For the implication, $P \rightarrow Q$, the negation can be written as.

   A. $\mathrm{Q} \rightarrow \mathrm{P}$
   B. $\neg \mathrm{Q} \rightarrow \neg \mathrm{P}$
   C. $\mathrm{P} \wedge \neg \mathrm{Q}$
   D. None of the above

2. Which of the following functions provides a counterexample to the following proposition?

   "If $f'(x) > 0$ for all $x \in \mathbb{R}$, then $f(x) > 0$ for all $x \in \mathbb{R}$."

   A. $f(x) = x^2 + 1$
   B. $f(x) = x^2 - 1$
   C. $f(x) = 1 - x$
   D. $f(x) = x^3 + x + 1$

3. Which of the following is the negation of the statement "all warlords are dishonourable people"?

   A. If a person is honourable, they are not a warlord.
   B. Some people are either dishonourable or they are warlords.
   C. Some warlords are honourable people
   D. A person is honourable if they're not a warlord.

4. The value of $\int \frac{1}{(\sin\theta)^2(\cos\theta)^2}\, d\theta$ is

   A. $\tan\theta + \cot\theta + c$
   B. $\tan\theta\cot\theta + c$
   C. $\tan\theta - \cot\theta + c$
   D. $\tan\theta - \cot 2\theta + c$

5. The value of $\int \frac{dx}{x(x^2+1)}$ is

A. $\ln|x| + \tan^{-1} x + c$

B. $\ln|x| + \ln(1 + x^2) + c$

C. $\ln|x| - \tan^{-1} x + c$

D. $\ln|x| - \frac{1}{2}\ln(1 + x^2) + c$

Question 6 (15 marks)

a) Let S and P be the following propositions:

S ≡ [I have studied for this examination]

P ≡ [this examination is passable]

i) Express $P \vee \neg S$ as an English sentence. (1 mark)

ii) ii) Write the statement given in (a) as an implication involving S and P, and express it also in English. (2 marks)

b) i) For all a, b $\in$ Z, show that 17 divides 10a+b if and only if 17 divides a-5b. (3 marks)

ii) Deduce, using part (i), that 17 divides 4080. (1 mark)

c) Using proof by contradiction, show that for all positive integers a, b and c,

if $a^2 + b^2 = c^2$, then either a is even or b is even. (3 marks)

d) i) Let z=2+i and ω=1-2i.

α) Find zω. (1 mark)

β) Find the angle between zω and ω, leaving your answer to 2 decimal places. (2 mark)

ii) Let $z = re^{i\frac{\pi}{3}}$ where r $\in R^+$ and ω = (3-3i)z.

If w^k lies on a unit circle centred at the origin for all integer values of k, find the value of r. (2 marks)

Question 7 (start a new page) (15 marks)

a) i) Find a, b and c such that $\int x^2 e^{2x} dx = (ax^2 + bx + c)e^{2x}$. (2marks)

ii) Show that $\int_0^{\pi} \frac{x \sin x}{1+(\cos x)^2} dx = \frac{\pi^2}{4}$. (2marks)

iii) Evaluate $\int \frac{\sin x}{\sin(x+a)} dx,\ a > 0$. (2marks)

(iv))Find $\frac{d}{dx}\left(\int_0^{\cos ax} \sqrt{1-t^2}\ dt\right)$. (2marks)

b) i) Show by taking the contrapositive, that for all positive integers n, n is even if n^2 is

even. (3 marks)

ii) Hence, using proof by contradiction, show that $\sqrt{8n-2}$ is irrational. (4 marks)

Question 8 (Start a new Page) (15 marks)

a) $5 - 2i$ is a solution to the equation $z^2 + az + b = 0$ where $a, b \in \mathbb{R}$. Find the value of a and b. (2 marks)

b) Let $z = 2(cos2\theta + isin2\theta)$ where $0 < \theta < 45°$. Find the modulus and argument of $z + 2$, expressing your answers in terms of θ. (2 marks)

c) Consider the complex number $z = \frac{\omega_1}{\omega_2}$ where $\omega_1 = \sqrt{2} + \sqrt{6}i$ and $\omega_2 = 3 + \sqrt{3}i$.

i) Express z in modulus-argument form. (3 marks)

ii) Find the smallest positive integer value of n such that z^n is a real number. (1 mark)

d) Sketch the region in the complex plane where the following inequalities hold simultaneously. (3 marks)

$$-\frac{\pi}{4} < Arg(z - 2 + 3i) < \frac{\pi}{4} \text{ and } |z - 2 + 3i| \leq 2$$

(Continue with Q8)

e) For the equation $z^3 + 64 = 0, z \in \mathbb{C}$.

i) Find the three distinct roots of the equation in modulus-argument form. (3marks)

ii) The roots of the equation represent the vertices of a triangle. Find the area of the\

triangle formed. (1 mark)

Question 9 (Start a New Page) (15 marks)

a) i) $\int \frac{dx}{\cos\alpha + \cos x}$, $0 < \alpha < \pi$ (3 marks)

ii) $\int \frac{3x-2}{(x+1)^2(x+3)} dx$ (3 marks)

b) Consider the complex numbers $z_1 = -1 + i$ and $z_2 = \frac{1}{\sqrt{2}}(\cos\left(\frac{\pi}{3}\right) + isin\left(\frac{\pi}{3}\right))$.

i) Write $z_1 z_2$ in the form $a + ib$ where $a, b \in \mathbb{R}$. (2marks)

ii) Hence or otherwise, find the value of $\tan\left(\frac{\pi}{12}\right)$ in the form $c + d\sqrt{3}$ where $c, d \in \mathbb{Z}$.

(3marks)

c) Given that $z^5 - 1 = (z-1)(z^4 + z^3 + z^2 + z + 1)$.

Let ω be the complex root with the smallest positive argument to the solution to

$z^5 - 1 = 0$ where $\omega \in \mathbb{C}$.

i) Prove that $1 + \omega^2 + \omega^4 = -(\omega + \omega^3)$ (1mark)

ii) Hence show that $\cos\left(\frac{2\pi}{5}\right) - \cos\left(\frac{\pi}{5}\right) = -\frac{1}{2}$ (3marks)

Question 10 (Start a New Page) (15 marks)

a) Use integration, find the area of the region in the first quadrant enclosed by the x-axis,

the line $y = x$ and the circle $x^2 + y^2 = 32$. (3 marks)

b) Given that $I_n = \int (x^2 - a^2)^{\frac{2n-1}{2}} dx$

i) Differentiate $x + \sqrt{x^2 - a^2}$, (1mark)

ii) Hence show that $I_0 = \ln|x + \sqrt{x^2 - a^2}|$ (2 marks)

iii) Use integration by parts to show that $I_n = \frac{1}{2n} x(x^2 - a^2)^{\frac{2n-1}{2}} - \frac{2n-1}{2n} a^2 I_{n-1}$

(3 marks)

c) Given that f is an integrable function over the closed interval $[0, a]$.

i) Prove that $\int_0^a f(x)dx = \int_0^{\frac{a}{2}} f(x)dx + \int_0^{\frac{a}{2}} f(a - x)dx$ (2 marks)

ii) Hence, or otherwise show that $\int_0^{\pi} \ln(\sin\theta)\, d\theta = -\pi \ln 2$. (4 marks)

END of PAPER

Ext 2 2021 Yr 12 Task 2

MATHEMATICS Extension 2: Question MC

Suggested Solutions	Marks	Marker's Comments
1. D		
2. D		
3. C		
4. C		

$$\int \frac{1}{\sin^2\theta\cos^2\theta}\,d\theta = \int \frac{\sec^2\theta}{\sin^2\theta}\,d\theta$$

$$= \int \frac{\tan^2\theta+1}{\sin^2\theta}\,d\theta$$

$$= \int\left(\frac{\frac{\sin^2\theta}{\cos^2\theta}}{\sin^2\theta} + \frac{1}{\sin^2\theta}\right)d\theta$$

$$= \int \sec^2\theta + \operatorname{cosec}^2\theta\,d\theta$$

$$= \tan\theta - \cot\theta + C$$

5. D

$$\frac{1}{x(x^2+1)} \equiv \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$1 = A(x^2+1) + (Bx+C)(x)$$

when $x=0$, $A=1$

$x=1$, $1 = 2+B+C$

$x=-1$, $1 = 2+B-C$

$\therefore\ 2 = 4+2B$

$B=-1$

$\therefore\ C=0$

$$\therefore \int \frac{1}{x(x^2+1)}\,dx = \int\left(\frac{1}{x} - \frac{x}{x^2+1}\right)dx$$

$$= \ln|x| - \frac{1}{2}\ln(x^2+1) + C$$

MATHEMATICS Extension 2: Question 6

Suggested Solutions	Marks	Marker's Comments
a)		
i I have not studied for this examination	1	
ii $S \longrightarrow P$	1	
If I have studied for this examination, then this examination is passable	1	It must be an "if, then" meaning! <u>NOT</u> "S happened, therefore P".
b) i let $10a+b = 17k_1$, where $k_1 \in \mathbb{Z}$		
$\therefore b = 17M_1 - 10a$		
$\therefore a - 5b = a - 5(17k_1 - 10a)$		1 for attempting to prove the implication from both directions
$= a - 85k_1 + 50a$		
$= 51a - 85k_1$		1 for correctly proving first implication
$= 17(3a_1 - 5k_1)$		
$= 17M_1$ where $M_1 \in \mathbb{Z}$ (Since $a_1, k_1 \in \mathbb{Z}$)		1 for correctly proving second implication
let $a - 5b = 17k_2$ where $k_2 \in \mathbb{Z}$,		
$\therefore a = 17k_2 + 5b$		
$\therefore 10a + b = 10(17k_2 + 5b) + b$		
$= 170k_2 + 50b + b$		

MATHEMATICS Extension 2: Question……..

Suggested Solutions	Marks	Marker's Comments
$= 170k_2 + 51b$		
$= 17(10k_2 + 3b)$		
$= 17M_2$ where $M_2 \in \mathbb{Z}$ (Since $k_2, b \in \mathbb{Z}$)		
$\therefore$ 17 divides $10a+b$ iff 17 divides $a-5b$		
See next page for b) ii		
c) Assume that $a^2+b^2=c^2$ where $a, b, c \in \mathbb{Z}$ and a, b, are odd	1	Students DO NOT get this mark if they attempt to assume both a, b are even and both a, b are odd
let $a = 2k_1+1$ $b = 2k_2+1$		
$\therefore c^2 = a^2+b^2$		
$= (2k_1+1)^2 + (2k_2+1)^2$		
$= (4k_1^2+4k_1+1) + (4k_2^2+4k_2+1)$		
$= 2(2k_1^2+2k_1+2k_2^2+2k_2+1)$		
$\therefore c^2$ is even.	1	
$\therefore c$ is even (property of an even number)		
$\therefore c = 2p$.		
$\therefore 4p^2 = 2(2k_1^2+2k_1+2k_2^2+2k_2+1)$		
$2p^2 = 2k_1^2+2k_1+2k_2^2+2k_2+1$		
LHS is even, RHS = odd		

$\therefore$ Original statement is proven by contradiction. 1

MATHEMATICS Extension 2: Question……..		
Suggested Solutions	**Marks**	**Marker's Comments**
b) ii let $4080 = 10 \times 408 + 0$		
$\therefore a = 408$ and $b = 0$ for $10a+b$		
$a - 5b = 408$		
$408 \div 17 = 24$		
$\therefore 17$ divides 408		
$\therefore 17$ divides 4080 (from part i)	1	

MATHEMATICS Extension 2: Question……..		
Suggested Solutions	**Marks**	**Marker's Comments**
d) i)		
α) $zw = (2+i)(1-2i)$		
$= 2-4i+i+2$		
$= 4-3i$	1	
β) $\arg(zw) - \arg(w)$		
$= \arg\left(\frac{zw}{w}\right)$		
$= \arg(z)$	1	
$= \tan^{-1}\left(\frac{1}{2}\right)$		
$= 26.57^\circ$ or 0.46 rad	1	
ii) $\lvert w^k \rvert = 1$		
$\therefore \lvert w \rvert = 1$	1	
$\therefore \lvert (3-3i)z \rvert = 1$		
$\lvert 3-3i \rvert \lvert z \rvert = 1$		
$\sqrt{18}\,\lvert z \rvert = 1$		
$\therefore \lvert z \rvert = \frac{1}{\sqrt{18}}$		
$= \frac{1}{3\sqrt{2}}$		
$\therefore r = \frac{1}{3\sqrt{2}}$ or $\frac{\sqrt{2}}{6}$	1	

Mathematics Ext 2 Question 7

Suggested Solutions	Marks	Marker's Comments
ai) Method 1		
$\frac{d}{dx}(ax^2+bx+c)e^{2x}=(2ax+b)e^{2x}+2(ax^2+bx+c)e^{2x}$		
$=(2ax^2+2(a+b)x+(b+2c))e^{2x}$	1	
Match with x^2e^{2x}		
$\begin{cases}2a=1\\2(a+b)=0\\b+2c=0\end{cases} \Rightarrow \begin{cases}a=\frac{1}{2}\\b=-\frac{1}{2}\\c=\frac{1}{4}\end{cases}$	1	
Method 2		
$u=x^2$, $\frac{dv}{dx}=e^{2x}$		
$du=2x\,dx$, $v=\frac{1}{2}e^{2x}$		
$\int x^2e^{2x}dx=\frac{1}{2}x^2e^{2x}-\int\frac{1}{2}e^{2x}2x\,dx$		
$=\frac{1}{2}x^2e^{2x}-\int e^{2x}x\,dx$	1	
$u=x$, $\frac{dv}{dx}=e^{2x}$		
$du=dx$, $v=\frac{1}{2}e^{2x}$		
$\int x^2e^{2x}dx=\frac{1}{2}x^2e^{2x}-(\frac{1}{2}xe^{2x}-\int\frac{1}{2}e^{2x}dx)$		The second integration by parts was done poorly.
$=\frac{1}{2}x^2e^{2x}-\frac{1}{2}xe^{2x}+\frac{1}{2}\int e^{2x}dx$		
$=\frac{1}{2}x^2e^{2x}-\frac{1}{2}xe^{2x}+\frac{1}{4}e^{2x}+C_1$		
$=(\frac{1}{2}x^2-\frac{1}{2}x+\frac{1}{4})e^{2x}+C_1$		
Match with $(ax^2+bx+c)e^{2x}$ — constant term, has nothing to do with e^{2x}		Many students forgot to write down the values for a, b, and c.
$a=\frac{1}{2}$, $b=-\frac{1}{2}$, $c=\frac{1}{4}$	1	

Mathematics Ext 2 Question 7

Suggested Solutions	Marks	Marker's Comments
a ii) $\int_0^{\pi} \frac{x \sin x}{1+(\cos x)^2}\,dx$		
$= \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)}\,dx$ (dummy variables).		some students used $\frac{\pi}{2}-x$ instend of $\pi - x$
$= \int_0^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x}\,dx$		
$= \int_0^{\pi} \frac{\pi \sin x}{1+\cos^2 x}\,dx - \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\,dx$	1	
$2\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\,dx = \int_0^{\pi} \frac{\pi \sin x}{1+\cos^2 x}\,dx$		
$= -\pi \int_0^{\pi} \frac{-\sin x}{1+\cos^2 x}\,dx$		
$-\sin x = \frac{d(\cos x)}{dx}$		
$= -\pi \int_0^{\pi} \tan^{-1}(\cos x)\,dx$		
$= -\pi\left[-\frac{\pi}{4} - \frac{\pi}{4}\right] = \frac{\pi^2}{2}$		
$\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\,dx = \frac{\pi^2}{4}$	1	

Mathematics Ext 2 Question 7

Suggested Solutions	Marks	Marker's Comments
(iii) Method 1 The goal is to make the denominator simply.		Some students substitute $u=\sin(x+a)$ but it doesn't make it simply
$\int \frac{\sin x}{\sin(x+a)}\,dx$ let $u=x+a$, $x=u-a$. $du=dx$		
$=\int \frac{\sin(u-a)}{\sin u}\,du$		→ cannot use dummy variable to swap it back to $\int \frac{\sin(x-a)}{\sin x}dx$
$=\int \frac{\sin u\cos a-\sin a\cos u}{\sin u}\,du$	1	
$=\int \cos a-\sin a\frac{\cos u}{\sin u}\,du$		
$=(\cos a)u-\sin a\ln\lvert\sin u\rvert+C$		
$=(\cos a)(x+a)-\sin a\ln\lvert\sin(x+a)\rvert+C$	1	Some students forgot to change back to x.
Method 2		
$\int \frac{\sin x}{\sin(x+a)}\,dx=\int \frac{\sin[(x+a)-a]}{\sin(x+a)}\,dx$		
$=\int \frac{\sin(x+a)\cos a-\cos(x+a)\sin a}{\sin(x+a)}\,dx$	1	
$=\int \cos a\,dx-\sin a\int \frac{\cos(x+a)}{\sin(x+a)}\,dx$		
$=(\cos a)x-\sin a\ln\lvert\sin(x+a)\rvert+C$	1	

Mathematics Ext 2 Question 7

Suggested Solutions	Marks	Marker's Comments
a iv) Method 1 Method 1 should not have calculation of lower bound 0.		
Using $\frac{d}{dx}\int_a^x f(t)dt = f(x)$		showing $f(\cos ax)$ or equivalent form and acknowledgement of $\frac{d\cos ax}{dx}$ in certain form, such as $\frac{f(\cos ax)}{a}$ or $\frac{f(\cos ax)}{a}(\sin ax)$ receives 1 mark
let $f(t)=\sqrt{1-t^2}$		
$I = \frac{d}{d(\cos ax)}\int_0^{\cos ax} f(t)dt \times \frac{d\cos ax}{dx}$		
$= f(\cos ax) \times (-a\sin ax)$	1	
$= \sqrt{1-\cos^2 ax} \times (-a\sin ax)$		
$= -a\sin^2 ax \quad (0\leq x\leq \frac{\pi}{2a}, 0\leq \sin ax\leq 1)$	1	
$\sqrt{1-\cos^2 ax} = \lvert \sin ax \rvert$		
Method 2		
Using $\int_a^b f(x)dx = F(b)-F(a)$		
let $f(t)=\sqrt{1-t^2}$, $F(t)=\int\sqrt{1-t^2}\,dt$.		Students need to define $F(t)$ and $f(t)$ if they use it during the calculation.
$\frac{d}{dx}\int_0^{\cos ax} f(t)dt$		
$= \frac{d}{dx}[F(\cos ax)-F(0)]$		
$= f(\cos ax)\frac{d\cos ax}{dx} - 0$	1	Bound calculation or its equivalent form is required for this method.
$= \sqrt{1-\cos^2 ax}\cdot(-a\sin(ax))$		
$= -a\sin^2(ax) \quad (0\leq x\leq\frac{\pi}{2a}, 0\leq \sin ax\leq 1)$	1	

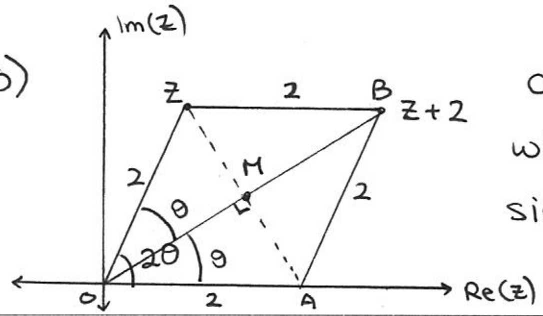
Mathematics Ext2 Question 7

Suggested Solutions	Marks	Marker's Comments
bi) RTP : n^2 is even $\Rightarrow n$ is even.		
Contrapositive: n is odd $\Rightarrow n^2$ is odd	1	many students get the order wrong.
If n is odd, let $n = 2k-1,\ k \in \mathbb{Z}^+$		
$n^2 = (2k-1)^2$	1	n is a positive integer, $n=2k+1$, $k\in\mathbb{Z}$ doesn't guarantee that.
$= 4k^2-4k+1$		
$= 2(2k^2-2k)+1$		
$\because 2k^2-2k \in \mathbb{Z}$ → many students didn't write this		
$\therefore n^2$ is odd.	1	
$\therefore$ contrapositive is proven.		
$\therefore$ the original statement is true.		

Mathematics Ext 2 Question 7

Suggested Solutions	Marks	Marker's Comments
bii) Suppose $\sqrt{8n-2} \in \mathbb{Q}$		
let $\sqrt{8n-2} = \frac{p}{q}$ ($p, q \in \mathbb{Z}$, $q \neq 0$, p, q coprime)	1	
$(8n-2) = \frac{p^2}{q^2}$		
$2(4n-1)q^2 = p^2$ ①		
$\because (4n-1)q^2 \in \mathbb{Z}$		Some students use $2(4n-1) = \frac{p^2}{q^2}$
$\therefore p^2$ is even.		$\frac{p^2}{q^2}$ is even $\Rightarrow \frac{p}{q}$ is even
$\therefore p$ is even (by bi) statement)	1	bi) only applies to positive integers not including fractions
let $p = 2k$ ($k \in \mathbb{Z}$) don't use the same letter that the question already used.		
from ① $(8n-2) \cdot q^2 = 4k^2$		
$(4n-1)q^2 = 2k^2$	1	
$\because 4n-1$ is odd		many students did not write this part.
$2k^2$ is even		
$\therefore q^2$ is even		common mistake: if two numbers are divisible by the same number, they are coprime.
$\therefore q$ is even (by b(i) statement)		
$\therefore$ both p and q are both even, which means that they aren't coprime. which is contradict with the initial assumption that p and q are coprime.		
$\therefore \sqrt{8n-2}$ is irrational	1	

MATHEMATICS Extension 2: Question 8

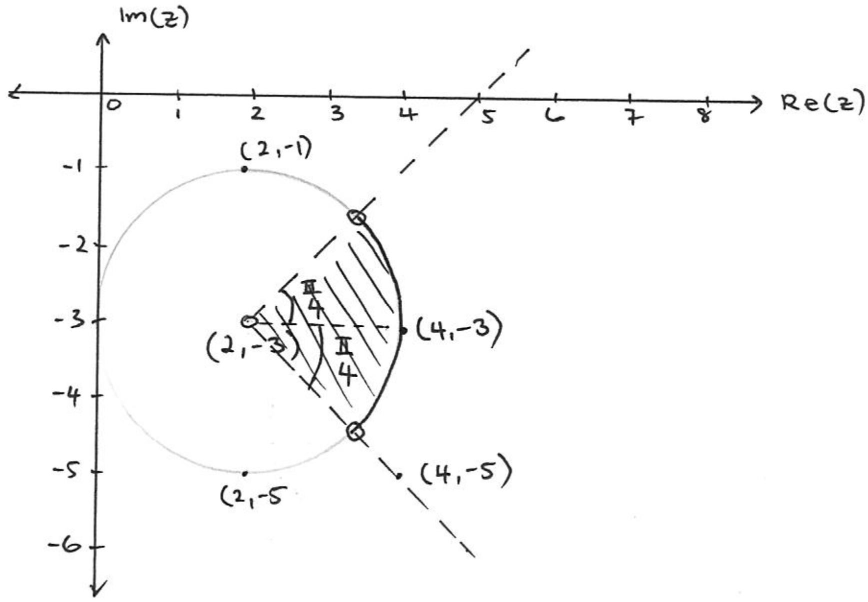
Suggested Solutions	Marks	Marker's Comments
a) Since $5-2i$ is a root, it will satisfy the equation.		
$\therefore (5-2i)^2 + a(5-2i) + b = 0$		
$25 - 20i - 4 + 5a - 2ai + b = 0$		
$21 + 5a + b - (20 + 2a)i = 0$		
equating real and imaginary parts.		
$\therefore 20 + 2a = 0$		
$a = -10$	1	
$\therefore 21 + 5a + b = 0$		
$21 + 5(-10) + b = 0$		
$b = 29$	1	
$\therefore a = -10,\ b = 29$		
OR since $5-2i$ is a root, then $5+2i$ is also a root as coefficients are real.		
sum of roots:		
$5 - 2i + 5 + 2i = -a$		
$\therefore a = -10$	1	
product of roots:		
$(5-2i)(5+2i) = b$		
$b = 25 + 4$		
$b = 29$	1	
$\therefore a = -10,\ b = 29$		
b)  OABZ is a rhombus with a pair of adjacent sides equal and opposite sides equal and parallel.		

MATHEMATICS Extension 2: Question …8…

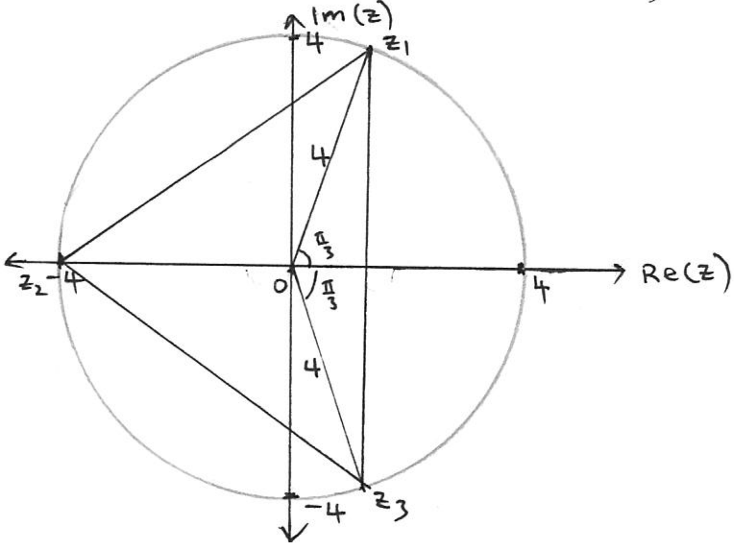
Suggested Solutions	Marks	Marker's Comments
$\therefore \arg(z+2) = \frac{1}{2}\arg z$		
$= \theta$	1	
In ΔOMA, $\cos\theta = \frac{OM}{2}$		
$OM = 2\cos\theta$		
$\|z+2\| = 2(OM)$		
$= 2(2\cos\theta)$		
$= 4\cos\theta.$	1	
OR $z+2 = 2(\cos 2\theta + i\sin 2\theta) + 2$		
$= 2(2\cos^2\theta - 1 + 2i\sin\theta\cos\theta) + 2$		
$= 4\cos^2\theta + 4i\sin\theta\cos\theta$		
$= 4\cos\theta(\cos\theta + i\sin\theta)$	1	
$\therefore \|z+2\| = 4\cos\theta$	1	
$\arg(z+2) = \theta$		
OR $z+2 = 2\cos 2\theta + 2 + 2i\sin 2\theta$		
$\|z+2\| = \sqrt{(2\cos 2\theta + 2)^2 + (2\sin 2\theta)^2}$		A lot of students forgot the square root for the modulus.
$= \sqrt{4\cos^2 2\theta + 8\cos 2\theta + 4 + 4\sin^2 2\theta}$		
$= \sqrt{4 + 8\cos 2\theta + 4}$		
$= \sqrt{8 + 8(2\cos^2\theta - 1)}$		
$= \sqrt{16\cos^2\theta}$		
$= 4\cos\theta$	1	
$\arg(z+2) = \tan^{-1}\left(\frac{2\sin 2\theta}{2\cos 2\theta + 2}\right)$		
$= \tan^{-1}\left(\frac{2\sin\theta\cos\theta}{2\cos^2\theta - 1 + 1}\right)$		
$= \tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right)$		
$= \tan^{-1}(\tan\theta)$		
$= \theta$	1	

Suggested Solutions	Marks	Marker's Comments
c) i) $z = \dfrac{\sqrt{2}+\sqrt{6}i}{3+\sqrt{3}i}$		
$= \dfrac{2\sqrt{2}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)}{2\sqrt{3}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)}$	1 1	
$= \frac{\sqrt{2}}{\sqrt{3}}\left(\cos\left(\frac{\pi}{3}-\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{3}-\frac{\pi}{6}\right)\right)$		
$= \frac{\sqrt{6}}{3}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$	1	
<u>OR</u> $z = \dfrac{\sqrt{2}+\sqrt{6}i}{3+\sqrt{3}i} \times \dfrac{3-\sqrt{3}i}{3-\sqrt{3}i}$		
$= \dfrac{3\sqrt{2} - \sqrt{6}i + 3\sqrt{6}i + \sqrt{18}}{9+3}$		
$= \dfrac{3\sqrt{2} + 2\sqrt{6}i + 3\sqrt{2}}{12}$		
$= \dfrac{6\sqrt{2} + 2\sqrt{6}i}{12}$		
$= \dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{6}}{6}i$	1	
$\therefore \lvert z\rvert = \sqrt{\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{6}}{6}\right)^2}$		
$= \sqrt{\frac{2}{3}}$	1	
$\arg(z) = \tan^{-1}\left(\dfrac{\frac{\sqrt{6}}{6}}{\frac{\sqrt{2}}{2}}\right)$		
$= \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$		
$= \frac{\pi}{6}$	1	
$\therefore z = \sqrt{\frac{2}{3}}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$		

MATHEMATICS Extension 2: Question…8…

Suggested Solutions	Marks	Marker's Comments
ii) for z^n to be real, $\text{Im}(z^n)=0$		
$z^n = \left(\frac{\sqrt{6}}{3}\right)^n\left(\cos\frac{n\pi}{6} + i\sin\frac{n\pi}{6}\right)$ (De moivre's Theorem)		
$\therefore \sin\frac{n\pi}{6} = 0$		
$\frac{n\pi}{6} = 0, \pi, 2\pi, \ldots$		
$n = 0, 6, 12, \ldots$		
$\therefore$ smallest positive integer is 6.	1	
d) 	1	circle - centre - radius - solid line
	1	lines - open circle at (2,-3) - dotted line - indicate angle or correct point indicated on each line
	1	correct shaded area.
e) $z^3 + 64 = 0$		
i) $z^3 = -64$		
let $z = r(\cos\theta + i\sin\theta)$		
$z^3 = r^3(\cos 3\theta + i\sin 3\theta)$ (De moivre's Theorem)		
$\therefore r^3(\cos 3\theta + i\sin 3\theta) = 64(\cos\pi + i\sin\pi)$	1	
$r^3 = 64$ $3\theta = \pi + 2k\pi$ for $k\in\mathbb{Z}$		
$\therefore r = 4$ $\therefore \theta = \frac{\pi + 2k\pi}{3}$	1	
$\therefore z = 4\left(\cos\left(\frac{\pi+2k\pi}{3}\right) + i\sin\left(\frac{\pi+2k\pi}{3}\right)\right)$		

MATHEMATICS Extension 2: Question 8

Suggested Solutions	Marks	Marker's Comments
for $k=0,\ z_1 = 4\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ $k=1,\ z_2 = 4(\cos\pi + i\sin\pi)$ $= -4$ $k=2,\ z_3 = 4\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$	1	
$\therefore$ the three roots are $4\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$, $4(\cos\pi + i\sin\pi)$ and $4\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$		
ii) 		
triangle is made of three smaller triangles $\therefore$ Area $= 3 \times \frac{1}{2} \times 4 \times 4 \times \sin\frac{2\pi}{3}$ $= 24 \times \frac{\sqrt{3}}{2}$ $= 12\sqrt{3}\ u^2$	1	

Question 9:

(a) (i) use the substitution $t = \tan\frac{x}{2}$,

find $\displaystyle\int \frac{dx}{\cos\alpha + \cos x}$, $0 < \alpha < \pi$

$t = \tan\frac{x}{2}$ $\therefore dt = \frac{1}{2}\left(1 + \tan^2\frac{x}{2}\right)dx$

$\frac{2\,dt}{1+t^2} = dx$, $\cos x = \frac{1-t^2}{1+t^2}$

$$\int \frac{dx}{\cos\alpha + \cos x} = \int \frac{1}{\cos\alpha + \frac{1-t^2}{1+t^2}} \times \frac{2dt}{1+t^2}$$ ✓

$$= \int \frac{2\,dt}{(1+t^2)\cos\alpha + 1 - t^2}$$

$$= \int \frac{2dt}{t^2(\cos\alpha - 1) + 1 + \cos\alpha}$$

$$= \frac{2}{\cos\alpha - 1}\int \frac{dt}{t^2 - \frac{1+\cos\alpha}{1-\cos\alpha}}$$ ✓

$1 + \cos\alpha = 2\cos^2\alpha/2$ and $1 - \cos\alpha = 2\sin^2\alpha/2$

$\therefore \frac{1+\cos\alpha}{1-\cos\alpha} = \cot^2\alpha/2$

$$\int \frac{dx}{\cos\alpha+\cos x} = \frac{2}{\cos\alpha - 1}\int \frac{dt}{t^2 - \cot^2(\alpha/2)} = \frac{2}{\cos\alpha - 1}\left[\int \frac{\frac{1}{2\cot\alpha/2}\,dt}{t - \cot\alpha/2} - \int \frac{\frac{1}{2\cot\alpha/2}\,dt}{t + \cot\alpha/2}\right]$$

$$= \frac{2}{\cos\alpha - 1} \times \frac{1}{2\cot\alpha/2}\left\{\ln\left|t - \cot\alpha/2\right| - \ln\left|t + \cot\alpha/2\right|\right\} + C$$

$$= \frac{1}{\sin\alpha}\ln\left|\frac{t + \cot\alpha/2}{t - \cot\alpha/2}\right| + C$$ ✓

$$= \frac{1}{\sin\alpha}\ln\left|\frac{\tan x/2 + \cot\alpha/2}{\tan x/2 - \cot\alpha/2}\right| + C$$

(ii) $\int \frac{3x-2}{(x+1)^2(x+3)}\,dx$

$$\frac{3x-2}{(x+3)(x+1)^2} = \frac{A}{x+3} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

multiply by $x+3$, then sub $x=-3$

$$A = \frac{3(-3)-2}{(-3+1)^2} = -11/4$$

Multiply by $(x+1)^2$, then sub $x=-1$

$$C = \frac{3(-1)-2}{(-1+3)} = -5/2$$

Sub $x=0$ $\therefore$ $\frac{-2}{3} = \frac{A}{3} + B + C$

$$-2 = A + 3B + 3C$$

$$-2 = \frac{-11}{4} + 3B - \frac{15}{2}$$

$$-8 = -11 + 12B - 30$$

$$33 = 12B \quad \therefore \quad B = \frac{33}{12} = 11/4$$

$$\int \frac{3x-2}{(x+1)^2(x+3)}\,dx = \frac{-11}{4}\int \frac{dx}{x+3} + 11/4\int \frac{dx}{x+1} - \frac{5}{2}\int \frac{dx}{(x+1)^2}$$

$$= \frac{-11}{4}\ln|x+3| + 11/4\ln|x+1| + \frac{5}{2}\,\frac{1}{x+1} + C$$

$$= \frac{11}{4}\ln\left|\frac{x+1}{x+3}\right| + \frac{5}{2(x+1)} + C$$

(b) $z_1 = -1+i$, $z_2 = \frac{1}{\sqrt{2}}\left(\cos\frac{\pi}{3} + i\sin \pi/3\right)$

$= \frac{1}{\sqrt{2}}\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{2}}{4} + i\frac{\sqrt{6}}{4}$

(i) $z_1 z_2 = (-1+i)\left(\frac{\sqrt{2}}{4} + i\frac{\sqrt{6}}{4}\right)$ ✓

$= -\frac{\sqrt{2}}{4} - i\frac{\sqrt{6}}{4} + i\frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4}$

$= -\frac{(\sqrt{2}+\sqrt{6})}{4} - i\frac{(\sqrt{6}-\sqrt{2})}{4}$ ✓ $\frac{-1-\sqrt{3}}{2\sqrt{2}} + \frac{1-\sqrt{3}}{2\sqrt{2}}i$

(ii) $z_1 = -1+i = \sqrt{2}\left(\frac{-1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

$z_1 z_2 = \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

$= \cos\left(\frac{\pi}{3} + \frac{3\pi}{4}\right) + i\sin\left(\pi/3 + \frac{3\pi}{4}\right)$

$= \operatorname{cis}\frac{13\pi}{12} = \operatorname{cis}\left(\pi + \frac{\pi}{12}\right)$ ✓

$\cos\frac{13\pi}{12} = -\frac{(\sqrt{2}+\sqrt{6})}{4}$, $\sin\frac{13\pi}{12} = \frac{\sqrt{2}-\sqrt{6}}{4}$

$\cos\frac{\pi}{12} = \frac{\sqrt{2}+\sqrt{6}}{4}$, $\sin\frac{\pi}{12} = \frac{\sqrt{6}-\sqrt{2}}{4}$ ✓

$\tan \pi/12 = \frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}+\sqrt{2}} = \frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{(\sqrt{3}-1)^2}{3-1} = \frac{4-2\sqrt{3}}{2} = 2-\sqrt{3}$ ✓

(c) $z^5 - 1 = (z-1)(z^4+z^3+z^2+z+1)$

(i) ω is the non-real root $\therefore$

$\omega^4+\omega^3+\omega^2+\omega+1=0 \therefore 1+\omega^2+\omega^4 = -\omega-\omega^3 = -(\omega+\omega^3)$ 1

(ii) divide by ω^2, we obtain

$$\frac{1}{\omega^2}+1+\omega^2 = -\left(\frac{1}{\omega}+\omega\right)$$

$$z^5-1=0 \therefore z^5 = 1 = e^{0+2k\pi i}$$

$$\therefore z = e^{2k\pi/5 \, i} \quad k=-2,-1,0,1,2$$ ✓

$$\frac{1}{\omega^2}+\omega^2 = 2\cos 2\theta = 2\cos\frac{4\pi}{5}, \quad \theta = \frac{2\pi}{5}.$$

$$\frac{1}{\omega}+\omega = 2\cos\theta = 2\cos\frac{2\pi}{5}$$

$$\frac{1}{\omega^2}+\omega^2 = -2\cos\pi/5.$$ ✓

$$-2\cos\pi/5 + 1 = -2\cos(2\pi/5)$$

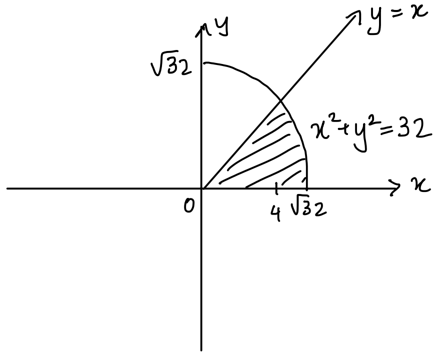
$$2\cos\left(\frac{2\pi}{5}\right) - 2\cos\frac{\pi}{5} = -1$$ ✓

$$\cos\left(\frac{2\pi}{5}\right) - \cos\frac{\pi}{5} = \frac{-1}{2}.$$

Question 10

Tuesday, 20 April 2021 11:45 am

a)



Find pt of intersection:

$x^2+y^2=32, \quad y=x$

$\therefore 2x^2=32$

$x^2=16$

$x=4, \quad x>0$

$y=\sqrt{32-x^2}$

Note: Sketch the diagram
Use integration
Check solution using
Must deal with $\sqrt{32}$ correctly for any marks.

$$\text{Area} = \int_0^4 x\,dx + \int_4^{\sqrt{32}} \sqrt{32-x^2}\,dx$$

① Integral expression for area

$$= \left[\frac{x^2}{2}\right]_0^4 + \int_{\pi/4}^{\pi/2} \sqrt{32}\cos\theta.\sqrt{32}\cos\theta\,d\theta$$

$x=\sqrt{32}\sin\theta$

$\frac{dx}{d\theta}=\sqrt{32}\cos\theta$

$x=4, \theta=\pi/4$

$x=\sqrt{32} \quad \theta=\pi/2$

$$= \frac{16}{2} - 0 + 32\int_{\pi/4}^{\pi/2} \cos^2\theta\,d\theta$$

$$= 8 + 16\int_{\pi/4}^{\pi/2} (1+\cos 2\theta)\,d\theta$$

① double angle conversion

$$= 8 + 16\left[\theta + \frac{\sin 2\theta}{2}\right]_{\pi/4}^{\pi/2}$$

$$= 8 + 16\left[\pi/2 + 0 - \left(\pi/4 + \frac{1}{2}\right)\right]$$

① correct numerical expression for area

$$= 8 + 16\left[\pi/4 - 1/2\right]$$

$$= 8 + 4\pi - 8$$

$$= 4\pi \text{ square units}$$

Question 10

Tuesday, 20 April 2021 11:58 am

b) (i) $\frac{d}{dx}\left(x+\sqrt{x^2-a^2}\right)$

$= 1 + \frac{1}{2\sqrt{x^2-a^2}} \times 2x$

$= 1 + \frac{x}{\sqrt{x^2-a^2}}$ ①

(ii) $I_0 = \int (x^2-a^2)^{-1/2}\, dx$

$= \int \frac{1}{\sqrt{x^2-a^2}}\, dx$

$= \int \frac{x+\sqrt{x^2-a^2}}{\sqrt{x^2-a^2}\left(x+\sqrt{x^2-a^2}\right)}\, dx$ ①

$= \int \frac{\frac{x}{\sqrt{x^2-a^2}} + 1}{x+\sqrt{x^2-a^2}}\, dx$ ①

$= \ln\left|x+\sqrt{x^2-a^2}\right| + C$

(iii) $I_n = \int (x^2-a^2)^{\frac{2n-1}{2}}\, dx$

$u = (x^2-a^2)^{\frac{2n-1}{2}}$ $\quad v' = 1$

$u' = \left(\frac{2n-1}{2}\right)(x^2-a^2)^{\frac{2n-3}{2}} 2x$ $\quad v = x$

$= (2n-1)(x^2-a^2)^{\frac{2n-3}{2}} x$

$I_n = x(x^2-a^2)^{\frac{2n-1}{2}} - \int (2n-1)\, x^2 (x^2-a^2)^{\frac{2n-3}{2}}\, dx$ ①

$= x(x^2-a^2)^{\frac{2n-1}{2}} - (2n-1)\int (x^2-a^2+a^2)(x^2-a^2)^{\frac{2n-3}{2}}\, dx$ ①

$= x(x^2-a^2)^{\frac{2n-1}{2}} - (2n-1)\int \left[(x^2-a^2)^{\frac{2n-1}{2}} + a^2(x^2-a^2)^{\frac{2n-3}{2}}\right] dx$

$= x(x^2-a^2)^{\frac{2n-1}{2}} - (2n-1) I_n - a^2(2n-1) I_{n-1}$

$(2n-1) I_n + I_n = x(x^2-a^2)^{\frac{2n-1}{2}} - a^2(2n-1) I_{n-1}$ ①

$I_n = \frac{1}{2n} x (x^2-a^2)^{\frac{2n-1}{2}} - \frac{(2n-1)}{2n} a^2 I_{n-1}$

Question 10

Tuesday, 20 April 2021 12:11 pm

c) (i) $\int_0^a f(x)\,dx = \int_0^{a/2} f(x)\,dx + \int_{a/2}^a f(x)\,dx$ ①

Consider $\int_{a/2}^a f(x)\,dx$ Let $x = a-u$

$\frac{dx}{du} = -1$

$dx = -du$

$\int_{a/2}^a f(x)\,dx = \int_{a/2}^0 f(a-u)(-du)$

$= \int_0^{a/2} f(a-u)\,du$ when $x = a/2,\ u = a/2$

$x = a,\ u = 0$

$= \int_0^{a/2} f(a-x)\,dx$ dummy variable ①

(or substituting x for u)

$\therefore \int_0^a f(x)\,dx = \int_0^{a/2} f(x)\,dx + \int_0^{a/2} f(a-x)\,dx$

ii, $\int_0^{\pi} \ln(\sin\theta)\,d\theta$

$$= \int_0^{\pi/2} \ln(\sin\theta)\,d\theta + \int_0^{\pi/2} \ln(\sin(\pi-\theta))\,d\theta$$

$$= \int_0^{\pi/2} \ln(\sin\theta)\,d\theta + \int_0^{\pi/2} \ln(\sin\theta)\,d\theta$$

$$= 2\int_0^{\pi/2} \ln(\sin\theta)\,d\theta$$

let $I = \int_0^{\pi/2} \ln(\sin\theta)\,d\theta$

$$= \int_0^{\pi/2} \ln\left(\sin\left(\tfrac{\pi}{2}-\theta\right)\right)d\theta \quad \text{using} \quad \int_0^{a} f(x)\,dx = \int_0^{a} f(a-x)\,dx$$

$$= \int_0^{\pi/2} \ln(\cos\theta)\,d\theta$$

$$\therefore \quad 2I = \int_0^{\pi/2} \left(\ln(\sin\theta) + \ln(\cos\theta)\right)d\theta$$

$$= \int_0^{\pi/2} \ln\left(\tfrac{1}{2}\sin 2\theta\right)d\theta$$

$$= \int_0^{\pi/2} \left(\ln(\sin 2\theta) - \ln 2\right)d\theta$$

$$= \int_0^{\pi/2} \ln(\sin 2\theta)\,d\theta - \int_0^{\pi/2} (\ln 2)\,d\theta$$

$$= \int_0^{\pi} \ln\sin(u) \times \tfrac{1}{2}\,du - \tfrac{\pi}{2}(\ln 2 - 0) \qquad \text{let} \quad \begin{aligned} u &= 2\theta \\ du &= -2d\theta \end{aligned}$$

$$2I = \tfrac{1}{2}\int_0^{\pi} \ln\sin(u)\,du - \tfrac{\pi}{2}\ln 2$$

$$\therefore \int_0^{\pi} \ln(\sin\theta)\,d\theta = 2I$$

$$= \tfrac{1}{2}\int_0^{\pi} \ln\sin(u)\,du - \tfrac{\pi}{2}\ln 2$$

$$\therefore \tfrac{1}{2}\int_0^{\pi} \ln(\sin\theta)\,d\theta = -\tfrac{\pi}{2}\ln 2$$

$$\int_0^{\pi} \ln(\sin\theta)\,d\theta = -\pi \ln 2$$