Student Name: ____________________

Maths class: ____________________

James Ruse Agricultural High School

2022 Task 1

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 1 hour and 15 minutes
- Write using black pen
- NESA approved calculators & templates may be used
- A reference sheet is provided
- In Question 6 – 10, show all relevant mathematical reasoning and/or calculations.
- Marks may not be awarded for careless or badly arranged working.

Total Marks: 45

Section I – 5 marks

- Attempt Questions 1–5
- Answer on the Multiple Choice answer sheet provided.
- Allow about 10 minutes for this section

Section II –40 marks

- Attempt Questions 6–10
- Answer on lined paper provided. Start a new page for each new question.
- Allow about 1 hour 5 minutes for this section.

The answers to all questions are to be returned in separate stapled bundles clearly labelled Question 6, Question 7, etc. Each question must show your Candidate Number.

Section I

5 marks
Attempt Questions 1 to 5
Allow approximately 15 minutes for this section

Answer on the Multiple Choice sheet provided.

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1. The point R represents the complex number z on the Argand diagram. Which of the following describes the locus of R specified by $|z-6| = |z|$? **1**

   (A) Circle center $(6,0)$ and radius $|z|$

   (B) Circle center the origin and radius $|z|$

   (C) Perpendicular bisector of $(0,0)$ and $(0,6)$

   (D) Perpendicular bisector of $(0,0)$ and $(6,0)$

2. Let $u = 4e^{\frac{3i\pi}{4}}$ and $v = r(\cos\theta + i\sin\theta)$. If $uv = 12(1+i\sqrt{3})$ **1**

   (A) $r = 3, \theta = \dfrac{5\pi}{12}$

   (B) $r = 6, \theta = \dfrac{-5\pi}{12}$

   (C) $r = 6, \theta = \dfrac{5\pi}{12}$

   (D) $r = 3, \theta = \dfrac{-5\pi}{12}$

3. If ω is an imaginary cube root of unity, then $\left(1+\omega-\omega^2\right)^{2023}$ is equal to **1**

   (A) $-2^{2023}\omega^2$ (B) $2^{2023}\omega$ (C) $-2^{2023}\omega$ (D) $2^{2023}\omega^2$

4. Consider $z = \frac{1}{2}\left(-1 + i\sqrt{3}\right)$, if a and b are constants, then $|a + bz|$ is **1**

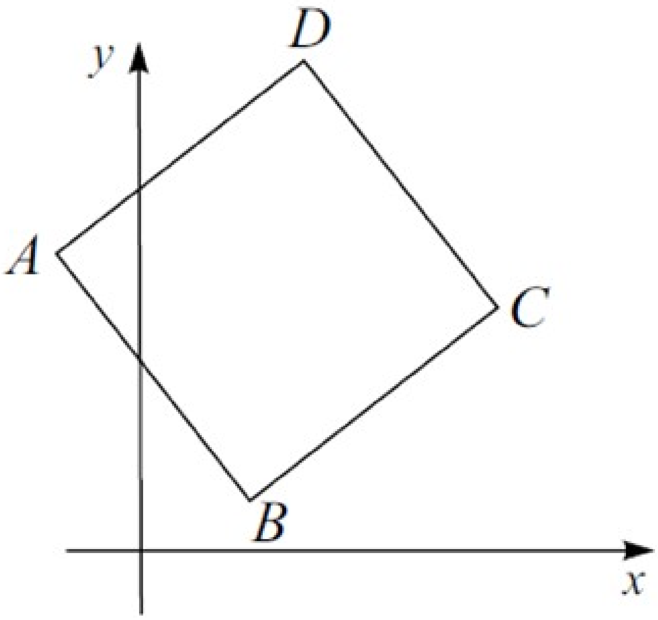
(A) $\sqrt{(a-b)^2 + 2ab}$

(B) $\sqrt{(a-b)^2 - ab}$

(C) $\sqrt{(a-b)^2 + ab}$

(D) $\sqrt{(a-b)^2 - 2ab}$

5. In the Argand diagram, $ABCD$ is a square and the vertices A and B correspond to the complex numbers u and v respectively. Which complex number corresponds to the diagonal BD?



(A) $(u-v)(1-i)$ (B) $(u-v)(1+i)$ (C) $(v-u)(1+i)$ (D) $(v-u)(1-i)$

End of Section I

Section II

40 marks
Attempt Questions 7 to 10
Allow approximately 1 hour for this section.

Write your answers on the paper supplied.
Your responses should include relevant mathematical reasoning and/or calculations.

Question 6 (8 Marks)

(a) Given that $z = \cos\dfrac{7\pi}{12} + i\sin\dfrac{7\pi}{12}$ and $\omega = 1 - i\sqrt{3}$

i. Find ω in modulus argument form. **1**

ii. Express $z\omega$ in the form $x + iy$. **2**

iii. Hence or otherwise find exact value of z in the form $x + iy$. **2**

(b) Two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ both satisfy

$$z + \bar{z} = 2|z - 1| \quad \text{and} \quad \arg(z_1 - z_2) = \frac{\pi}{4},$$

find $Im(z_1 + z_2)$. **3**

End of Question 6

Question 7 (8 Marks)

(a) i. Solve $\omega^2 = -11 - 60i$ for ω, writing your answer in the form $\omega = x + iy$, where x and y are real numbers. **2**

ii. Hence, or otherwise, solve the equation

$$z^2 - (1+4i)z - 1 + 17i = 0$$

2

(b) Let $P(z) = z^4 - 11z^3 + az^2 - 35z + b$, where a and b are real constants.

Given that $z = 5$ and $z = \dfrac{1}{2} + i\dfrac{\sqrt{3}}{2}$ are zeros of $P(z)$.

i. Explain why $z^2 - z + 1$ must be a factor of $P(z)$. **2**

ii. Hence, or otherwise, find a and b. **2**

End of Question 7

Question 8 (8 Marks)

(a) i. Sketch the locus of z showing all important features, given that

$$\arg\left(1-\frac{2}{z+1}\right)=\frac{\pi}{3}$$

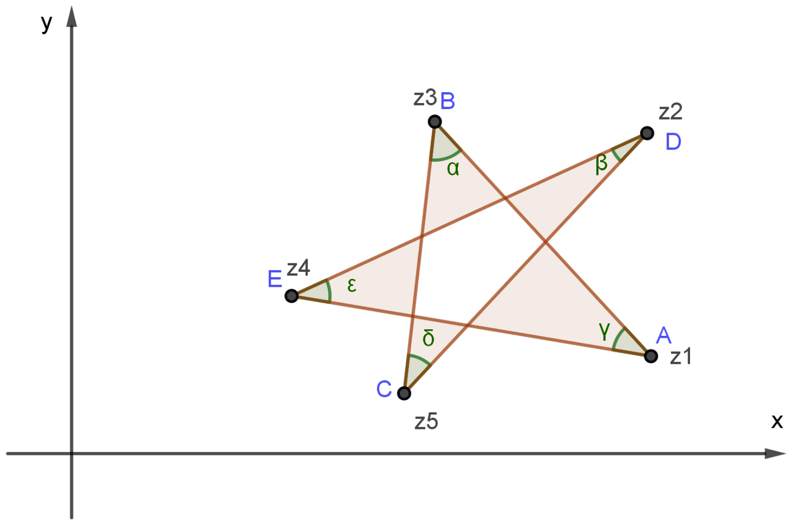
2

ii. Hence, show that the points on this locus also satisfy

$$(|z-1|-|z+1|)^2=4-|z^2-1|$$

2

(b) Given the complex numbers z_1, z_2, z_3, z_4 and z_5, see diagram below.



i. Prove that

$$\arg\left(\frac{z_4-z_1}{z_3-z_1}\right)+\arg\left(\frac{z_2-z_5}{z_2-z_4}\right)+\arg\left(\frac{z_3-z_1}{z_3-z_5}\right)+\arg\left(\frac{z_2-z_4}{z_4-z_1}\right)+\arg\left(\frac{z_3-z_5}{z_2-z_5}\right)$$
$$=\alpha+\beta+\gamma+\delta+\varepsilon-180^\circ.$$

3

ii. Hence, write the exact value of $\alpha+\beta+\gamma+\delta+\varepsilon$. **1**

End of Question 8

Question 9 (8 Marks)

(a) i. Find the roots of $z^7+1=0$, leaving your solutions in modulus-argument form. **2**

ii. Prove that the solutions of $z^6-z^5+z^4-z^3+z^2-z+1=0$ are the non real solutions of $z^7+1=0$. **1**

iii. Show that if $z^6-z^5+z^4-z^3+z^2-z+1=0$, where $z=\cos\theta+i\sin\theta$, then

$$8\cos^3\theta-4\cos^2\theta-4\cos\theta+1=0.$$

3

iv. Hence, find the exact value of

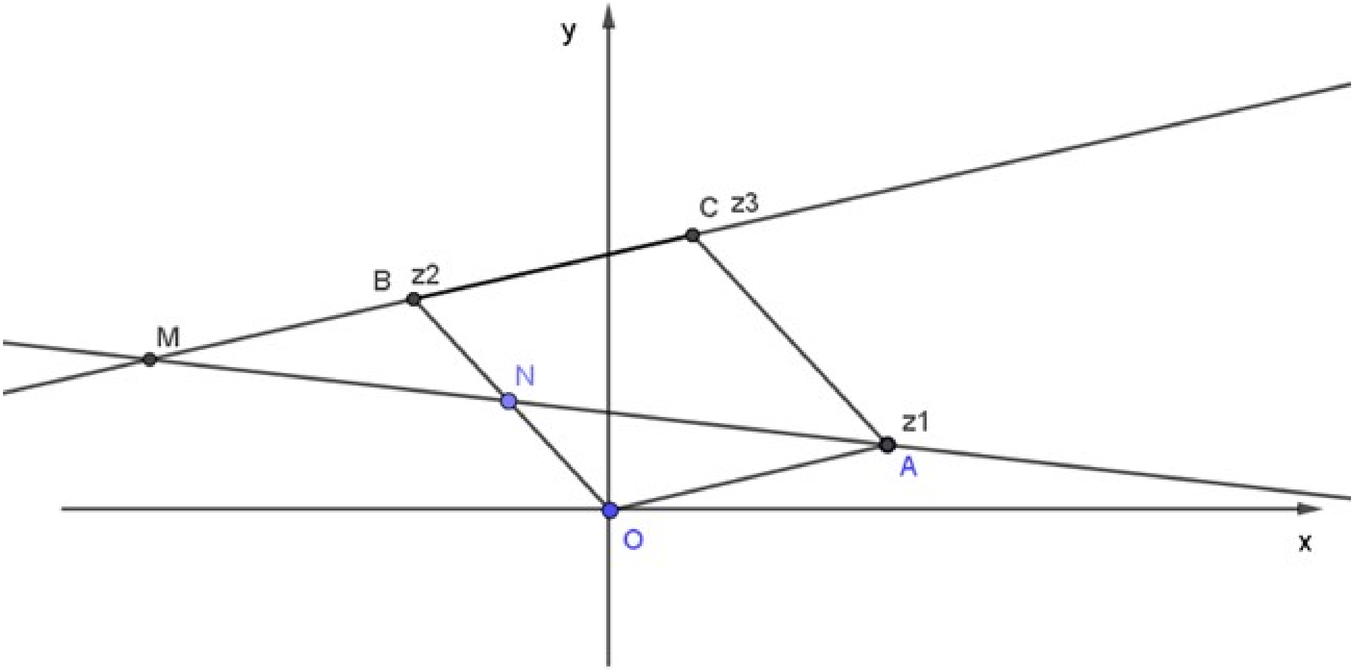
$$\cos^2\frac{\pi}{7}+\cos^2\frac{3\pi}{7}+\cos^2\frac{5\pi}{7}.$$

2

End of Question 9

Question 10 (8 Marks)

(a) Given z_1, z_2, and z_3 are the complex numbers in the Argand diagram (see diagram). Let $\omega = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$, so $z_2 = \omega z_1$ and $z_3 = (1+\omega)z_1$.



i. Show that $z_3^2 = z_1 z_2$ **2**

ii. Let N be the point on OB, where $ON = tOB$ for $0 < t < 1$ and t is a real number. The lines AN and BC intersect at point M which is represented by the complex number z. Show that $z = z_1\left(\omega + (1 - \frac{1}{t})\right)$ **3**

iii. Prove that the angle between the lines OM and CN is the argument of the complex number
$$\frac{te^{\frac{i\pi}{3}} - 1}{1 - \frac{1}{t}e^{\frac{-i\pi}{3}}}$$ **2**

iv. Hence, find $\arg\left(\frac{te^{\frac{i\pi}{3}} - 1}{1 - \frac{1}{t}e^{\frac{-i\pi}{3}}}\right)$ **1**

End of Question 10

End of paper.

Question 6

i) $\omega = 2\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$ 1 mark

ii) $z\omega = (1 \times 2)\left(\cos\left(\frac{7\pi}{12} - \frac{\pi}{3}\right) + i\sin\left(\frac{7\pi}{12} - \frac{\pi}{3}\right)\right)$ 1 mark

$= 2\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)$

$= \sqrt{2} + i\sqrt{2}$ 1 mark

iii) $z = \frac{z\omega}{\omega}$

$= \frac{\sqrt{2}+i\sqrt{2}}{1-i\sqrt{3}} \times \frac{1+i\sqrt{3}}{1+i\sqrt{3}}$ 1 mark

$= \frac{\sqrt{2}+i\sqrt{6}+i\sqrt{2}-\sqrt{6}}{4}$

$= \frac{\sqrt{2}-\sqrt{6}}{4} + i\left(\frac{\sqrt{2}+\sqrt{6}}{4}\right)$ 1 mark

iv) Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

$z_1 + \overline{z_1} = 2|z_1 - 1| \rightarrow 2x_1 = 2|x_1 - 1 + iy_1|$

$\therefore x_1 = \sqrt{(x_1 - 1)^2 + y_1^2}$ $- \textit{Equation 1}$

Similarly, $x_2 = \sqrt{(x_2 - 1)^2 + y_2^2}$ $- \textit{Equation 2}$ 1 mark

$\arg(z_1 - z_2) = \frac{\pi}{4} \rightarrow \frac{y_1 - y_2}{x_1 - x_2} = 1$

$\therefore y_1 - y_2 = x_1 - x_2$ $- \textit{Equation 3}$ 1 mark

$(\textit{Equation 1})^2 - (\textit{Equation 2})^2 \rightarrow x_1^2 - x_2^2 = (x_1 - 1)^2 - (x_2 - 1)^2 + y_1^2 - y_2^2$

$(x_1 + x_2)(x_1 - x_2) = (x_1 + x_2 - 2)(x_1 - x_2) + (y_1 - y_2)(y_1 + y_2)$

$(x_1 + x_2) = (x_1 + x_2 - 2) + (y_1 + y_2)$

$\therefore y_1 + y_2 = 2$

$\therefore Im(z_1 + z_2) = 2$ 1 mark

MATHEMATICS EXT 2: Question 7

Suggested Solutions	Marks	Marker's Comments
a i) $\omega^2 = -11-60i$		
let $\omega = x+iy$		
$(x+iy)^2 = -11-60i$		
$(x^2-y^2)+2xyi = -11-60i$		
$\therefore x^2-y^2 = -11$ ①		
$2xy = -60$		
$y = -\frac{30}{x}$ ②	①	Progress
sub ② into ①:		
$x^2 - \frac{900}{x^2} = -11$		
$x^4+11x^2-900=0$		
$(x^2+36)(x^2-25)=0$		
$\therefore x = \pm 5$ (since $x \in \mathbb{R}$)		
$x = \pm 5 \rightarrow y = \mp 6$		
$\therefore \omega = \pm(5-6i)$	①	Final answer
ii) $z^2-(1+4i)z - 1+17i = 0$		
$z = \frac{(1+4i) \pm \sqrt{(1+4i)^2-4(-1+17i)}}{2}$		
$= \frac{1+4i \pm \sqrt{-15+8i+4-68i}}{2}$		
$= \frac{1+4i \pm \sqrt{-11-60i}}{2}$	①	Progress
$= \frac{1+4i \pm (5-6i)}{2}$, from (i)		
$= 3-i, -2+5i$	①	Final

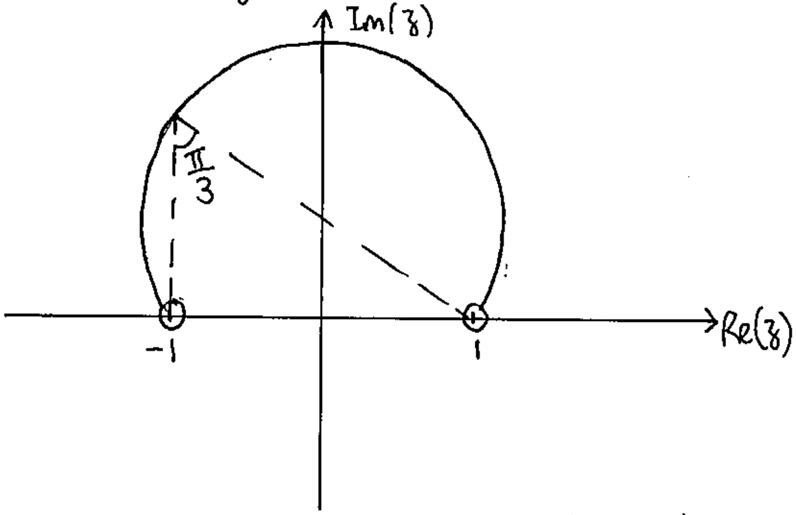
MATHEMATICS EXT 2: Question 7

Suggested Solutions	Marks	Marker's Comments
bi, $P(z)$ has real coefficients, and so all complex roots occur in conjugate pairs.		
$\therefore P\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)=0 \Rightarrow P\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)=0.$	1	Using CAT correctly
So $\left[z-\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\right]$ & $\left[z-\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)\right]$ are factors of $P(z)$.		
So, $\left[z-\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\right]\left[z-\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)\right]$		using the appropriate two roots to explain why z^2-z+1 is a factor
$=\left(z^2-2\mathrm{Re}\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)z+\left\|\frac{1}{2}+i\frac{\sqrt{3}}{2}\right\|^2\right)$	1	
$=(z^2-z+1)$ is also a factor		
$\therefore z^2-z+1$ is a factor of $P(z)$.		Students need to understand the difference between a root and a factor!!
ii, Consider sum of roots:		
$\alpha+5+\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)+\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)=-\frac{(-11)}{1}$		
$\alpha+5+1=11$		
$\alpha=5$		
Consider product of roots:		
$5^2\times\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)=\frac{b}{1}$		
$\therefore \boxed{b=25}$	1	
$P(5)=0 \Rightarrow 5^4-11\times5^3+a\times5^2-35\times5+25=0$		
$25a=900$		
$\boxed{a=36}$	1	

Maths Ext 2 Question 8

a) i. $\arg\left(1 - \dfrac{2}{z+1}\right) = \dfrac{\pi}{3}$

$\arg\left(\dfrac{z-1}{z+1}\right) = \dfrac{\pi}{3}$



(2) 1 for above the Real axis and showing a major arc of the circle, from -1, 1.
1 for angle between $z-1$ and $z+1$ and open circles at -1 and 1.

ii. From the diagram by cosine rule:

$$2^2 = |z-1|^2 + |z+1|^2 - 2|z-1||z+1|\cos\frac{\pi}{3}$$ (1)

Cosine rule with correct side lengths.

$$4 = |z-1|^2 + |z+1|^2 - |z-1||z+1|$$

$$= |z+1|^2 - 2|z-1||z+1| + |z-1|^2 + |z-1||z+1|$$

$$= (|z+1| - |z-1|)^2 + |z-1||z+1|$$

$$= (|z+1| - |z-1|)^2 + |z^2-1|$$

$$\therefore 4 - |z^2-1| = (|z+1| - |z-1|)^2$$ (1)

Hence show does not mean prove LHS = RHS.

b) From diagram:

$$\arg\left(\frac{z_4-z_1}{z_3-z_1}\right) = \angle EAB = \gamma$$

$$\arg\left(\frac{z_2-z_5}{z_2-z_4}\right) = \angle CDA = \beta$$ (1)

$$\arg\left(\frac{z_3-z_1}{z_3-z_5}\right) = \angle CBA = \alpha$$

$$\arg\left(\frac{z_3-z_5}{z_2-z_5}\right) = \angle BCD = \delta$$ (1)

$$\arg\left(\frac{z_4-z_1}{z_2-z_4}\right) = \arg(z_4-z_1) - \arg(z_2-z_4)$$

$$= \arg(z_4-z_1) - \arg(-1)(z_4-z_2)$$

$$= \arg\left(\frac{z_4-z_1}{z_4-z_2}\right) - \arg(-1)$$

$$= \angle AED - 180^\circ = \varepsilon - 180^\circ$$

$$\therefore \arg\left(\frac{z_4-z_1}{z_3-z_1}\right) + \arg\left(\frac{z_2-z_5}{z_2-z_4}\right) + \arg\left(\frac{z_3-z_1}{z_3-z_5}\right)$$
$$+ \arg\left(\frac{z_2-z_4}{z_4-z_1}\right) + \arg\left(\frac{z_3-z_5}{z_2-z_5}\right)$$
$$= \gamma + \beta + \alpha + \varepsilon - 180^\circ + \delta$$ (1)

ii. Since

$$\arg\left(\frac{z_4-z_1}{z_3-z_1}\right) + \arg\left(\frac{z_2-z_5}{z_2-z_4}\right) + \arg\left(\frac{z_3-z_1}{z_3-z_5}\right)$$
$$+ \arg\left(\frac{z_2-z_4}{z_4-z_1}\right) + \arg\left(\frac{z_3-z_5}{z_2-z_5}\right)$$
$$= \arg\left[\frac{z_4-z_1}{z_3-z_1} \times \frac{z_2-z_5}{z_2-z_4} \times \frac{z_3-z_1}{z_3-z_5} \times \frac{z_2-z_4}{z_4-z_1} \times \frac{z_3-z_5}{z_2-z_5}\right]$$
$$= \arg(1)$$
$$= 0$$
$$\therefore \alpha + \beta + \gamma + \delta + \varepsilon = 180^\circ$$ (1)

For the full mark you needed LHS = 0 somewhere as well as 180°.

If you used geometry in part i) then you would reference that for 180°.

For the geometrical proof:

1 for some evidence of proof

2 for comprehensive understanding of proof and clearly communicated

3 connected to the LHS arguments.

2023 Yr 12 Ext. 2 Task 1

Question 9

a, i, Let $z = r(\cos\theta + i\sin\theta)$

$z^7 + 1 = 0$

$z^7 = -1$

$[r(\cos\theta + i\sin\theta)]^7 = -1$

$r^7(\cos 7\theta + i\sin 7\theta) = -1$ by De Moivre's Theorem

$= e^{i(2k+1)\pi}$, $k \in \mathbb{Z}$ 1

$\therefore r = 1$, $7\theta = (2k+1)\pi$

$\theta = \frac{(2k+1)\pi}{7}$

For $k=0$, $z = 1(\cos\frac{\pi}{7} + i\sin\frac{\pi}{7})$

$k=1$, $z = 1(\cos\frac{3\pi}{7} + i\sin\frac{3\pi}{7})$

$k=2$, $z = 1(\cos\frac{5\pi}{7} + i\sin\frac{5\pi}{7})$

$k=3$, $z = 1(\cos\pi + i\sin\pi)$

$k=-3$, $z = 1(\cos(-\frac{5\pi}{7}) + i\sin(-\frac{5\pi}{7}))$

$k=-2$, $z = 1(\cos(-\frac{3\pi}{7}) + i\sin(-\frac{3\pi}{7}))$

$k=-1$, $z = 1(\cos(-\frac{\pi}{7}) + i\sin(-\frac{\pi}{7}))$ 1

ii, $z^7 + 1 = 0$

$(z+1)(z^6 - z^5 + z^4 - z^3 + z^2 - z + 1) = 0$

$\therefore z = -1$ $\quad z^6 - z^5 + z^4 - z^3 + z^2 - z + 1 = 0$

Since $z = -1$ is the only real solution of $z^7 + 1 = 0$, the solutions of $z^6 - z^5 + z^4 - z^3 + z^2 - z + 1 = 0$ are the non-real solutions of $z^7 + 1 = 0$ 1 iii, $z^6 - z^5 + z^4 - z^3 + z^2 - z + 1 = 0$

$-\frac{1}{z} - \left(-\frac{1}{z^2}\right) + \left(-\frac{1}{z^3}\right) - z^3 + z^2 - z + 1 = 0$

as $z^7 = -1$

$-\left(z^3 + \frac{1}{z^3}\right) + \left(z^2 + \frac{1}{z^2}\right) - \left(z + \frac{1}{z}\right) + 1 = 0$ 1

$-2\cos 3\theta + 2\cos 2\theta - 2\cos\theta + 1 = 0$

where $z = 1(\cos\theta + i\sin\theta)$

Now, $\cos 3\theta = \cos(2\theta + \theta)$

$= \cos 2\theta \cos\theta - \sin 2\theta \sin\theta$

$= (2\cos^2\theta - 1)\cos\theta - 2\sin^2\theta\cos\theta$

$= 2\cos^3\theta - \cos\theta - 2(1-\cos^2\theta)\cos\theta$

$= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta$

$= 4\cos^3\theta - 3\cos\theta$

1

$-2[4\cos^3\theta - 3\cos\theta] + 2[2\cos^2\theta - 1] - 2\cos\theta + 1 = 0$

$-8\cos^3\theta + 6\cos\theta + 4\cos^2\theta - 2 - 2\cos\theta + 1 = 0$

$\therefore\ 8\cos^3\theta - 4\cos^2\theta - 4\cos\theta + 1 = 0$ 1

iv, From part iii, $8\cos^3\theta - 4\cos^2\theta - 4\cos\theta + 1 = 0$

for $\theta = \pm\frac{\pi}{7}, \pm\frac{3\pi}{7}, \pm\frac{5\pi}{7}$

$\therefore$ The solutions of $8x^3 - 4x^2 - 4x + 1 = 0$ are

$\cos\frac{\pi}{7}, \cos\frac{3\pi}{7}, \cos\frac{5\pi}{7}$

$\cos^2\frac{\pi}{7} + \cos^2\frac{3\pi}{7} + \cos^2\frac{5\pi}{7}$

1

$= \left(\cos\frac{\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{5\pi}{7}\right)^2 - 2\left(\cos\frac{\pi}{7}\cos\frac{3\pi}{7} + \cos\frac{3\pi}{7}\cos\frac{5\pi}{7} + \cos\frac{\pi}{7}\cos\frac{5\pi}{7}\right)$

$= \left[-\left(\frac{-4}{8}\right)\right]^2 - 2\left(\frac{-4}{8}\right)$

$= \frac{5}{4}$ 1

Question 10 :

z_1, ωz_1 and $(1+\omega)z_1$

(i) $z_3^2 = (1+\omega)^2 z_1^2$

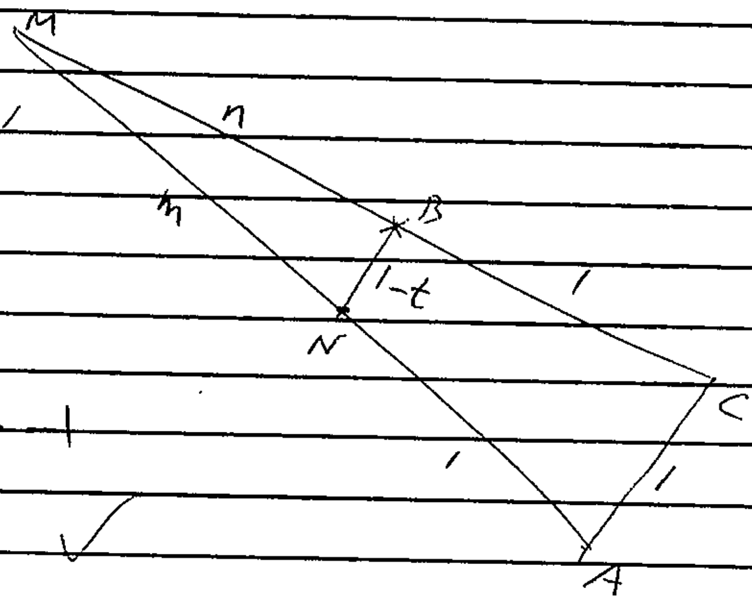
$= (1+2\omega+\omega^2) z_1^2$ as $1+\omega+\omega^2 = 0$.

$= \omega z_1^2$ ✓

$= \omega z_1 \times z_1$

$= z_1 z_2$ ✓

(ii) Let z be the complex number representing M.



$\frac{n}{n+1} = \frac{m}{m+1} = \frac{1-t}{1}$ ✓

$n = (n+1)(1-t)$

$= (n+1) - (n+1)t$

$t = \frac{1}{n+1}$ or $n = \frac{1}{t} - 1$

$\boxed{m = n}$ ✓

$\vec{OM} = (n+1)\vec{ON} - n\vec{OA}$

$z = (n+1)t z_2 + \left(1 - \frac{1}{t}\right) z_1$

$= \omega z_1 + \left(1 - \frac{1}{t}\right) z_1$ ✓

$= \left(\omega + 1 - \frac{1}{t}\right) z_1$

(iii) The angle between OM and CN is the argument of the complex number

$$\frac{tz_2 - z_3}{z} = \frac{twz_1 - (1+w)z_1}{\left[w + 1 - \frac{1}{t}\right]z_1} \checkmark$$

$$= \frac{tw - (1+w)}{w + 1 - \frac{1}{t}}$$

$$= \frac{tw - e^{i\pi/3}}{e^{i\pi/3} - \frac{1}{t}}$$

$$= \frac{te^{i\pi/3} - 1}{1 - \frac{1}{t}e^{-i\pi/3}} \checkmark$$

(iv) $$\frac{te^{i\pi/3} - 1}{1 - \frac{1}{t}e^{-i\pi/3}} = te^{i\pi/3}\left(\frac{te^{i\pi/3} - 1}{te^{i\pi/3} - 1}\right) = te^{i\pi/3}$$

$$\arg\left(\frac{te^{i\pi/3} - 1}{1 - \frac{1}{t}e^{-i\pi/3}}\right) = \arg\left(te^{i\pi/3}\right) = \pi/3 \checkmark$$