Student Name: ____________________

Maths class: ____________________

James Ruse Agricultural High School

2023 **YEAR 12 Task 1 HSC Examination**

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 75 minutes
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- In Questions 6–10, show relevant mathematical reasoning and/ or calculations

Total marks: 45

Section I – 5 marks

- Attempt Questions 1–5
- Allow about 7 minutes for this section

Section II – 40 marks

- Attempt Questions 6–10
- Allow about 68 minutes for this section

Start your answer to each question on a separate piece of paper.

Section I

5 marks
Attempt Questions 1-5
Allow about 7 for this section
Write your answers on the multiple choice answer sheet provided.

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1. Evaluate $i^{2022} + i^{2023} + i^{2024}$

   (A) i

   (B) $-i$

   (C) 1

   (D) -1

2. A complex number z is defined such that $Re(z) > 0$ and $Im(z) < 0$. Which of the following is a suitable range for $\arg(z^3)$?

   (A) $\left(\frac{\pi}{2}, 2\pi\right)$

   (B) $\left(0, \frac{\pi}{2}\right)$

   (C) $(0,2\pi)$

   (D) $\left(-\frac{\pi}{2}, 0\right)$

3. If the complex numbers ω and z represent two diagonally opposite vertices of a square, then the other two vertices could be given by the complex numbers:

   (A) $\frac{1}{2}(z - w) + \frac{1}{2}i(z - w)$ and $\frac{1}{2}(z - w) - \frac{1}{2}i(w - z)$

   (B) $\frac{1}{2}(\omega + z) + \frac{1}{2}i(\omega + z)$ and $\frac{1}{2}(\omega + z) - \frac{1}{2}i(\omega + z)$

   (C) $\frac{1}{2}(\omega - z) + \frac{1}{2}i(\omega - z)$ and $\frac{1}{2}(\omega - z) - \frac{1}{2}i(\omega - z)$

   (D) $\frac{1}{2}(\omega + z) + \frac{1}{2}i(\omega - z)$ and $\frac{1}{2}(\omega + z) - \frac{1}{2}i(\omega - z)$

4. Let $a, b, c \in \mathbb{R}$. If the equation $x^4 + ax^3 + bx^2 + cx + 1 = 0$ has no real roots and if at least one root is of modulus one, then which of the following is true?

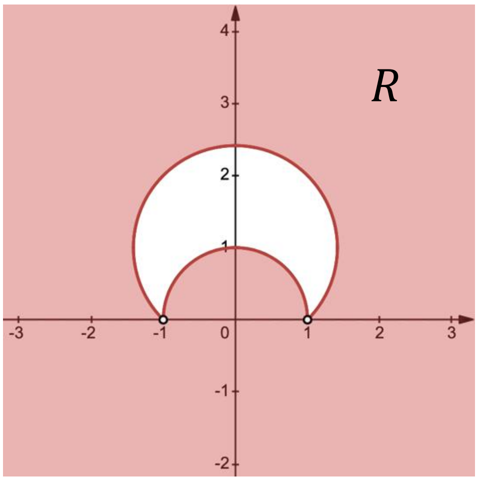
   (A) $b = c$

   (B) $a = b$

   (C) $a = c$

   (D) None of the above

5. Which of the following options represents the shaded region, R, below?



(A) $\frac{\pi}{2} \leq \operatorname{Arg}\left(\frac{z-1}{z+1}\right) \leq \frac{3\pi}{4}$

(B) $\frac{\pi}{4} \leq \operatorname{Arg}\left(\frac{z-1}{z+1}\right) \leq \frac{\pi}{2}$

(C) $\operatorname{Arg}\left(\frac{z-1}{z+1}\right) \leq \frac{\pi}{2}$ and $\operatorname{Arg}\left(\frac{z-1}{z+1}\right) \geq \frac{3\pi}{4}$

(D) $\operatorname{Arg}\left(\frac{z-1}{z+1}\right) \leq \frac{\pi}{4}$ and $\operatorname{Arg}\left(\frac{z-1}{z+1}\right) \geq \frac{\pi}{2}$

Section II begins on the next page

Section II

40 marks
Attempt Questions 6-10
Allow about 68 minutes for this section
Start each question on a new page. Extra paper is available.
In Questions 6-10, your responses should include relevant mathematical reasoning and/ or calculations.

Question 6 (8 marks) **Start a new page**.

(a) If $z = 3 + 4i$ and $\omega = 2 - i$, simplify the following, giving your answer in the form $x + iy$

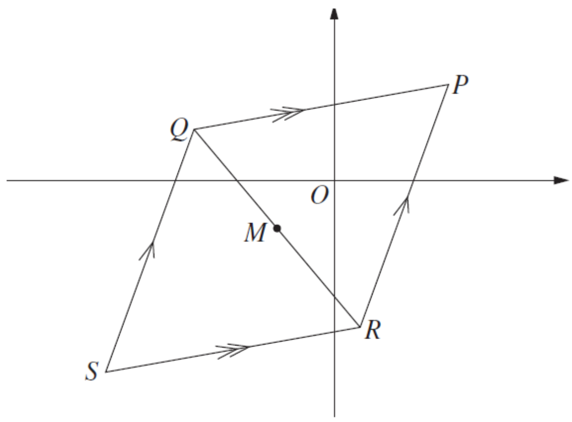
(i) $z - 2\omega$ **1**

(ii) $z\bar{z} - \omega\bar{\omega}$ **1**

(iii) $\frac{z}{\omega}$ **2**

(b) Given two complex numbers $z = \sqrt{3} + i$ and $\omega = 6\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$. Express $\frac{z\omega}{4i}$ in the form $r(\cos\theta + i\sin\theta)$. **2**

(c) The point P on the Argand diagram represents the complex number z. The points Q and R represent the points ωz and $\bar{\omega}z$ respectively, where $\omega = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$. The point M is the midpoint of QR.



(i) Find the complex number representing M in terms of z. **1**

(ii) The point S is chosen so that $PQSR$ is a parallelogram. Find the complex number represented by S. **1**

Question 7 (8 marks) **Start a new page**.

(a) The complex number z is such that $|z| = 1$ and $\arg(z) = \frac{\pi}{4}$.

(i) Find the coordinates of points A representing z^2, and B representing $z^2 - \bar{z}$ **2**

respectively and plot the points on a common half-page Argand diagram. **2**

(ii) Prove that $|z^2 - \bar{z}| = |z + 1|$.

(b) (i) Find the square roots of $-63 + 16i$ **2**

(ii) Hence or otherwise, solve for z in the equation $2z^2 + (\sqrt{63})iz - 2i = 0$. Write your answers in the form $x + iy$. **2**

Question 8 (8 marks) **Start a new page**.

(a) (i) Show that $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$ **1**

(ii) Hence or otherwise, prove that the solution to θ in the equation $\cos\theta = 2$ is that of $e^{i\theta} = 2 \pm \sqrt{3}$ **1**

(b) The complex number z satisifies the relationship $5(z + i)^n = (4 + 3i)(1 + iz)^n, n \in \mathbb{R}$. Show that z is a real number. **3**

(c) Find all complex numbers z satisfying both $\text{Arg}(z^2 - 4) = \frac{5\pi}{6}$ and $\text{Arg}(z^2 + 4) = \frac{\pi}{3}$ **3**

Question 9 (8 marks) **Start a new page**.

(a) (i) Let $a \in \mathbb{C}$ and $b \in \mathbb{R},\ b \geq 0$. Show that the equation **2**

$$z\bar{z} = a\bar{z} + \bar{a}z + b$$

for $z \in \mathbb{C}$ can be written in the form $|z - a| = r$, where $r \geq 0$.

Let A and B be two points on the complex plan representing $2 + 3i$ and $1 + 2i$ respectively. The point P, representing the complex number z, moves such that $PA = \sqrt{2}PB$.

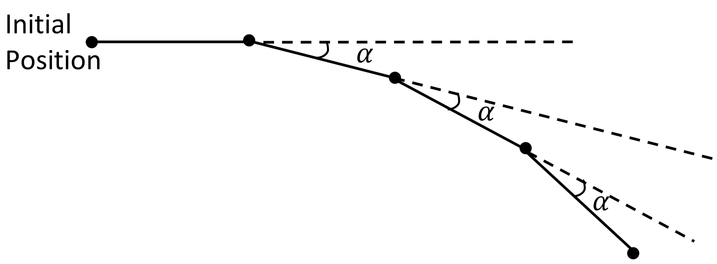
(ii) Show that the equation of the path traced out by P is a circle with equation $C_1: z\bar{z} = i\bar{z} - iz + 3$. State it's centre and radius. **3**

Let Q, representing ω, be a point on the circle C_1.

(iii) Show that the circle C_2 given by $\left|z - \left(\omega + \frac{\omega - i}{|\omega - i|}\right)\right| = 1$ touches C_1 at Q externally. **3**

Question 10 (8 marks) **Start a new page**.

(a) A student is in the initial stages of programming a simple robotic car. So far, the car only moves in a straight line 1 unit at a time. After each 1 unit move forward, it changes direction by turning to the right through an acute angle of α radians and then moving forward another 1 unit. An example of a possible path is shown below with 4 forward moves.



During a test run, the car starts from the origin and initially moves to the right. It moves forward a total of 24 times before finally stopping.

(i) Explain why the final position of the car can be represented by the expression **1**

$$1 + e^{-ia} + e^{-i2a} + \cdots + e^{-i23a}.$$

(ii) If the car's final position is the same as its initial position, find all possible values of α. **2**

(iii) If the **only** time the car reaches its initial position during its path is when it arrives at its final position, find all possible values of α, giving reasons. **2**

The test run is considered to be *successful* if the car stops within a one unit distance from its starting point.

(iv) Show that for the test to be considered *successful*, then α must satisfy $|\sin(12\alpha)| \leq \left|\sin\left(\frac{\alpha}{2}\right)\right|$. **3**

END OF EXAMINATION

2023 Year 12 Task 1 Extension 2

1. Evaluate $i^{2022} + i^{2023} + i^{2024}$

A) i

B) $-i$

C) 1

D) -1

2. If the complex numbers ω and z represent two diagonally opposite vertices of a square, then the other two vertices could be given by the complex numbers:

(A) $\frac{1}{2}(z-w)+\frac{1}{2}i(z-w)$ and $\frac{1}{2}(z-w)-\frac{1}{2}i(w-z)$
(B) $\frac{1}{2}(\omega+z)+\frac{1}{2}i(\omega+z)$ and $\frac{1}{2}(\omega+z)-\frac{1}{2}i(\omega+z)$
(C) $\frac{1}{2}(\omega-z)+\frac{1}{2}i(\omega-z)$ and $\frac{1}{2}(\omega-z)-\frac{1}{2}i(\omega-z)$
(D) $\frac{1}{2}(\omega+z)+\frac{1}{2}i(\omega-z)$ and $\frac{1}{2}(\omega+z)-\frac{1}{2}i(\omega-z)$

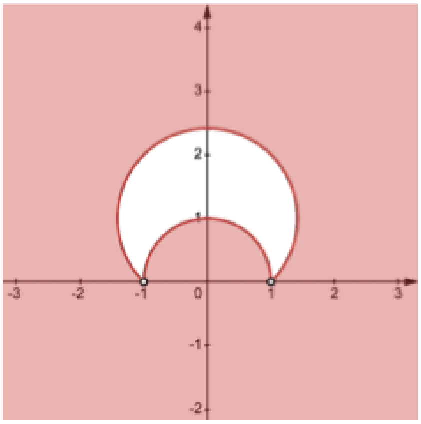
3. A complex number z_1 is defined such that $Re(z_1) > 0$ and $Im(z_1) < 0$. Which of the following is a suitable range for $\arg(z_1^3)$?

A) $\left(\frac{\pi}{2}, 2\pi\right)$
B) $\left(0, \frac{\pi}{2}\right)$
C) $(0, 2\pi)$
D) $\left(-\frac{\pi}{2}, 0\right)$

4. Let $a, b, c \in \mathbb{R}$. If the equation $x^4 + ax^3 + bx^2 + cx + 1 = 0$ has no real roots and if at least one root is of modulus one, then which of the following is true?

(A) $b = c$
(B) $a = b$
(C) $a = c$
(D) None of the above.

5. Which of the following options represents the shaded region below?



(A) $\frac{\pi}{2} \leq \operatorname{Arg}\left(\frac{z-1}{z+1}\right) \leq \frac{3\pi}{4}$
(B) $\frac{\pi}{4} \leq \operatorname{Arg}\left(\frac{z-1}{z+1}\right) \leq \frac{\pi}{2}$
(C) $\operatorname{Arg}\left(\frac{z-1}{z+1}\right) \leq \frac{\pi}{2}$ and $\operatorname{Arg}\left(\frac{z-1}{z+1}\right) \geq \frac{3\pi}{4}$
(D) $\operatorname{Arg}\left(\frac{z-1}{z+1}\right) \leq \frac{\pi}{4}$ and $\operatorname{Arg}\left(\frac{z-1}{z+1}\right) \geq \frac{\pi}{2}$

Student Number ______________

Question 6

(a) $Z = 3+4i$, $\omega = 2-i$

(i) $Z - 2\omega = 3+4i-2(2-i)$

$= 3+4i-4+2i$

$= -1+6i$

(ii) $Z\bar{Z} - \omega\bar{\omega} = |Z|^2 - |\omega|^2$

$= 3^2+4^2-(2^2+(-1)^2)$

$= 9+16-4-1$

$= 20$

(iii) $\frac{Z}{\omega} = \frac{Z\bar{\omega}}{|\omega|^2} = \frac{(3+4i)(2+i)}{2^2+(-1)^2} = \frac{6+3i+8i-4}{5} = \frac{2}{5}+\frac{11}{5}i$

(b) $Z = \sqrt{3}+i$ and $\omega = 6\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$

$= 2\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right) = 2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$

$\frac{Z\omega}{4i}$

$= \frac{2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)\times 6\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)}{4\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)}$

$= 3\left(\cos\left(\frac{\pi}{6}+\frac{2\pi}{3}-\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{6}+\frac{2\pi}{3}-\frac{\pi}{2}\right)\right)$

$= 3\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$

Student Number ____________

(c)

(i) the complex number representing M

is $\frac{\omega z + \bar{\omega} z}{2} = \frac{(\omega + \bar{\omega}) z}{2}$

$= \frac{2 \cos 2\pi/3 \; z}{2}$

$= \cos \frac{2\pi}{3} z$

$= \frac{-z}{2}$ ✓

(ii) $SQ = RP$

let α the complex number representing S

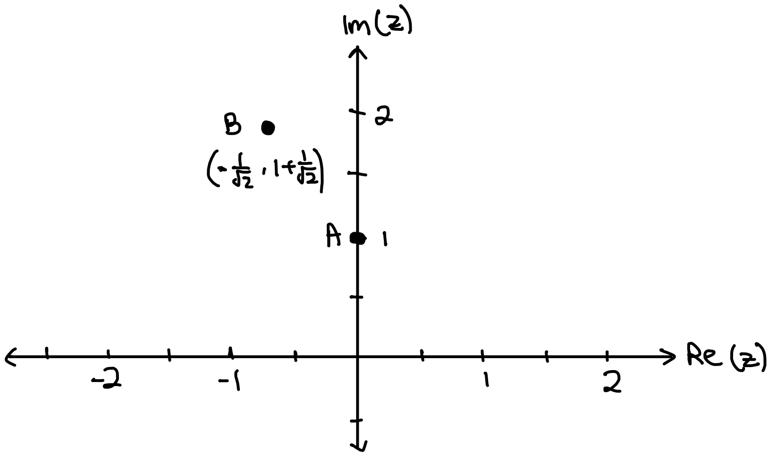
$\omega z - \alpha = z - \bar{\omega} z$

$\alpha = (\omega + \bar{\omega}) z - z$

$= -z - z$

$= -2z.$ ✓

MATHEMATICS Extension 2: Question…7…

Suggested Solutions	Marks	Marker's Comments
a) i) $z^2 = \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^2$		
$= \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$ (De Moivre's)		
$= i$		
$z^2 - \bar{z} = i - \left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$		
$= i - \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$		
$= -\frac{1}{\sqrt{2}} + i\left(1 + \frac{1}{\sqrt{2}}\right)$	1	for values of z^2 and $z^2 - \bar{z}$
	1	for plotting on Argand diagram
ii) $\lvert z^2 - \bar{z} \rvert = \lvert z + 1 \rvert$		
LHS $= \left\lvert -\frac{1}{\sqrt{2}} + i\left(1 + \frac{1}{\sqrt{2}}\right)\right\rvert$		
$= \sqrt{\left(-\frac{1}{\sqrt{2}}\right)^2 + \left(1 + \frac{1}{\sqrt{2}}\right)^2}$	1	must show the $\left(-\frac{1}{\sqrt{2}}\right)^2$
$= \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(1 + \frac{1}{\sqrt{2}}\right)^2}$		
RHS $= \left\lvert \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i + 1 \right\rvert$		
$= \sqrt{\left(\frac{1}{\sqrt{2}} + 1\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2}$		
= LHS	1	
$\therefore \lvert z^2 - \bar{z} \rvert = \lvert z + 1 \rvert$		

MATHEMATICS Extension 2: Question…7…

Suggested Solutions	Marks	Marker's Comments
b) i) $\sqrt{-63+16i} = \sqrt{1+2(8i)+64i^2}$	1	
$= \sqrt{1+2(8i)+(8i)^2}$		
$= \sqrt{(1+8i)^2}$		
$= \pm(1+8i)$	1	
ii) $z = \dfrac{-\sqrt{63}i \pm \sqrt{(\sqrt{63}i)^2-4(2)(-2i)}}{2(2)}$		
$= \dfrac{-3\sqrt{7}i \pm \sqrt{-63+16i}}{4}$	1	
$= \dfrac{-3\sqrt{7}i \pm (1+8i)}{4}$		
$= \dfrac{1}{4} + \dfrac{8-3\sqrt{7}}{4}i$ or $-\dfrac{1}{4} - \dfrac{8+3\sqrt{7}}{4}i$	1	

MATHEMATICS EXT 2: Question 8

Suggested Solutions	Marks	Marker's Comments

a i) $e^{i\theta} = \cos\theta + i\sin\theta$ (1)

$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta)$

$e^{-i\theta} = \cos\theta - i\sin\theta$ (2)

(1) + (2): $e^{i\theta} + e^{-i\theta} = 2\cos\theta$

$\therefore \cos\theta = \dfrac{e^{i\theta}+e^{-i\theta}}{2}$ — **1**

ii) $\cos\theta = 2$

$\dfrac{e^{i\theta}+e^{-i\theta}}{2} = 2$

$e^{i\theta} + e^{-i\theta} = 4$

$e^{i\theta} - 4 + e^{-i\theta} = 0$

$(e^{i\theta})^2 - 4e^{i\theta} + 1 = 0$

$e^{i\theta} = \dfrac{4 \pm \sqrt{16 - 4\times 2\times 1}}{2}$

$= \dfrac{4 \pm \sqrt{12}}{2}$

$e^{i\theta} = 2 \pm \sqrt{3}$ — **1**

b) $5(z+i)^n = (4+3i)(1+iz)^n$

$|5(z+i)^n| = |(4+3i)(1+iz)^n|$ — **1** Recognising to take modulus of both sides

$5|z+i|^n = 5|1+iz|^n$

$|z+i|^n = |1+iz|^n$

$|z+i| = |1+iz|$

$|z+i| = |i||z-i|$

$|z+i| = |z-i|$ — **1** Use modulus properties to simplify expression

$\therefore$ z lies on the perpendicular bisector of i and $-i$, which is the real axis, i.e. z must be a real number. — **1** Valid explanation

MATHEMATICS EXT 2: Question 8		
Suggested Solutions	Marks	Marker's Comments

c, ALGEBRAIC APPROACH:

Let $z^2 = x + iy$

$\therefore \operatorname{Arg}(x + iy - 4) = \frac{5\pi}{6}$

$\Rightarrow \tan^{-1}\left(\frac{y}{x-4}\right) = \frac{5\pi}{6}$

$\Rightarrow \frac{y}{x-4} = -\frac{1}{\sqrt{3}}$

$\Rightarrow y\sqrt{3} = 4 - x$ ①

$\therefore \operatorname{Arg}(x + iy + 4) = \frac{\pi}{3}$

$\Rightarrow \tan^{-1}\left(\frac{y}{x+4}\right) = \pi/3$

$\Rightarrow \frac{y}{x+4} = \sqrt{3}$

$\Rightarrow y = \sqrt{3}x + 4\sqrt{3}$ ②

(1)

Sub ② into ①:

$\sqrt{3}(\sqrt{3}x + 4\sqrt{3}) = 4 - x$

$3x + 12 = 4 - x$

$4x = -8$

$x = -2$

$\therefore y = 2\sqrt{3}$

$\therefore z^2 = -2 + 2\sqrt{3}i$ (1)

$= 1 + 2 \times 1 \times \sqrt{3}i + (\sqrt{3}i)^2$

$z^2 = (1 + \sqrt{3}i)^2$

$\therefore z = \pm(1 + \sqrt{3}i)$ (1)

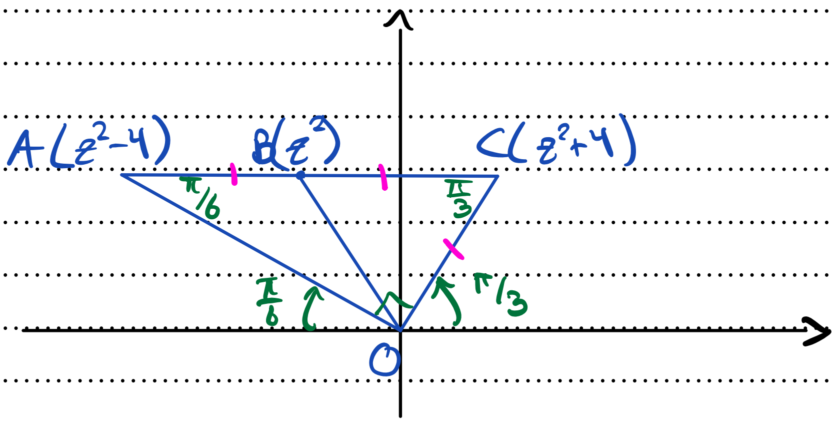
MATHEMATICS EXT 2: Question 8

Suggested Solutions	Marks	Marker's Comments
<u>Note:</u> Letting $z = x + iy$ can work, but it is very ugly. Since it COULD work, one mark was assigned for getting to		
$\frac{2xy}{x^2 - y^2 - 4} = -\frac{1}{\sqrt{3}}$ & $\frac{2xy}{x^2 - y^2 + 4} = \sqrt{3}$		

MATHEMATICS EXT 2: Question 8

Suggested Solutions	Marks	Marker's Comments
GEOMETRIC METHOD:		

$\operatorname{Arg}(z^2-4) = \frac{5\pi}{6}$ & $\operatorname{Arg}(z^2+4) = \frac{\pi}{3}$



$\angle OCB = \pi/3$ (alternate angles)
$\angle OAC = \pi/6$ (alternate angles)

Now $\frac{OC}{AC} = \sin\frac{\pi}{6}$

$\frac{OC}{AC} = \frac{1}{2}$

$\therefore OC = \frac{1}{2} AC$
$= 4$

But $BC = 4 = OC$

$\therefore \triangle OBC$ is isosceles with included angle of $\pi/3$

$\therefore \triangle OBC$ is equilateral

$\therefore OB = 4$

$\Rightarrow z^2 = 4e^{\frac{2\pi i}{3}}$

$\Rightarrow z = \pm 2e^{\frac{\pi i}{3}}$

$= \pm(1+\sqrt{3}i)$

Question 9

a)

i. $z\bar{z} = a\bar{z} + \bar{a}z + b$

Let $z = x + iy$ and $a = c + id$

$x^2 + y^2 = (c + id)(x - iy) + (c - id)(x + iy) + b$

$= (cx - cyi + dxi + dy) + (cx + cyi - dxi + dy) + b$

$= 2(cx + dy) + b$ 1 mark

$x^2 - 2cx + y^2 - 2dy = b$

$(x^2 - 2cx + c^2) + (y^2 - 2dy + d^2) = b + c^2 + d^2$

$(x - c)^2 + (y - d)^2 = b + c^2 + d^2$

$|(x - c) + (y - d)i|^2 = b + c^2 + d^2$

$|z - a|^2 = b + c^2 + d^2$

$|z - a| = \sqrt{b + |a|^2}$

$= r$ (Since $b, c, d \in \mathbb{R}$)

$|z - a| = r$ where $r = \sqrt{b + |a|^2}$ 1 mark

ii. Let $P(x, y)$ represent the complex number $x + iy$

$PA = \sqrt{2}PB \Rightarrow |PA|^2 = 2|PB|^2$

$(x - 2)^2 + (y - 3)^2 = 2(x - 1)^2 + 2(y - 2)^2$ 1 mark for interpreting condition algebraically

(Note: It was given $PA = \sqrt{2}PB$, NOT $\overrightarrow{PA} = \sqrt{2}\overrightarrow{PB}$, if you equate the two without a square root or having squared both sides you don't receive the first mark)

$(x^2 - 4x + 4) + (y^2 - 6y + 9) = (2x^2 - 4x + 2) + (2y^2 - 8y + 8)$

$x^2 + y^2 - 2y = 3$

$x^2 + (y^2 - 2y + 1) = 4$

$x^2 + (y - 1)^2 = 4$

$\therefore$ The vector equation of this circle is given by $|z - i| = 2$

which can be expressed as $z\bar{z} = i\bar{z} - iz + b$ 1 mark

Since $r = \sqrt{b + |a|^2}$ (proven in part i)

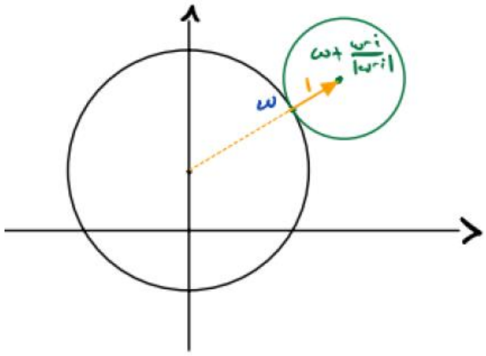
$2 = \sqrt{b + 1^2}$

$b + 1 = 4$

$b = 3$

$\therefore$ P traces out a path given by $z\bar{z} = i\bar{z} - iz + 3$ which is a circle with radius 2 and centre $(0,1)$. 1 mark

iii. The circles are externally tangential if the distance between the centres are equal to the sum of the two radii. That is,



1 mark for interpreting what is required geometrically

C_2 is given by: $\left|z-\left(\omega+\dfrac{\omega-i}{|\omega-i|}\right)\right|=1$

ω lies on $C_1 \Rightarrow |\omega-i|=2$

$\therefore \left|z-\left(\omega+\dfrac{\omega-i}{2}\right)\right|=1$ 1 mark for simplifying equation of C_2 using the fact that ω lies on C_1

$\left|z-\left(\dfrac{3\omega-i}{2}\right)\right|=1$

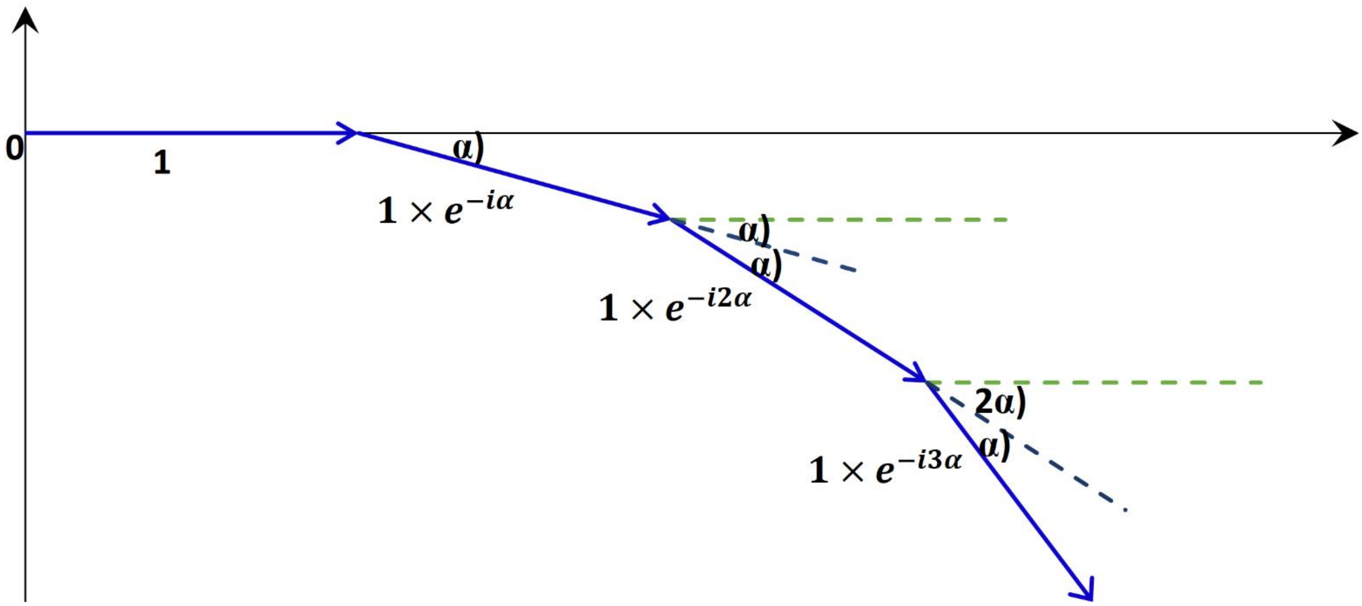
$\therefore$ Distance between the two centres is given by:

$$\begin{aligned}\left|i-\left(\frac{3\omega-i}{2}\right)\right| &= \left|\frac{2i-3\omega+i}{2}\right| \\ &= \left|\frac{3i-3\omega}{2}\right| \\ &= \frac{3}{2}|i-\omega| \\ &= \frac{3}{2}\times 2 \\ &= 3\end{aligned}$$

1 mark for completing the proof

Question 10

i. The path of the car is shown below.



Each forward movement has a distance of 1 unit and turns to the right by α each time. This is equivalent to multiply by $\boldsymbol{e^{-i\alpha}}$ each time, where $0 < \alpha < \frac{\pi}{2}$.

Therefore, the final position is given by :

$$1 + 1 \times e^{-i\alpha} + 1 \times \left(e^{-i\alpha}\right)^2 + 1 \times \left(e^{-i\alpha}\right)^3 + \cdots 1 \times \left(e^{-i\alpha}\right)^{23}$$
$$= 1 + e^{-i\alpha} + e^{-i2\alpha} + e^{-i3\alpha} + \cdots + e^{-i23\alpha}$$

(1 mark for providing correct explanation: stating each movement has a distance of 1 unit, and turns to the right by α is to multiply by $\boldsymbol{e^{-i\alpha}}$)

ii. The car's final position is the same as its initial position, which means:

$$1 + e^{-i\alpha} + e^{-i2\alpha} + e^{-i3\alpha} + \cdots + e^{-i23\alpha} = 0$$

That is:

$$\frac{1 - e^{-24i\alpha}}{1 - e^{-i\alpha}} = 0 \qquad (1\ mark)$$

$$e^{-24i\alpha} = 1,$$

(The denominator can not be zero since $0 < \alpha < \frac{\pi}{2}$)

Therefore:

$$24\alpha = 2k\pi$$

$$\alpha = \frac{2k\pi}{24}$$

Where:

$$0 < \frac{2k\pi}{24} < \frac{\pi}{2}, \qquad k \in Z$$

$$0 < k < 6$$

$$\therefore \qquad k = 1, 2, 3, 4, 5$$

Therefore, possible values of α are

$$\frac{\pi}{12}, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{10\pi}{24} \qquad (1\ mark)$$

iii. In order for the car to reach the original position for the first time, we need values of k which has no factors in common with 24 except 1.

That is $k = 1, \; 5$

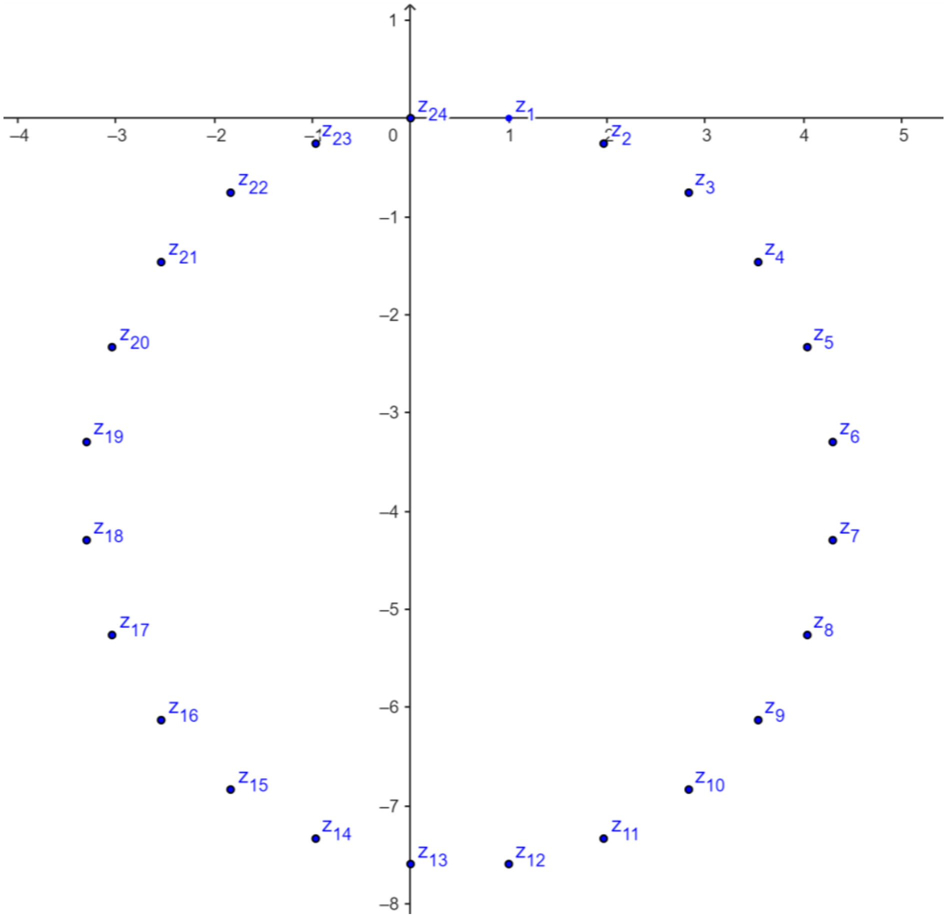
$$\therefore \; k = 1, \qquad \alpha = \frac{\pi}{12}$$

$$k = 5, \qquad \alpha = \frac{10\pi}{24}$$

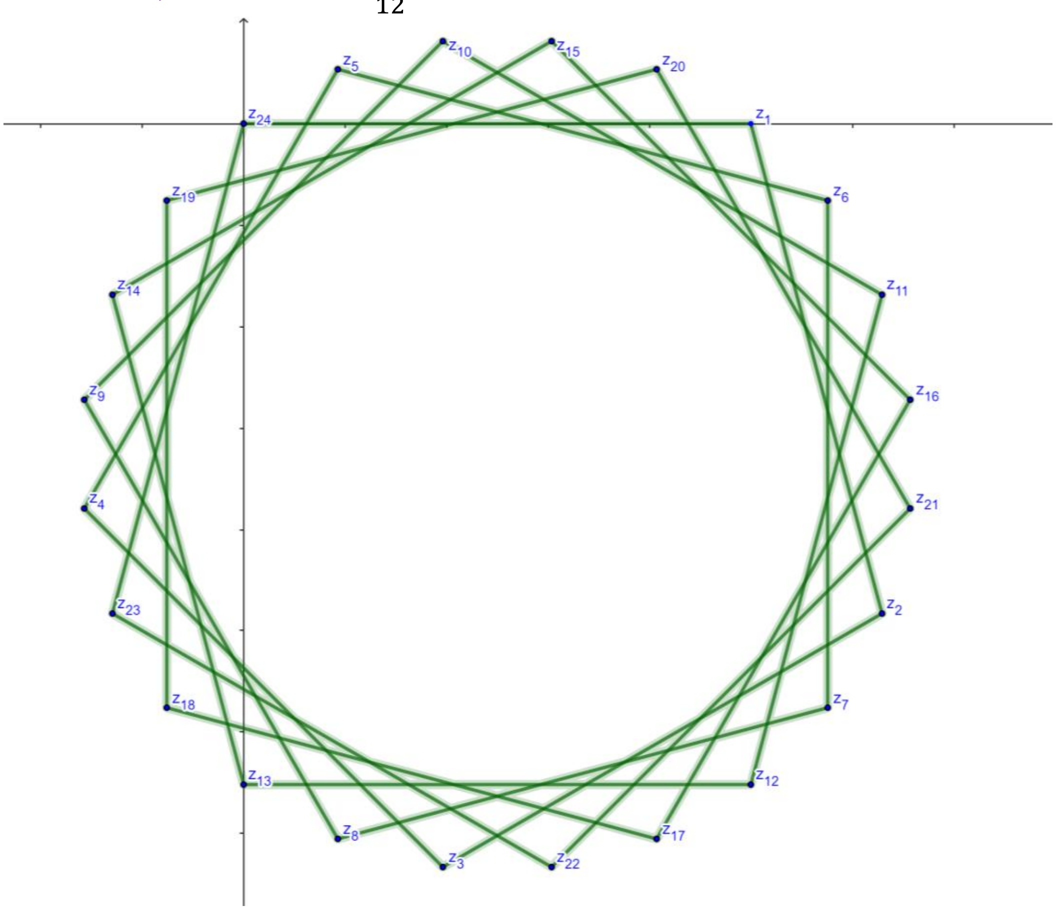
(1 mark for providing correct reason, 1 mark for correct answer)

(Common mistakes: many students assume that the path of the car must be regular polygons. The paths for three cases are shown below for your reference.)

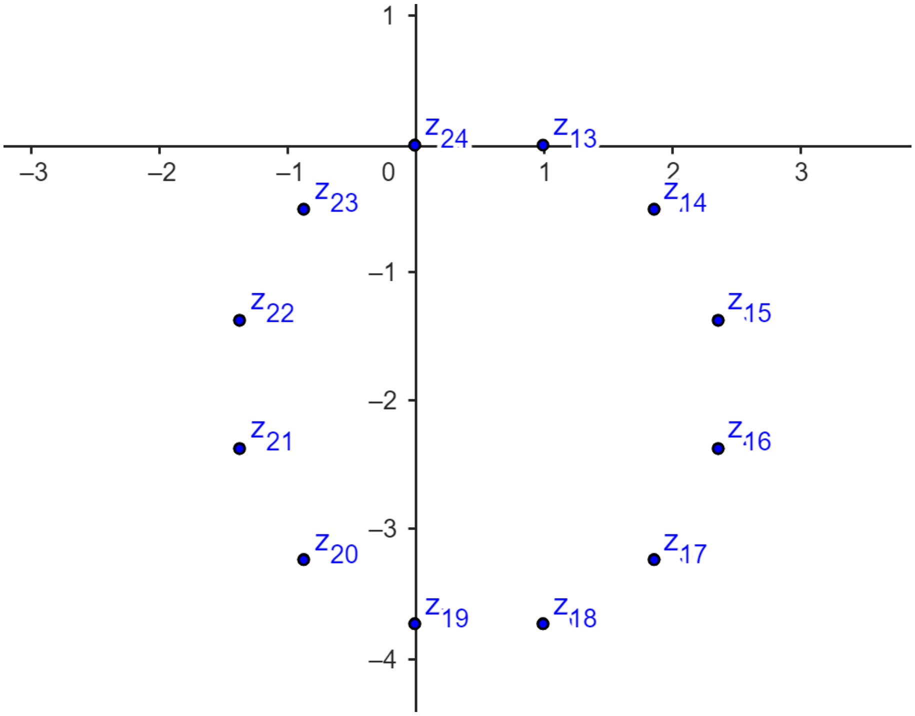
For case 1, $k = 1,\ \alpha = \frac{\pi}{12}$



For case 2, $k = 5,\ \alpha = \frac{5\pi}{12}$



For case 3, $k = 2, \ \alpha = \frac{\pi}{6}$



iv. For the test to be successful, the following condition must be met:

$$|1 + e^{-i\alpha} + e^{-i2\alpha} + e^{-i3\alpha} + \cdots + e^{-i23\alpha}| \leq 1$$

$$1 + e^{-i\alpha} + e^{-i2\alpha} + e^{-i3\alpha} + \cdots + e^{-i23\alpha} = \frac{1 - e^{-24i\alpha}}{1 - e^{-i\alpha}}$$

$$= \frac{e^{-12i\alpha}(e^{12i\alpha} - e^{-12i\alpha})}{e^{-i\frac{\alpha}{2}}(e^{i\frac{\alpha}{2}} - e^{-i\frac{\alpha}{2}})}$$

1 mark

$$= e^{-\frac{23}{2}i\alpha} \times \frac{2i\,sin(12\alpha)}{2i\,sin(\frac{\alpha}{2})}$$

$$\therefore \quad \left| e^{-\frac{23}{2}i\alpha} \times \frac{2i\,sin(12\alpha)}{2i\,sin(\frac{\alpha}{2})} \right| \leq 1$$

1 mark

$$\left| e^{-\frac{23}{2}i\alpha} \right| \times \left| \frac{2i\,sin(12\alpha)}{2i\,sin(\frac{\alpha}{2})} \right| \leq 1$$

$$Since \qquad \left|e^{-\frac{23}{2}i\propto}\right| = 1$$

$$\therefore \quad \left|\frac{2i\,sin(12\alpha)}{2i\,sin(\frac{\alpha}{2})}\right| \le 1$$

1 mark

$$|2i\,sin(12\alpha)| \le \left|2i\,sin(\frac{\alpha}{2})\right|$$

That is:

$$|sin(12\alpha)| \le \left|sin(\frac{\alpha}{2})\right|$$

■

1 mark