Student Name: ______________________

Maths class: ______________________

James Ruse Agricultural High School

2023 **YEAR 12 Task 3 HSC Examination**

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 95 minutes
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- In Questions 6–10, show relevant mathematical reasoning and/ or calculations

Total marks: 55

Section I – 5 marks (pages 2–3)

- Attempt Questions 1–5
- Allow about 7 minutes for this section

Section II – 50 marks (pages 4–7)

- Attempt Questions 6-10
- Allow about 1 hour and 28 minutes for this section

Section I

5 marks

Attempt Questions 1-5

Allow about 7 minutes for this section

Write your answers on the Multiple Choice sheet provided

---

1. Let P be the point $(2, 5, -3)$. What is the distance of P from the yz-plane?

   A. 2

   B. $\sqrt{29}$

   C. $\sqrt{34}$

   D. $\sqrt{38}$

2. Given that $a > b > 0$, which of the following is not always true?

   A. $a^2 + b^2 > 2ab$

   B. $a^2 + b^2 > 4ab$

   C. $(a + b)^2 > 4ab$

   D. $a + b > 2\sqrt{ab}$

3. If x, y and z are any real numbers and $x > y$, which of the following statements must be true?

   A. $x^2 > y^2$

   B. $\frac{1}{x} < \frac{1}{y}$

   C. $x + z > y + z$

   D. $xz > yz$

4. Consider the vector equations of the lines below.

$$\underset{\sim}{r} = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 4 \\ -3 \end{pmatrix} \text{ and } \underset{\sim}{r} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ 8 \\ 1 \end{pmatrix}$$

Which of the following statements is correct?

A. The lines are parallel with no point of intersection.

B. The lines are not parallel with no point of intersection.

C. The lines are parallel with a point of intersection.

D. The lines are not parallel with a point of intersection.

5. Let A, B, C be distinct, non-collinear points in the plane with position vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ respectively. P has position vector $\underset{\sim}{v}$ and is a variable, distinct fourth point in the plane. Consider the following statements:

I. $(\underset{\sim}{v} - \underset{\sim}{a}) \cdot (\underset{\sim}{b} - \underset{\sim}{c}) = 0$

II. $(\underset{\sim}{v} - \underset{\sim}{b}) \cdot (\underset{\sim}{c} - \underset{\sim}{a}) = 0$

III. $(\underset{\sim}{v} - \underset{\sim}{c}) \cdot (\underset{\sim}{a} - \underset{\sim}{b}) = 0$

Which of the following conclusions can be made about these statements?

A. Either all statements are true or all statements are false.

B. If two statements are true, then the third is also true.

C. Two statements are true if and only if the other one is true.

D. None of the above.

End of Section I

Section II

50 marks

Attempt Questions 6-10

Allow about 1 hour and 28 minutes for this section

Start each question on a new page. Extra paper is available.

Include relevant mathematical reasoning and/or calculations.

Question 6 (10 marks) **Start a new page**.

(a) For any integer x, where $x > 1$,

(i) Show that $\frac{1}{x^2} < \frac{1}{x(x-1)}$ **1**

(ii) Given that $\frac{1}{x^2-x} = \frac{1}{x-1} - \frac{1}{x}$, show that, for a positive integer n **3**

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 2$$

(b) A sequence of numbers is defined by **3**

$$a_0 = 5$$

$$a_n = 2a_{n-1} + 1 \text{ for } n \geq 1$$

Prove by induction that, for integers $n \geq 0$,

$$a_n = 2^{n+2} + 2^{n+1} - 1$$

(c) Find the shortest distance of the point (1, 2, 3) from the line that passes through the origin and the point (−2, 6, 5). **3**

Question 7 (10 marks) **Start a new page**.

(a) Show that $x^4 + y^4 + z^2 + 1 \geq 2x(xy^2 - x + z + 1)$ **2**

(b) A particle moves such that at any time t, its position vector is given by $\underset{\sim}{r} = \begin{pmatrix} -2t^2 + 3t \\ 4t^2 \\ 6t - 4 \end{pmatrix}$. **2**

Find the speed of the particle at time after 3 seconds.

(c) The sphere $\mathcal{S}$ has equation $(x-2)^2 + (y+1)^2 + (z+3)^2 = 169$.

(i) Write equation of $\mathcal{S}$ in vector form, stating its centre and radius. **2**

(ii) A line segment in three-dimensional space has equation **4**

$$\underset{\sim}{r} = (7\underset{\sim}{i} + \underset{\sim}{j} + 12\underset{\sim}{k}) + \lambda(2\underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k}) \quad \text{for} \quad 0 < \lambda < 3.$$

Find the point(s) of intersection between this line segment and the sphere $\mathcal{S}$.

Question 8 (10 marks) **Start a new page**.

(a) A function $f(x)$ is defined such that $f(x) > 0$ for all real numbers x and $f(a+b) = f(a) \times f(b)$ for any real numbers a and b.

(i) Show that $f(0) = 1$ and deduce that $f(-x) = \frac{1}{f(x)}$ **3**

(ii) By using mathematical induction, show that $f(nx) = [f(x)]^n$ for $n \in \mathbb{Z}^+$. **2**

(b) Let $\underset{\sim}{a}$ and $\underset{\sim}{b}$ be the position vectors of the fixed points A and B respectively and let $\underset{\sim}{v}$ be the position vector of the variable point P. **3**

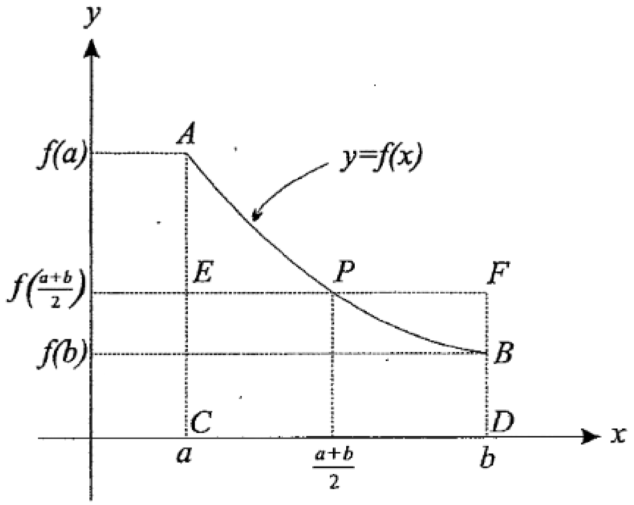
(i) Given that $|\underset{\sim}{a} - \underset{\sim}{v}| = |\underset{\sim}{b} - \underset{\sim}{v}|$, find the value of $(\underset{\sim}{b} - \underset{\sim}{a}) \cdot \left(\underset{\sim}{v} - \frac{\underset{\sim}{a} + \underset{\sim}{b}}{2}\right)$. **1**

(ii) If $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{v}$ are vectors in the plane, give a geometrical interpretation for the set of all possible points P. **1**

(iii) Hence, geometrically interpret the set of all possible points P if $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{v}$ are vectors in three-dimensional space.

Question 9 (10 marks) **Start a new page**.

(a) The diagram below shows the curve $y = f(x)$ for $a \leq x \leq b$. It is given that $f(x)$ is concave up for $a \leq x \leq b$.



(i) Use areas to explain why **3**

$$(b-a)f\left(\frac{a+b}{2}\right) < \int_a^b f(x)\,dx < (b-a)\frac{f(a)+f(b)}{2}$$

(ii) Let $f(x) = \frac{1}{x^2}$, $a = n-1$ and $b = n$, where $n \in \mathbb{Z}, n > 1$. Hence show that **2**

$$\frac{4}{(2n-1)^2} < \frac{1}{n-1} - \frac{1}{n} < \frac{1}{2}\left(\frac{1}{(n-1)^2} + \frac{1}{n^2}\right)$$

(iii) Deduce that **2**

$$4\left(\frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right) < 1 < \frac{1}{2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

(iv) Show that **1**

$$\frac{1}{2}\left(\frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots\right) < \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$$

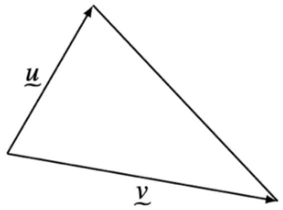
(v) Hence show that **2**

$$\frac{3}{2} < \sum_{n=1}^{\infty} \frac{1}{n^2} < \frac{7}{4}$$

Question 10 (10 marks) **Start a new page**.

(a) Find the Cartesian equation of the curve defined parametrically by $\underset{\sim}{r} = \begin{pmatrix} \cos t\,(1+\tan t) \\ \sin t\,(1+\tan t) \end{pmatrix}$, stating any restrictions necessary. **4**

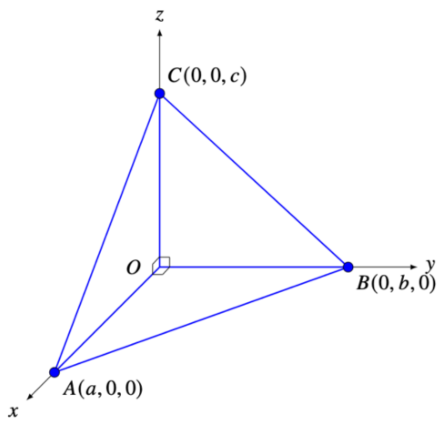
(b) (i) The triangle formed by vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$ is shown below. **3**



By considering the projection of $\underset{\sim}{u}$ onto $\underset{\sim}{v}$, show that the area of the triangle is given by

$$A = \frac{1}{2}\sqrt{|\underset{\sim}{u}|^2|\underset{\sim}{v}|^2 - (\underset{\sim}{u}\cdot\underset{\sim}{v})^2}.$$

(ii) Now consider the tetrahedron $OABC$ below with points A, B and C lying on the x, y and z axes respectively. **3**



Let $|XYZ|$ denote the area of ΔXYZ. Using vector methods and the result in part (i), show that

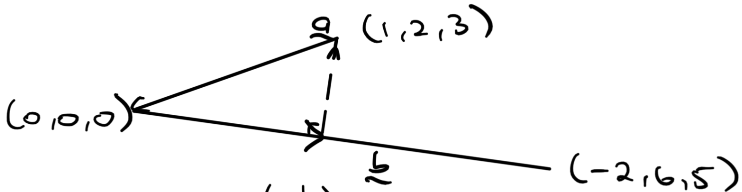
$$|AOB|^2 + |AOC|^2 + |BOC|^2 = |ABC|^2.$$

END OF PAPER

MATHEMATICS Extension 2: Question 6		
Suggested Solutions	**Marks**	**Marker's Comments**
a) i) RTP: $\frac{1}{x^2} < \frac{1}{x(x-1)}$ ie. $\frac{1}{x(x-1)} - \frac{1}{x^2} > 0$ LHS $= \frac{x-(x-1)}{x^2(x-1)}$ $= \frac{1}{x^2(x-1)}$ > 0 as $x^2 > 0$ and $x > 1$ $\therefore x-1>0$ OR $x > x-1$ as $x > 1$ $x^2 > x^2 - x$ $x^2 > x(x-1)$ $\frac{1}{x^2} < \frac{1}{x(x-1)}$ as $x>1$, $x^2>0$, $x-1>0$		1 must have reasoning to support your claim.
ii) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2} < 2$ LHS $= \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2}$ $< \frac{1}{1^2} + \frac{1}{2(2-1)} + \frac{1}{3(3-1)} + \frac{1}{4(4-1)} + \dots + \frac{1}{n(n-1)}$ from (i) $= 1 + \frac{1}{2-1} - \frac{1}{2} + \frac{1}{3-1} - \frac{1}{3} + \frac{1}{4-1} - \frac{1}{4} + \dots + \frac{1}{n-1} - \frac{1}{n}$ given $\frac{1}{x^2-x} = \frac{1}{x-1} - \frac{1}{x}$ $= 1 + 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{1}{n-1} - \frac{1}{n}$	1	

Suggested Solutions	Marks	Marker's Comments
$= 2 - \frac{1}{n}$	1	
< 2 as $n \in \mathbb{Z}^+$ $\therefore \frac{1}{n} > 0$	1	need to have reasoning
b) $a_0 = 5$		
$a_n = 2a_{n-1} + 1$		
$a_n = 2^{n+2} + 2^{n+1} - 1$		
for $n = 0$		
$a_0 = 2^{0+2} + 2^{0+1} - 1$	1	
$= 4 + 2 - 1$		
$= 5$		
$\therefore$ true for $n = 0$		
Assume true for $n = k$, $k \in \mathbb{Z}, k \geq 0$		1 for setting out of induction question
ie. $a_k = 2^{k+2} + 2^{k+1} - 1$		
Prove true for $n = k+1$		- $k \in \mathbb{Z}, k \geq 0$
ie. $a_{k+1} = 2^{(k+1)+2} + 2^{(k+1)+1} - 1$		- statement for $n = k$
LHS $= a_{k+1}$		
$= 2a_k + 1$ (by definition)		- statement for $n = k+1$
$= 2(2^{k+2} + 2^{k+1} - 1) + 1$ (by assumption)		- by assumption
		- conclusion
$= 2 \times 2^{k+2} + 2 \times 2^{k+1} - 2 + 1$	1	
$= 2^{(k+1)+2} + 2^{(k+1)+1} - 1$		
$=$ RHS		
$\therefore$ true for $n = k+1$		
$\therefore$ the statement is true by the principle of mathematical induction.		

MATHEMATICS Extension 2: Question 6

Suggested Solutions	Marks	Marker's Comments
c) 		
let $\underset{\sim}{a} = \begin{pmatrix}1\\2\\3\end{pmatrix}$ $\underset{\sim}{b} = \begin{pmatrix}-2\\6\\5\end{pmatrix}$		
$\therefore$ shortest distance $= \left\lvert \text{proj}_{\underset{\sim}{b}}\underset{\sim}{a} - \underset{\sim}{a} \right\rvert$		
$= \left\lvert \left(\dfrac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{\lvert\underset{\sim}{b}\rvert^2}\right)\underset{\sim}{b} - \underset{\sim}{a} \right\rvert$	1	
$= \left\lvert \left(\dfrac{-2+12+15}{4+36+25}\right)\begin{pmatrix}-2\\6\\5\end{pmatrix} - \begin{pmatrix}1\\2\\3\end{pmatrix} \right\rvert$		
$= \left\lvert \dfrac{5}{13}\begin{pmatrix}-2\\6\\5\end{pmatrix} - \begin{pmatrix}1\\2\\3\end{pmatrix} \right\rvert$	1	correct $\text{proj}_{\underset{\sim}{b}}\underset{\sim}{a}$
$= \left\lvert \begin{pmatrix}-\frac{23}{13}\\ \frac{4}{13}\\ -\frac{14}{13}\end{pmatrix} \right\rvert$		
$= \sqrt{\left(\dfrac{-23}{13}\right)^2 + \left(\dfrac{4}{13}\right)^2 + \left(\dfrac{-14}{13}\right)^2}$		
$= \sqrt{\dfrac{57}{13}}$	1	

MATHEMATICS Extension 2: Question……..

Suggested Solutions	Marks	Marker's Comments

OR

let the line passing through origin and point $(-2,6,5)$ be:

$$\underset{\sim}{r} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 6 \\ 5 \end{pmatrix} \text{ where } \lambda \in \mathbb{R}$$

$\therefore$ any point can be written as:

$$\begin{pmatrix} -2\lambda \\ 6\lambda \\ 5\lambda \end{pmatrix}$$

$\therefore$ perpendicular vector through $\underset{\sim}{r}$ and $(1,2,3)$ is:

$$\begin{pmatrix} -2\lambda - 1 \\ 6\lambda - 2 \\ 5\lambda - 3 \end{pmatrix}$$

as they are perpendicular:

$$\begin{pmatrix} -2\lambda - 1 \\ 6\lambda - 2 \\ 5\lambda - 3 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 6 \\ 5 \end{pmatrix} = 0$$ 1

$$(-2\lambda-1)(-2) + (6\lambda-2)(6) + (5\lambda-3)(5) = 0$$

$$4\lambda + 2 + 36\lambda - 12 + 25\lambda - 15 = 0$$

$$65\lambda = 25$$

$$\lambda = \frac{5}{13}$$ 1

$\therefore$ vector is:

$$\begin{pmatrix} -2\left(\frac{5}{13}\right) - 1 \\ 6\left(\frac{5}{13}\right) - 2 \\ 5\left(\frac{5}{13}\right) - 3 \end{pmatrix} = \begin{pmatrix} -\frac{23}{13} \\ \frac{4}{13} \\ -\frac{14}{13} \end{pmatrix}$$

$\therefore$ shortest distance is:

$$\sqrt{\left(-\frac{23}{13}\right)^2 + \left(\frac{4}{13}\right)^2 + \left(\frac{-14}{13}\right)^2} = \sqrt{\frac{57}{13}}$$ 1

Year 12 Ext. 2 Task 3 Q7 solutions

(a) Method I

$$x^4 + y^4 \geq 2x^2y^2 \quad ① \quad (AM-GM)$$

$$x^2 + z^2 \geq 2xz \quad ②$$

$$x^2 + 1 \geq 2x \quad ③$$

1 mark

$$①+②+③: \quad x^4 + y^4 + 2x^2 + z^2 + 1 \geq 2x^2y^2 + 2xz + 2x$$

$$x^4 + y^4 + z^2 + 1 \geq 2x^2y^2 - 2x^2 + 2xz + 2x$$

$$\therefore x^4 + y^4 + z^2 + 1 \geq 2x(xy^2 - x + z + 1)$$

1 mark

Method II

$$(x^2 - y^2)^2 + (x-z)^2 + (x-1)^2 \geq 0$$

1 mark

$$x^4 - 2x^2y^2 + y^4 + x^2 - 2xz + z^2 + x^2 - 2x + 1 \geq 0$$

$$x^4 + y^4 + z^2 + 1 \geq 2x^2y^2 - 2x^2 + 2xz + 2x$$

$$\therefore x^4 + y^4 + z^2 + 1 \geq 2x(xy^2 - x + z + 1)$$

1 mark

(b)

$$\underset{\sim}{r} = \begin{pmatrix} -2t^2 + 3t \\ 4t^2 \\ 6t - 4 \end{pmatrix}$$

$$\underset{\sim}{v} = \begin{pmatrix} -4t + 3 \\ 8t \\ 6 \end{pmatrix}$$

1 mark for finding velocity vector correctly

$$\underset{\sim}{v}(3) = \begin{pmatrix} -9 \\ 24 \\ 6 \end{pmatrix}$$

$$\therefore \text{speed} = \sqrt{(-9)^2 + (24)^2 + (6)^2}$$

$$= \sqrt{693}$$

$$= 3\sqrt{77}$$

1 mark for correct working and answer

(c)

$$(x-2)^2 + (y+1)^2 + (z+3)^2 = 169$$

i) centre $= (2, -1, -3)$ 1 mark for finding BOTH centre and radius correctly

radius $= 13$

$\therefore$ vector form : $\left| \underset{\sim}{r} - \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix} \right| = 13$ 1 mark for correct vector form of sphere

ii) $\underset{\sim}{r} = \begin{pmatrix} 7 \\ 1 \\ 12 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} \qquad 0 < \lambda < 3$

Sub into sphere S

$\left| \begin{pmatrix} 7 \\ 1 \\ 12 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix} \right| = 13$ 1 mark for substituting correctly

$$\left| \begin{pmatrix} 5 \\ 2 \\ 15 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} \right| = 13$$

$(5 + 2\lambda)^2 + (2 - 2\lambda)^2 + (15 + 3\lambda)^2 = 169$ 1 mark for getting a quadratic equation of lambda

$$(25 + 20\lambda + 4\lambda^2) + (4 - 8\lambda + 4\lambda^2) + (225 + 90\lambda + 9\lambda^2) = 169$$

$$17\lambda^2 + 102\lambda + 85 = 0$$

$$\lambda^2 + 6\lambda + 5 = 0$$

$$(\lambda + 1)(\lambda + 5) = 0$$

$\therefore \lambda = -1, -5$ 1 mark for correctly solving the quadratic equation

but $0 < \lambda < 3$

$\therefore$ no point of intersection 1 mark for rejection of solutions and correct conclusion

Question 8

(a) $f(x) > 0$ for all $x \in \mathbb{R}$

$f(a+b) = f(a) \times f(b)$ for all $a \in \mathbb{R}$ and $b \in \mathbb{R}$

(i) show that $f(0) = 1$ and deduce that $f(-x) = \frac{1}{f(x)}$

$$f(0+0) = f(0) \times f(0)$$
$$f(0) = f(0)^2$$
$$f(0)(1 - f(0)) = 0 \quad \text{but } f(x) > 0 \text{ for all } x$$
$$\therefore f(0) \neq 0 \text{ and } 1 - f(0) = 0 \therefore f(0) = 1$$

$$f(x + (-x)) = f(x) \times f(-x)$$
$$f(0) = f(x) \times f(-x)$$
$$1 = f(x) \times f(-x)$$
$$f(-x) = \frac{1}{f(x)}$$

(ii) By Mathematical induction, show that $f(nx) = [f(x)]^n$ for $n \in \mathbb{Z}^+$

for $n = 1$ $\quad f(x) = f(x)^1 = f(x)$

True for $n = 1$

~~for n~~ Assume it is true for $k \in \mathbb{Z}^+$

$$f(kx) = [f(x)]^k \quad (*)$$

and prove it true for $n = k+1$

i.e. $f((k+1)x) = [f(x)]^{k+1}$

Now $f((k+1)x) = f(kx + x) = f(kx) \times f(x)$

But $f(kx) = [f(x)]^k$

Hence $f((k+1)x) = (f(x))^k \times f(x) = [f(x)]^{k+1}$

By mathematical induction $f(nx) = [f(x)]^n$ for all $n \in \mathbb{Z}$

(b) $\underset{\sim}{a}$ and $\underset{\sim}{b}$ position vectors of A and B
$\underset{\sim}{v}$ position vector of P

(i) Given that $|\underset{\sim}{a}-\underset{\sim}{v}| = |\underset{\sim}{b}-\underset{\sim}{v}|$
Find the value of
$$(\underset{\sim}{b}-\underset{\sim}{a})\cdot\left(\underset{\sim}{v}-\frac{\underset{\sim}{a}+\underset{\sim}{b}}{2}\right)$$

$$(\underset{\sim}{b}-\underset{\sim}{a})\cdot\left(\underset{\sim}{v}-\frac{\underset{\sim}{a}+\underset{\sim}{b}}{2}\right) = (\underset{\sim}{b}-\underset{\sim}{a})\cdot\underset{\sim}{v} - \frac{(\underset{\sim}{b}-\underset{\sim}{a})(\underset{\sim}{a}+\underset{\sim}{b})}{2} \quad \checkmark$$

$$|\underset{\sim}{a}-\underset{\sim}{v}| = |\underset{\sim}{b}-\underset{\sim}{v}| \quad \therefore \quad |\underset{\sim}{a}-\underset{\sim}{v}|^2 = |\underset{\sim}{b}-\underset{\sim}{v}|^2$$

$$\therefore \quad (\underset{\sim}{a}-\underset{\sim}{v})\cdot(\underset{\sim}{a}-\underset{\sim}{v}) = (\underset{\sim}{b}-\underset{\sim}{v})\cdot(\underset{\sim}{b}-\underset{\sim}{v}) \quad \checkmark$$

$$\underset{\sim}{a}\cdot\underset{\sim}{a} - 2\underset{\sim}{a}\cdot\underset{\sim}{v} + \underset{\sim}{v}\cdot\underset{\sim}{v} = \underset{\sim}{b}\cdot\underset{\sim}{b} - 2\underset{\sim}{b}\cdot\underset{\sim}{v} + \underset{\sim}{v}\cdot\underset{\sim}{v}$$

$$2(\underset{\sim}{b}-\underset{\sim}{a})\cdot\underset{\sim}{v} - (\underset{\sim}{b}\cdot\underset{\sim}{b} - \underset{\sim}{a}\cdot\underset{\sim}{a}) = 0$$

$$\therefore \quad (\underset{\sim}{b}-\underset{\sim}{a})\cdot\underset{\sim}{v} - \frac{1}{2}(\underset{\sim}{b}-\underset{\sim}{a})\cdot(\underset{\sim}{a}+\underset{\sim}{b}) = 0 \quad \checkmark$$

Note.
$$(\underset{\sim}{b}-\underset{\sim}{a})\cdot(\underset{\sim}{a}+\underset{\sim}{b}) = \underset{\sim}{b}\cdot\underset{\sim}{a} + \underset{\sim}{b}\cdot\underset{\sim}{b} - \underset{\sim}{a}\cdot\underset{\sim}{a} - \underset{\sim}{a}\cdot\underset{\sim}{b}$$
$$= \underset{\sim}{b}\cdot\underset{\sim}{b} - \underset{\sim}{a}\cdot\underset{\sim}{a}$$

(ii) $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{v}$ are vectors in the plane, give a geometrical interpretation for the set of all possible points P.

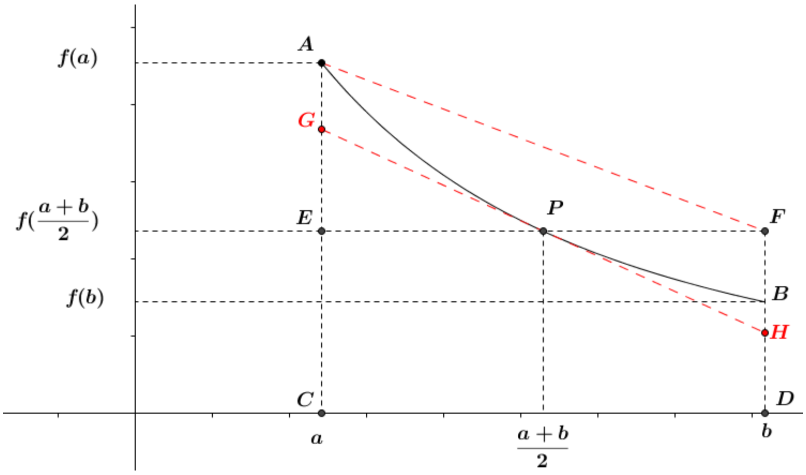
P lies on the perpendicular bisector of AB. $\checkmark$

(iii) Hence, geometrically interpret the set of all possible points P if $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{v}$ are vectors in 3D space.

P lies on the plane which passes by the midpoint of AB and $\overrightarrow{AB}$ is Normal to the plane. $\checkmark$

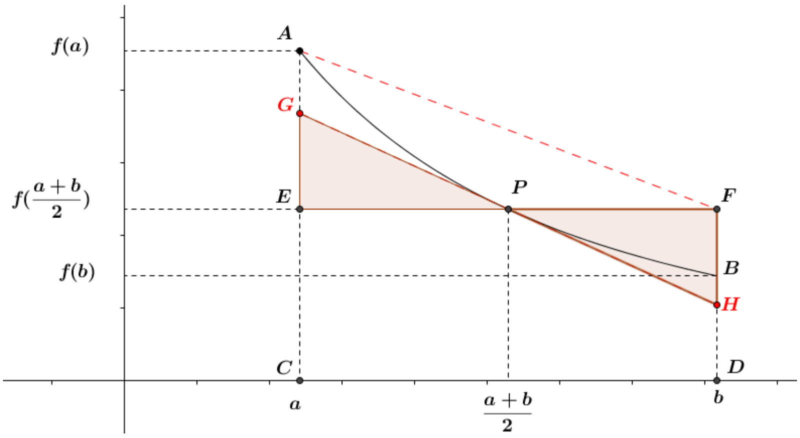
Question 9

i. Construct a tangent at P to meet AC and FD at G and H respectively.



Area Trapezium$_{GHDC}$ < Area under $y = f(x)$ between $x = a$ and $x = b$ < Area Trapezium$_{AFDC}$

$\Delta GEP \equiv \Delta HFP \Rightarrow \text{Area}_{\Delta GEP} = \text{Area}_{\Delta HFP}$

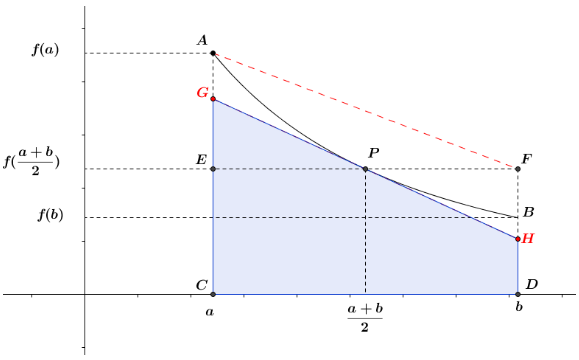


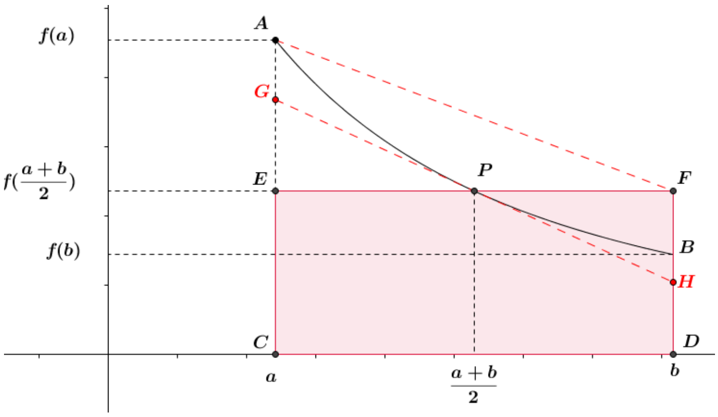
1 mark for identifying what each expression represents in terms of area (i.e. Rectangle, Trapezium and area under the curve).

1 mark for establishing one side of the inequality with sufficient reasons.

1 mark for establishing the other side of the inequality with sufficient reasons.

$\therefore$ Area rectangle$_{\text{ECDF}}$ = Area Trapezium$_{GHDC}$





Area rectangle$_{\text{ECDF}}$ < Area under $y = f(x)$between $x = a$ and $x = b$ < Area Trapezium$_{AFDC}$

$$EC \times CD < \int_a^b f(x)\,dx < \frac{1}{2}(AC + FD) \times CD$$

$$(b-a)f\left(\frac{a+b}{2}\right) < \int_a^b f(x)\,dx < (b-a) \times \frac{f(a)+f(b)}{2}$$

Note:

- If you define $(b-a) \times \frac{f(a)+f(b)}{2}$ as the area of a rectangle (or 2 rectangles) you're almost certain to get 0.
- "The curve is concave up" argument is only sufficient to prove $\int_a^b f(x)\,dx <$ Area Trapezium$_{GHDC}$

ii.

$$f(x) = \frac{1}{x^2}, a = n-1, b = n, n > 1, n \in Z$$

$$\therefore [n-(n-1)] \times \frac{1}{\left(n-\frac{1}{2}\right)^2} < \int_{n-1}^{n} \frac{1}{x^2}dx < [n-(n-1)] \times \frac{\frac{1}{n^2}+\frac{1}{(n-1)^2}}{2}$$

$$\frac{1}{\left(n-\frac{1}{2}\right)^2} < \left[-\frac{1}{x}\right]_{n-1}^{n} < \frac{1}{2} \times \left(\frac{1}{(n-1)^2}+\frac{1}{n^2}\right)$$

1 mark for getting the middle expression through integration.

$$4 \times \frac{1}{(2n-1)^2} < \left[\frac{1}{x}\right]_{n}^{n-1} < \frac{1}{2}\left[\frac{1}{(n-1)^2}+\frac{1}{n^2}\right]$$

$$\frac{4}{(2n-1)^2} < \frac{1}{n-1} - \frac{1}{n} < \frac{1}{2}\left[\frac{1}{(n-1)^2}+\frac{1}{n^2}\right]$$

1 mark for establishing the other two expressions by substitution and showing all necessary working out.

iii.

1 mark for establishing a pattern correctly starting from $n = 2$ for all 3 expressions.

$$n = 2 \Rightarrow \frac{4}{3^2} < 1 - \frac{1}{2} < \frac{1}{2}\left[\frac{1}{1^2}+\frac{1}{2^2}\right] - \text{Equation 1}$$

$$n = 3 \Rightarrow \frac{4}{5^2} < \frac{1}{2} - \frac{1}{3} < \frac{1}{2}\left[\frac{1}{2^2}+\frac{1}{3^2}\right] - \text{Equation 2}$$

$$n = 4 \Rightarrow \frac{4}{7^2} < \frac{1}{4} - \frac{1}{3} < \frac{1}{2}\left[\frac{1}{3^2}+\frac{1}{4^2}\right] - \text{Equation 3}$$

$\vdots$

1 mark for correct addition of equations AND clear working out. You <u>**MUST SHOW**</u> necessary working <u>in addition to</u> just stating "sum all the equations together" before getting to the required expression.

$$\therefore \sum_{n=2}^{\infty} \frac{4}{(2n-1)^2} < \sum_{n=2}^{\infty} \frac{1}{n-1} - \frac{1}{n} < \sum_{n=2}^{\infty} \frac{1}{2}\left[\frac{1}{(n-1)^2}+\frac{1}{n^2}\right]$$

$$\frac{4}{3^2}+\frac{4}{5^2}+\frac{4}{7^2}+\cdots < \left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots < \frac{1}{2}\left[\frac{1}{1^2}+\frac{1}{2^2}\right]+\frac{1}{2}\left[\frac{1}{2^2}+\frac{1}{3^2}\right]+\frac{1}{2}\left[\frac{1}{3^2}+\frac{1}{4^2}\right]+\cdots$$

$$4\left(\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots\right) < \lim_{n\to\infty}\left(1-\frac{1}{n}\right) < \frac{1}{2}\left[\frac{1}{1^2}+\frac{1}{2^2}\right]+\frac{1}{2}\left[\frac{1}{2^2}+\frac{1}{3^2}\right]+\frac{1}{2}\left[\frac{1}{3^2}+\frac{1}{4^2}\right]+\cdots$$

$$4\left(\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots\right) < 1 < \frac{1}{2}\left(1+2\left(\frac{1}{2^2}\right)+2\left(\frac{1}{3}\right)^2+\cdots\right)$$

$$4\left(\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots\right) < 1 < \frac{1}{2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots$$

iv.

$$\frac{1}{2}\left(\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\cdots\right) = \frac{1}{2}\left(\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots\right)+\frac{1}{2}\left(\frac{1}{4^2}+\frac{1}{6^2}+\frac{1}{8^2}+\cdots\right)$$

$$< \frac{1}{2}\left(\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots\right)+\frac{1}{2}\left(\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots\right) \left(\text{since} \frac{1}{(n-1)^2} > \frac{1}{n^2}\ \forall n > 1\right)$$

$$= \frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots$$

$$\therefore \frac{1}{2}\left(\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\cdots\right) < \left(\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots\right)$$

1 mark for getting to required line showing all necessary working.

v.

$$4\left(\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots\right)<1<\frac{1}{2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots \quad \text{(proven in iii)}$$

$$1<\frac{1}{2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots \Rightarrow \left(1+\frac{1}{2}\right)<\frac{1}{2}+\left(\frac{1}{2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right)$$

$$\frac{3}{2}<1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots$$

$$=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots$$

1 mark for getting the inequality on the left with necessary working out.

$$\therefore \frac{3}{2}<\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$4\left(\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots\right)<1 \Rightarrow \left(\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots\right)<\frac{1}{4} \quad \text{(Proven in iii.)}$$

$$\frac{1}{2}\left(\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\cdots\right)<\frac{1}{4} \quad \left(\frac{1}{2}\left(\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\cdots\right)<\left(\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots\right) \text{ proven in iv.}\right)$$

$$\left(\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\cdots\right)<\frac{1}{2}$$

$$\left(\frac{1}{1^2}+\frac{1}{2^2}\right)+\left(\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\cdots\right)<\frac{1}{2}+\left(\frac{1}{1^2}+\frac{1}{2^2}\right)$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2}<1+\frac{1}{2}+\frac{1}{4}$$

$$=\frac{7}{4}$$

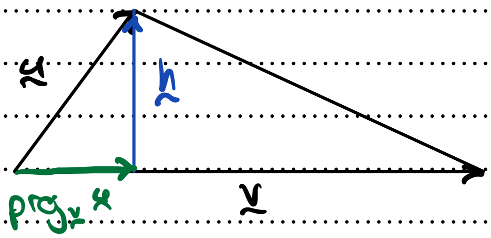
1 mark for getting the inequality on the right with necessary working out.

$$\therefore \frac{3}{2}<\sum_{n=1}^{\infty}\frac{1}{n^2}<\frac{7}{4}$$

MATHEMATICS EXT 2: Question 10

Suggested Solutions	Marks	Marker's Comments
a, $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos t\,(1+\tan t) \\ \sin t\,(1+\tan t) \end{pmatrix}$		
$x = \cos t\,(1+\tan t)$ ① $y = \sin t\,(1+\tan t)$ ②		
② ÷ ① : $\dfrac{y}{x} = \tan t$	①	
①² + ②² : $x^2+y^2 = \cos^2 t(1+\tan t)^2 + \sin^2 t(1+\tan t)^2$ $= (1+\tan t)^2(\cos^2 t+\sin^2 t)$ $x^2+y^2 = (1+\tan t)^2$	①	
$\therefore x^2+y^2 = \left(1+\dfrac{y}{x}\right)^2$	①	Eliminating t completely
But as stated above, $(0,0)$ is not included. However, subbing $t=\frac{3\pi}{4}$ into the original parametric eqn shows that $(0,0)$ is valid, so we must express our cartesian eqn in such a way that it includes $(0,0)$.		
$\therefore x^2(x^2+y^2) = (x+y)^2$	①	Including origin.
Notes: • Learn the common tricks for parametrics ⇒ squaring & adding ⇒ multiplying / dividing ⇒ make use of trig identities. • Few students got ③ marks, let alone ④.		

MATHEMATICS EXT 2: Question 10

Suggested Solutions	Marks	Marker's Comments
b i)		
		
$A = \frac{1}{2} b h$		
$= \frac{1}{2} \lvert v \rvert \lvert h \rvert$		
$A = \frac{1}{2} \lvert v \rvert \lvert \text{proj}_v u - u \rvert$	1	
$A^2 = \frac{1}{4} \lvert v \rvert^2 \left\lvert \frac{u \cdot v}{\lvert v \rvert^2} v - u \right\rvert^2$		
$= \frac{1}{4} \lvert v \rvert^2 \left(\frac{u \cdot v}{\lvert v \rvert^2} v - u \right) \cdot \left(\frac{u \cdot v}{\lvert v \rvert^2} v - u \right)$	1	Using $\lvert a \rvert^2 = a \cdot a$
$= \frac{1}{4} \lvert v \rvert^2 \left(\left(\frac{u \cdot v}{\lvert v \rvert^2} \right)^2 v \cdot v - \frac{(u \cdot v)^2}{\lvert v \rvert^2} - \frac{(u \cdot v)^2}{\lvert v \rvert^2} + u \cdot u \right)$		
$= \frac{1}{4} \lvert v \rvert^2 \left(\frac{(u \cdot v)^2}{\lvert v \rvert^4} \lvert v \rvert^2 - \frac{2(u \cdot v)^2}{\lvert v \rvert^2} + \lvert u \rvert^2 \right)$		
$= \frac{1}{4} \lvert v \rvert^2 \left(\lvert u \rvert^2 - \frac{(u \cdot v)^2}{\lvert v \rvert^2} \right)$		
$= \frac{1}{4} \left(\lvert u \rvert^2 \lvert v \rvert^2 - (u \cdot v)^2 \right)$		
$\therefore A = \frac{1}{2} \sqrt{\lvert u \rvert^2 \lvert v \rvert^2 - (u \cdot v)^2}$	1	Getting to final with sufficient steps shown.

MATHEMATICS EXT 2: Question 10		
Suggested Solutions	**Marks**	**Marker's Comments**
ii, ΔAOB is formed by $\vec{OA}$ and $\vec{OB}$		
$\lvert AOB\rvert = \frac{1}{2}\sqrt{\lvert\vec{OA}\rvert^2\lvert\vec{OB}\rvert^2-(\vec{OA}\cdot\vec{OB})^2}$		
$= \frac{1}{2}\sqrt{a^2b^2-0}$		
$\lvert AOB\rvert^2 = \frac{1}{4}a^2b^2$		
similarly, $\lvert AOC\rvert^2 = \frac{1}{4}a^2c^2$ & $\lvert BOC\rvert = \frac{1}{4}b^2c^2$	1	
ΔABC is formed by $\vec{AC}$ and $\vec{AB}$		
$\lvert ABC\rvert = \frac{1}{2}\sqrt{\lvert\vec{AC}\rvert^2\lvert\vec{AB}\rvert^2-(\vec{AC}\cdot\vec{AB})^2}$		
$= \frac{1}{2}\sqrt{(a^2+c^2)(a^2+b^2)-a^4}$	1	progress
$\lvert ABC\rvert^2 = \frac{1}{4}(\cancel{a^4}+a^2b^2+a^2c^2+b^2c^2-\cancel{a^4})$		
$= \frac{1}{4}(a^2b^2+a^2c^2+b^2c^2)$		
$= \frac{1}{4}a^2b^2+\frac{1}{4}a^2c^2+\frac{1}{4}b^2c^2$		
$\lvert ABC\rvert^2 = \lvert AOB\rvert^2+\lvert AOC\rvert^2+\lvert BOC\rvert^2$	1	Getting to final.
Notes: • If you didn't MEANINGFULLY use the projection, no marks awarded • Too many students confused themselves by introducing $\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}$ without defn. They confused the vectors with already defined scalars a, b, c. • You need to SHOW non-trivial steps!		