KILLARA HIGH SCHOOL

TERM 2 2003

EXTENSION 2 MATHEMATICS EXAMINATION

YEAR 12

Instructions:

Time Allowed: 70 minutes

Approved calculators may be used.

Show ALL working clearly.

Marks may not be awarded for careless or badly arranged work.

Begin each question on a new page.

Question 1 **Marks**

i) Use integration by parts to evaluate $\int_1^3 x^2 \ln x \; dx$ 2

ii) Evaluate $\int_3^4 \frac{3x}{2+3x} dx$ 2

iii) Find $\int_0^1 \frac{1}{x^2+4x+5} dx$ 2

iv) (a) If $I_n = \int \cot^n \theta d\theta$ show that $I_n = \frac{-1}{n-1} \cot^{n-1}\theta - I_{n-2}$ 3

(b) Evaluate $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^4 \theta d\theta$ 2

v) Use the substitution $x = a - t$ where a is a constant, to prove that
$\int_o^a f(x)dx = \int_o^a f(a-t)dt$
Hence, or otherwise, show that $\int_o^1 x(1-x)^{99} dx = \frac{1}{10100}$ 4

Question 2

i) The equation $8x^3 - 54x + m = 0$ has a double root. Find the possible values of m. 3

ii) Let α, β, γ be the roots of the equation $x^3 + qx + r = 0$ where $r \neq 0$. Obtain as functions of q, r in their simplest forms, the co-efficients of the cubic equations whose roots are:

(a) $\alpha^2, \beta^2, \gamma^2$ 2

(b) $\alpha^{-1}, \beta^{-1}, \gamma^{-1}$ 2

(c) $\alpha^{-2}, \beta^{-2}, \gamma^{-2}$ 2

Question 3

a) Prove using mathematical induction that for $n \geq 1$

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \; \ldots\ldots \; + \frac{1}{n^2} \leq 2 - \frac{1}{n}$$ 4

b) Hence prove that $1 \cdot 45 \leq 1 + \frac{1}{2^2} + \frac{1}{3^2} + \; \ldots\ldots \; + \frac{1}{99^2} \leq 1 \cdot 99$ 2

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i) $\int_1^3 x^2 \ln x \, dx$ 15

$= \int_1^3 \ln x \frac{d}{dx}\left(\frac{x^3}{3}\right) dx$

$= \left[\frac{x^3}{3}\ln x\right]_1^3 - \int_1^3 \frac{x^3}{3} \times \frac{1}{x} \, dx$ 1

$= \left[\frac{x^3}{3}\ln x\right]_1^3 - \int_1^3 \frac{x^2}{3} \, dx$

$= \left[\frac{x^3}{3}\ln x - \frac{x^3}{9}\right]_1^3$

$= 9\ln 3 - 3 - \left(0 - \frac{1}{9}\right)$

$= 9\ln 3 - 2\frac{8}{9}$ 1

ii) $\int_3^4 \frac{3x}{2+3x} \, dx$

$= \int_3^4 \left(\frac{2+3x}{2+3x} - \frac{2}{2+3x}\right) dx$ 1

$= \int_3^4 \left(1 - \frac{2}{2+3x}\right) dx$

$= \left[x - \frac{2}{3}\ln|2+3x|\right]_3^4$

$= \left(4 - \frac{2}{3}\ln 14\right) - \left(3 - \frac{2}{3}\ln 11\right)$

$= 1 - \frac{2}{3}\ln\left(\frac{14}{11}\right)$ 1

iii) $\int_0^1 \frac{1}{x^2+4x+5} \, dx$

$= \int_0^1 \frac{1}{(x+2)^2+1} \, dx$ 1

$= \left[\tan^{-1}(x+2)\right]_0^1$

$= \tan^{-1} 3 - \tan^{-1} 2$

$= (0.142 \quad 3 \text{ dp})$ 1

iv) a) $I_n = \int \cot^n\theta \, d\theta$

$= \int \cot^{n-2}\theta \cot^2\theta \, d\theta$ 1

$= \int \cot^{n-2}\theta (\csc^2\theta - 1) \, d\theta$ 1

$= \int \cot^{n-2}\theta \csc^2\theta - I_{n-2}$

$= \frac{-1}{n-1}\cot^{n-1}\theta - I_{n-2}$

b) $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^4\theta \, d\theta = \left[-\frac{1}{3}\cot^3\theta - I_2\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$

$+1 - \frac{\pi}{4}$

$= \left[-\frac{1}{3}(0^3 - 1^3) - (1 - I_0)\right]$

$= \frac{1}{3} - 1 + \frac{\pi}{4} = \frac{\pi}{4} - \frac{2}{3}$ 1

Since $I_2 = \left[-\frac{1}{1}\cot\theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} - I_0$

$= [0 + 1] - I_0$

$I_0 = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\theta$

$= \frac{\pi}{2} - \frac{\pi}{4}$

$= \frac{\pi}{4}$ 1

) $x = a - t$

$dx = -dt$

At $x = 0 \quad t = a$

$x = a \quad t = 0$

$\therefore \int_0^a f(x)\,dx = \int_a^0 f(a-t)(-dt)$ 1

$= -\int_a^0 f(a-t)\,dt$ 1

$= \int_0^a f(a-t)\,dt$

$\therefore$ if $a = 1$ and $x = 1 - t$

and $f(x) = x(1-x)^{99}$

then $f(1-t) = (1-t)(t)^{99}$

$\therefore \int_0^1 x(1-x)^{99} = \int_0^1 (1-t)t^{99}\,dt$ 1

$= \int_0^1 (t^{99} - t^{100})\,dt$

$= \left[\frac{t^{100}}{100} - \frac{t^{101}}{101}\right]_0^1$

$= \frac{1}{100} - \frac{1}{101}$

$= \frac{1}{10100}$ 1

2/ i) $P(x) = 8x^3 - 54x + m$

$P(\alpha) = 0$

$P'(\alpha) = 0$ since double root

$P'(x) = 24x^2 - 54$ 1

$P'(\alpha) = 24\alpha^2 - 54$

$= 0$

$\alpha^2 = \frac{54}{24}$

$= \frac{27}{12}$

$= \frac{9}{4}$

$\alpha = \pm\frac{3}{2}$ 1

$P(\alpha) = 0$

$\therefore 8\alpha^3 - 54\alpha + m = 0$

$m = 54\alpha - 8\alpha^3$

$= (54 - 8\alpha^2)\alpha$

$= \left(54 - 8\times\frac{9}{4}\right)\left(\pm\frac{3}{2}\right)$

$= (54 - 18)\times\pm\frac{3}{2}$

$= 36\times\pm\frac{3}{2}$

$= \pm 54$ 1

) Let $y = x^2$ where x satisfies the equation

$$x^3 + qx + r = 0$$

then $\quad x(x^2+q) = -r$

$\therefore \quad x^2(x^2+q)^2 = r^2$

ie $\quad y(y+q)^2 = r^2$

ie $\quad y^3 + 2qy + q^2y - r^2 = 0$

Thus if $x^3+qx+r=0$ has roots α, β, γ then

$y^3 + 2qy + q^2y - r^2 = 0$ has roots $\alpha^2, \beta^2, \gamma^2$

$\therefore$ coeffts are $1, 2q, q^2, -r^2$

or $\quad \alpha+\beta+\gamma = 0$

$\alpha\beta + \beta\gamma + \gamma\alpha = q$

$\alpha\beta\gamma = -r$

$$\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\gamma\alpha)$$

$$= -2q$$

$\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2$

$= (\alpha\beta+\beta\gamma+\gamma\alpha)^2 - 2\alpha\beta\gamma(\alpha+\beta+\gamma)$

$= q^2$

$\alpha^2\beta^2\gamma^2 = (\alpha\beta\gamma)^2 = r^2$

$\therefore$ eqn satisfied by $\alpha^2, \beta^2, \gamma^2$

is $\quad y^3 + 2qy^2 + q^2y - r^2 = 0$

b) Let $y = x^{-1}$ where x satisfies the equation $x^3+qx+r=0$

then $\quad \frac{1}{y^3} + q\frac{1}{y} + r = 0$

ie $\quad 1 + qy^2 + ry^3 = 0$

$\therefore \quad y^3 + \frac{q}{r}y^2 + \frac{1}{r} = 0$

$\therefore$ if $x^3+qx+r=0$ has roots α, β, γ then

$y^3 + \frac{q}{r}y^2 + \frac{1}{r} = 0$ has roots $\alpha^{-1}, \beta^{-1}, \gamma^{-1}$

$\therefore$ coeffts are $1, \frac{q}{r}, 0, \frac{1}{r}$

or $\quad \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma+\gamma\alpha+\alpha\beta}{\alpha\beta\gamma} = -\frac{q}{r}$

$\frac{1}{\alpha\beta} + \frac{1}{\beta}\frac{1}{\gamma} + \frac{1}{\gamma}\frac{1}{\alpha} = \frac{\alpha+\beta+\gamma}{\alpha\beta\gamma} = 0$

$\frac{1}{\alpha\beta\gamma} = -\frac{1}{r}$

etc.

c) From a)

$y^3 + 2qy^2 + q^2y - r^2 = 0$ has roots $\alpha^2, \beta^2, \gamma^2$

Let $z = y^{-1}$ then

$$\frac{1}{z^3} + 2q\frac{1}{z^2} + q^2\frac{1}{z} - r^2 = 0$$

ie $1 + 2qz + q^2z^2 - r^2z^3 = 0$

ie $z^3 - \frac{q^2}{r^2}z^2 - \frac{2q}{r^2}z - \frac{1}{r^2} = 0$

$\therefore$ coeffts of eqn with roots $\alpha^{-2}, \beta^{-2}, \gamma^{-2}$ are

$1, -\frac{q^2}{r^2}, -\frac{2q}{r^2}, -\frac{1}{r^2}$

) For $n=1$

$LHS = 1$

$RHS = 2 - \frac{1}{1}$
$= 1$

$\therefore$ true for $n=1$

Assume true for $n=k$

ie $1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{k^2} \leq 2 - \frac{1}{k}$ 1

Prove true for $n = k+1$

ie $1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{k^2} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k+1}$

Now $LHS \leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2}$ by assumption

$$= 2 - \left[\frac{(k+1)^2 - k}{k(k+1)^2}\right]$$

$$= 2 - \frac{k^2+k+1}{k(k+1)^2}$$

$$= 2 - \frac{1}{(k+1)} \times \frac{k^2+k+1}{k(k+1)}$$ 2

$$= 2 - \frac{1}{k+1}\left[1 + \frac{1}{k^2+k}\right]$$

$$< 2 - \frac{1}{k+1} \times 1$$

$$= 2 - \frac{1}{k+1}$$

$\therefore$ If it is true for $n=1$ it is true for $n=2$ and so on by induction for all +ve n.

b) $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} < 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{99^2} \leq 2 - \frac{1}{99}$

$\therefore \quad 1\cdot4636\dot{1} \leq 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{99^2} \leq 1\cdot\dot{9}\dot{8}$

but $1\cdot45 < 1\cdot4636\dot{1}$ and $1\cdot\dot{9}\dot{8} < 1\cdot99$

$\therefore \quad 1\cdot45 < 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{99^2} < 1\cdot99$