KILLARA HIGH SCHOOL

EXTENSION 2 MATHEMATICS

YEAR 12
MARCH 2006

Time Allowed: 3 hours

Instructions:

* You may attempt ALL questions
* All necessary working should be shown
* Start each question on a new page
* Approved calculators may be used
* All questions are of equal value
* Full marks may not be awarded for careless or badly arranged work
* Standard Integrals are printed at the end of the paper

Question 1.

(a) Let $z=a+ib$ where a and b are real numbers and $a\neq0$.

(i) Express $|z|$ and $\tan(\arg z)$ in terms of a and b. (2 marks)

(ii) Express $\dfrac{z}{3+4i}$ in the form $x+iy$ where x and y are real. (2 marks)

(b) (i) If $w=\dfrac{1+i\sqrt{3}}{2}$ show that $w^3=-1$ (2 marks)

(ii) Hence calculate w^{16} (1 mark)

(c) If $z=5-5i$ write z, z^2 and $\dfrac{1}{z^2}$ in modulus argument form. (3 marks)

(d) Let u and v be two complex numbers where $u=-2+i$ and v is defined by $|v|=3$ and $\arg v=\dfrac{\pi}{3}$

On an Argand diagram:

(i) Plot the points A and B representing the complex numbers u and v respectively. (2 marks)

(ii) Plot the points C and D represented by the complex numbers u-v and iu respectively. Indicate any geometric relationships between the four points A, B, C and D. (3 marks)

Question 2 over page ...

Question 2.

(a) Find real values of a, b and c to make the following true. (3 marks)

$$\frac{2x^2-x+1}{(x-1)(x^2+1)}=\frac{a}{x-1}+\frac{bx+c}{x^2+1}$$

(b) Let $x=\alpha$ be a root of the quartic polynomial $P(x)=x^4+Ax^3+Bx^2+Ax+1$, where A and B are real but α may be complex.

(i) Show that $\alpha\neq 0$ (1 mark)

(ii) Show that $x=\alpha$ is also a root of $Q(x)=x^2+\frac{1}{x^2}+A(x+\frac{1}{x})+B$. (2 marks)

(iii) Let $u=x+\frac{1}{x}$. Show that $Q(x)$ becomes $R(u)=u^2+Au+(B-2)$. (2 marks)

(c) Consider the polynomial equation $x^4+ax^3+bx^2+cx+d=0$ where a, b, c and d are all integers. Suppose the equation has a root of the form ki, where k is real and $k\neq 0$.

(i) State why the conjugate $-ki$ is also a root. (1 mark)

(ii) Show that $c=k^2a$ (2 marks)

(iii) Show that $c^2+a^2d=abc$ (2 marks)

(iv) If 2 is also a root of the equation, and $b=0$, show that c is even. (2 marks)

Continued over page ...

Question 3.

(a) (i) Write down the relations which hold between the roots α, β, γ of the equation $ax^3+bx^2+cx+d=0$, where $a\neq 0$ and the co-efficients a, b, c, d are real. (3 marks)

(ii) Consider the equation $36x^3-12x^2-11x+2=0$. You are given that the roots α, β, γ of this equation satisfy $\alpha=\beta+\gamma$. Use Part (i) to find α. (3 marks)

(iii) Suppose the equation $x^3+px^2+qx+r=0$ has roots λ, μ, τ which satisfy $\lambda=\mu+\tau$. Show that $p^3-4pq+8r=0$. (3 marks)

(b) None of the roots α, β, γ of the equation $x^3+3mx+n=0$ is zero.

(i) Obtain the monic equation whose roots are $\frac{\beta\gamma}{\alpha}$, $\frac{\alpha\gamma}{\beta}$ and $\frac{\alpha\beta}{\gamma}$, expressing its co-efficients in terms of m and n. (3 marks)

(ii) Hence show that $\gamma=\alpha\beta$ if and only if $(3m-n)^2+n=0$. (3 marks)

Question 4.

The hyperbola H has equation $xy = 16$.

(a) Sketch this hyperbola and indicate on your diagram the positions and coordinates of all points at which the curve intersects the axes of symmetry. (3 marks)

(b) $P(4p,\frac{4}{p})$, where $p>0$ and $q(4q,\frac{4}{q})$ where $q>0$ are two distinct arbitrary points on H. Find the equation of the chord PQ. (2 marks)

(c) Prove that the equation of the tangent at P is $x+p^2y=8p$. (2 marks)

(d) The tangents at P and Q intersect at T. Find the coordinates of T. (3 marks)

(e) The chord PQ produced passes through the point $N(0,8)$.

(i) Find the equation of the locus of T. (3 marks)

(ii) Give the geometrical description of this locus. (2 marks)

Continued over page ...

Question 5.

(a) (i) Suppose that k is a triple root of the polynomial $P(x) = 0$. Show that $P''(k) = 0$. (2 marks)

(ii) What feature does the graph of a polynomial have at a root of multiplicity 3? (1 mark)

(iii) The polynomial $Q(x) = x^4 + 4x^3 - 16x - 16$ has a root of multiplicity 3. Solve $Q(x) = 0$. (2 marks)

(iv) Let $R(x) = 1 + \frac{x}{1} + \frac{x^2}{1\times2} + \frac{x^3}{1\times2\times3} + \frac{x^4}{1\times2\times3\times4} + \frac{x^5}{1\times2\times3\times4\times5}$.

Prove that $R(x)$ has no triple root. (2 marks)

(b) The cubic equation $x^3 + px + q = 0$ has 3 non zero roots α, β, γ.
Find in terms of the constants p and q the values of

(i) $\alpha^2 + \beta^2 + \gamma^2$ (2 marks)

(ii) $\alpha^2\beta^2 + \beta^2\gamma^2 + \alpha^2\gamma^2$ (2 marks)

(iii) Show that the condition for the roots α, β, γ to be in geometric progression is that $p = 0$. (2 marks)

(iv) Prove that these roots cannot be in Arithmetic Progression. (2 marks)

Continued over page ...

Question 6.

(a) Evaluate i^{2006} (1 mark)

(b) Let $z = \dfrac{18+4i}{3-i}$

(i) Simplify $(18+4i)\ \overline{(3-i)}$ (1 mark)

(ii) Express z in the form $a+ib$ where a and b are real numbers. (2 marks)

(iii) Hence or otherwise find $|z|$ and $\arg(z)$. (2 marks)

(c) Sketch the region where the inequalities $|z-3+i| \le 5$ and $|z+1| \le |z-1|$ both hold. (3 marks)

(d) Let $w = \dfrac{3+4i}{5}$ and $z = \dfrac{5+12i}{13}$

(i) Show that $|w| = |z| = 1$ (2 marks)

(ii) Find wz and $w\,\overline{z}$ in the form $a+ib$ (2 marks)

(iii) Hence find two distinct ways of writing 65^2 as the sum p^2+q^2 where p and q are integers and $0<p<q$. (2 marks)

Question 7.

The ellipse $E : (\frac{x}{5})^2 + (\frac{y}{3})^2 = 1$ has foci $S(4,0)$ and $S'(-4,0)$

(i) Sketch the ellipse E indicating its foci S, S' and its directrices. (3 marks)

(ii) Show that the tangent at $P(x_1, y_1)$ on the ellipse E has equation $9x_1x + 25y_1y = 225$. (3 marks)

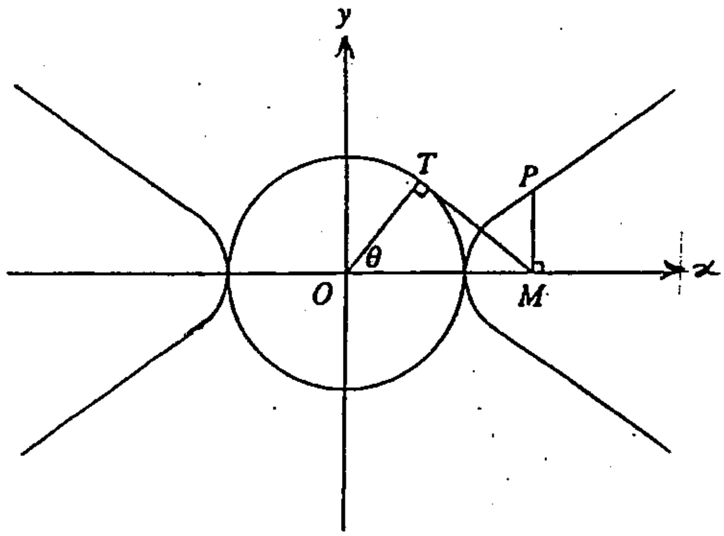
(iii) The line joining $P(x_1, y_1)$ to $Q(x_2, y_2)$ passes through S. Show that $4(y_2 - y_1) = x_1y_2 - x_2y_1$. (3 marks)

(iv) It is also known that $Q(x_2, y_2)$ lies on E. Show that the tangents at P and Q on the ellipse intersect on the directrix corresponding to S. (3 marks)

(v) Find the equation of the normal to E at P and decide under what circumstances, if any, it passes through S or S'. (3 marks)

Question 8.

(a)



The figure shows the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and the circle $x^2+y^2=a^2$, where $a, b>0$. The point T lies on the circle with $\angle TOx=\theta$, where $0<\theta<\frac{\pi}{2}$. The tangent to the circle at T meets the x axis at M; MP is perpendicular to Ox and P is a point on the hyperbola in the first quadrant.

(i) Show that P has coordinates $(a\sec\theta, b\tan\theta)$. (2 marks)

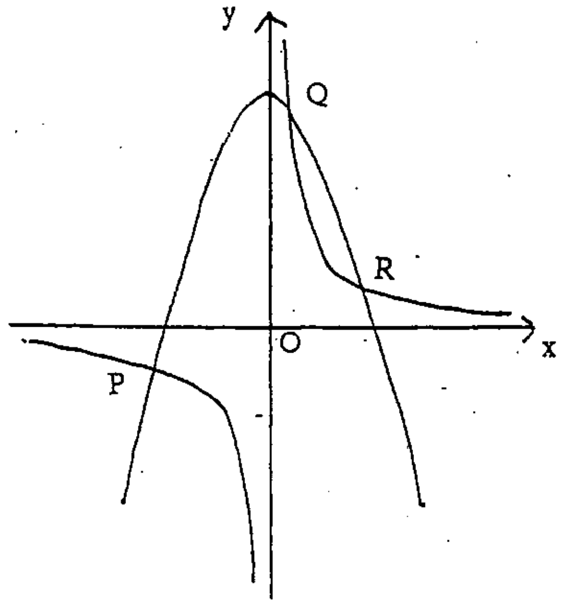
(ii) Suppose that Q is a point on the hyperbola with coordinates $(a\sec\varphi, b\tan\varphi)$. If $\theta+\varphi=\frac{\pi}{2}$, and $\theta\neq\frac{\pi}{4}$, show that the chord PQ has equation $ay=b(\cos\theta+\sin\theta)x-ab$. (3 marks)

(iii) Show that every such chord PQ passes through a fixed point and find its coordinates. (2 marks)

(iv) Show that as θ approaches $\frac{\pi}{2}$ the chord PQ approaches a line parallel to an asymptote. (2 marks)

Question 8(b) over page...

(b)



The curves $y=k-x^2$, for some real number k, and $y=\frac{1}{x}$ intersect at the points P, Q and R where $x=\alpha$, $x=\beta$ and $x=\gamma$.

(i) Show that the monic cubic equation with coefficients in terms of k whose roots are α^2, β^2 and γ^2 is given by $x^3-2kx^2+k^2x-1=0$. (2 marks)

(ii) Find the monic cubic equation with coefficients in terms of k whose roots are $\frac{1}{a^2}$, $\frac{1}{\beta^2}$ and $\frac{1}{\gamma^2}$ (2 marks)

(iii) Hence show that $OP^2+OQ^2+OR^2=k^2+2k$, where O is the origin. (2 marks)

KILLARA HIGH SCHOOL YEAR 12 EXAM

EXTENSION 2 MATHEMATICS MARCH 2006.

QUESTION 1.

(a) (i) $z = a+ib$

$|z| = \sqrt{a^2+b^2}$, $\tan(\arg z) = \frac{b}{a}$ 1

1

(ii) $\frac{z}{3+4i} = \frac{(a+ib)(3-4i)}{3^2+4^2}$

$= \frac{3a+4b}{25} + i\frac{(-4a+3b)}{25}$ 1 1

(b) (i) $\omega = \frac{1+i\sqrt{3}}{2} = 1(\cos\pi/3 + i\sin\pi/3)$

$\omega^3 = 1(\cos\pi + i\sin\pi)$ 1

$= -1.$ 1

(ii) $\omega^{16} = \omega^{15}\omega$

$= (-1)^5 \frac{1+i\sqrt{3}}{2}$

$= -\frac{1}{2} - i\sqrt{3}/2$ 1

Accept $-\text{cis}\,\pi/3$.

(c) $z = 5-5i$

1 $= 5\sqrt{2}\left(\cos(-\pi/4) + i\sin(-\pi/4)\right)$

1 $z^2 = 50\left[\cos(-\pi/2) + i\sin(-\frac{\pi}{2})\right]$

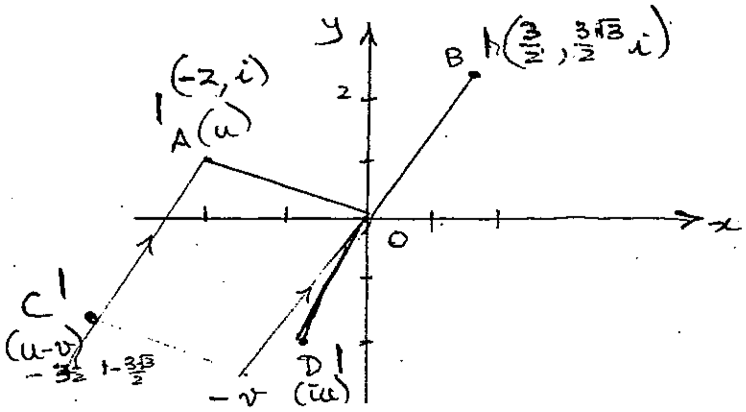
1 $\frac{1}{z^2} = (z^2)^{-1}$

$= \frac{1}{50}\left[\cos(\pi/2) + i\sin(\pi/2)\right]$

$= \frac{i}{50}$

(d) $u = -2+i$ $\quad v = 3(\cos\pi/3 + i\sin\pi/3)$

$= 3\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$



ABOC is a parallelogram.

OD ⊥ OA. 1

QUESTION 2.

(a) $\frac{2x^2-x+1}{(x-1)(x^2+1)} = \frac{a}{x-1} + \frac{bx+c}{x^2+1}$

$\therefore 2x^2-x+1 = a(x^2+1) + (bx+c)(x-1)$

$= a(x^2+1) + bx^2 - bx + cx - c$

$= x^2(a+b) + x(-b+c) + a - c$

$\therefore a+b = 2$ ——— ①

$-b+c = -1$ ——— ②

$a-c = 1$ ——— ③

①+② $a+c = 1$ ——— ④

③+④ $2a = 2 \quad \therefore a = 1$ 1

Sub in ① $1+b = 2 \quad \therefore b = 1$ 1

Sub in ③ $1-c = 1 \quad \therefore c = 0$ 1

$a=1,\ b=1,\ c=0$

(b) $P(x) = x^4 + Ax^3 + Bx^2 + Ax + 1$

$P(\alpha) = \alpha^4 + A\alpha^3 + B\alpha^2 + A\alpha + 1$

true if α is a root $P(\alpha) = 0$

but $P(0) = 1 \therefore \alpha = 0$ is not a root

1

QUEST 2 (b)(ii)

$Q(x) = x^2+\frac{1}{x^2} + A(x+\frac{1}{x}) + B$

$= \frac{1}{x^2}[x^4+1+Ax^2(x+\frac{1}{x})+Bx^2]$ 1

$= \frac{1}{x^2}[x^4+Ax^3+Bx^2+Ax+1]$

$= \frac{P(x)}{x^2}$

$\therefore Q(\alpha) = \frac{P(\alpha)}{\alpha^2}$ 1 Note $\alpha \neq 0$

$= 0$

$\therefore x=\alpha$ is a root of $Q(x)$

(iii) Let $u = x+\frac{1}{x}$

$(x+\frac{1}{x})^2 = (x^2+\frac{1}{x^2}) + 2.1$

$\therefore Q(x) = (x+\frac{1}{x})^2+A(x+\frac{1}{x})+B-2$

$= u^2 + A(u)+B-2$ 1

$= R(u)$.

(c) $x^4+ax^3+bx^2+cx+d=0$

(i) For polynomials with real co-efficients, complex roots occur in conjugate pairs. 1
i.e. $(a-ib)$ a factor then $(a+ib)$ is a factor $\Rightarrow a^2+b^2$ a real factor

(ii) $P(x) = x^4+ax^3+bx^2+cx+d$

$P(ki) = k^4-ak^3i-bk^2+cki+d$

if factor $\therefore 0 = P(ki)$ 1

$\therefore$ Imag $0 = -ak^3+ck$

$\therefore c = k^2a$. 1

(iii) Equating Real Parts

$k^4-bk^2+d = 0$

But $k^2 = \frac{c}{a}$ 1

$\therefore (\frac{c}{a})^2 - b(\frac{c}{a}) + d = 0$

$c^2 - abc + a^2d = 0$ 1

$c^2 + a^2d = abc$

(iv)

$\therefore 16+8a+4b+2c+d=0$

From $b=0$

$16+8a+2c+d=0$

and $c^2+a^2d = a\times 0 \times c$

$\therefore d = -c^2/a^2$

$\therefore 16+8a+2c-c^2/a^2=0$

1 i.e. $c^2-2a^2c-8a^3-16a^2=0$

Solve for c

$c = \frac{2a^2 \pm \sqrt{4a^4+4(8a^3+16a^2)}}{2}$

$= a^2 \pm \sqrt{a^4+8a^3+16a^2}$

$= a^2 \pm a\sqrt{(a+4)^2}$

i.e. $c = (a^2+a^2+4a)$ or a^2-a^2-4a

$= 2(a^2+4a)$ or $2(-2a)$

1 $\therefore$ is even. (a is an integer)

OR $P(x)$ is degree 4 with roots $ki, -ki, 2$ and α.

$\therefore \sum\alpha = 2+\alpha = -a$

$\sum\alpha\beta = k^2+2\alpha = 0$

$\therefore k^2 = -2\alpha$

$= 2(a+2)$ 1

But $k^2 = c/a \therefore \frac{c}{a} = 2(a+2)$

$c = 2(a^2+2a)$

$=$ even. 1

QUESTION 3.

a) (i) $\Sigma\alpha = \alpha+\beta+\gamma = -b/a$ 1

$\Sigma\alpha\beta = \alpha\beta+\beta\gamma+\gamma\alpha = c/a$ 1

$\alpha\beta\gamma = \quad = -d/a.$ 1

(ii) $36x^3 - 12x^2 - 11x + 2 = 0$

Has roots $\beta, \gamma, (\beta+\gamma)$

$\therefore 2\beta+2\gamma = \frac{12}{36} = \frac{1}{3}$ — 1 ①

$\alpha = \beta+\gamma = \frac{1}{6}$ 1

(iii) $x^3+px^2+qx+r=0$

Roots λ, μ, τ

$\therefore \lambda = \mu+\tau$

$\Sigma\alpha$: $2\mu+2\tau = -p.$

$\lambda = \mu+\tau = -p/2.$ 1

$\cancel{\Sigma\alpha\beta: (\mu+\tau)\mu+\mu\tau+(\mu+\tau)\tau = q}$

NOT NEEDED $\cancel{(\mu+\tau)(\mu+\tau)+\mu\tau = q}$

$\cancel{p^2/4 + \mu\tau = q.}$

But if λ is a root

$\lambda^3+p\lambda^2+q\lambda+r=0$

$(-p/2)^3+p\left(-\frac{p}{2}\right)^2+q(-p/2)+r=0$ 1

$-p^3/8 + p^3/4 - pq/2 + r = 0$

$p^3-4pq+8r=0$ 1

Results used here are also in Q5(b)

(b) $x^3+3mx+n=0$

(i) $\Sigma\alpha=0$ $\Sigma\alpha\beta=3m$ $\alpha\beta\gamma=-n.$

$\frac{\beta\gamma}{\alpha}+\frac{\alpha\gamma}{\beta}+\frac{\alpha\beta}{\gamma} = \frac{(\alpha\beta)^2+(\alpha\gamma)^2+(\gamma\beta)^2}{\alpha\beta\gamma}$

Sum 1 at a time =

Now $(\alpha\beta+\alpha\gamma+\beta\gamma)^2$

$= \alpha^2\beta^2+\alpha^2\gamma^2+\beta^2\gamma^2+\alpha(\alpha\beta\gamma)+\beta(\alpha\beta\gamma)$

$+\alpha(\alpha\beta\gamma)+\gamma(\alpha\beta\gamma)+\beta(\alpha\beta\gamma)+\gamma(\alpha\beta\gamma)$

i.e. $\alpha^2\beta^2+\alpha^2\gamma^2+\beta^2\gamma^2$

$= (\alpha\beta+\alpha\gamma+\beta\gamma)^2 - 2\alpha\beta\gamma\,\Sigma\alpha$

$= (3m)^2 - 2\times n\times 0 = 9m^2$

$\therefore$ Sum 1 at a time $= -\frac{9m^2}{n}.$ 1

and considering 2 at a time

$\left(\frac{\alpha\beta}{\gamma}\times\frac{\alpha\gamma}{\beta}\right)+\left(\frac{\alpha\beta}{\gamma}\times\frac{\beta\gamma}{\alpha}\right)+\left(\frac{\beta\gamma}{\alpha}\times\frac{\alpha\gamma}{\beta}\right)$

$= \alpha^2+\beta^2+\gamma^2$

$= (\alpha+\beta+\gamma)^2 - 2\Sigma\alpha\beta$

$= 0^2 - 2\times 3m$

$= -6m$ 1

and 3 at a time

$\frac{\alpha\beta}{\gamma}\times\frac{\alpha\gamma}{\beta}\times\frac{\beta\gamma}{\alpha} = \alpha\beta\gamma$

$= -n.$ 1

$\therefore$ Polynomial (must be monic) is

$x^3+\frac{9m^2}{n}x^2-6mx+n=0$

(ii) If $\gamma = \alpha\beta$ $\therefore$ the root $\frac{\alpha\beta}{\gamma}=1$ 1

$\therefore 1+\frac{9m^2}{n}-6m+n=0$ 1

$n+9m^2-6mn+n^2=0$

$(3m-n)^2+n=0.$ 1

b(i)

Method 2: $\alpha\beta\gamma=-n.$ But

But $\frac{\alpha\beta}{\gamma} = \frac{\alpha\beta\gamma}{\gamma^2} = \frac{-n}{\gamma^2}$

Let $Y = -\frac{n}{x^2}$ $\therefore$ $x^2 = -\frac{n}{Y}$

$P(x) = x^3+3mx+n=0$

$x(x^2+3m) = -n$

$x^2(x^2+3m)^2 = +n^2$

$-\frac{n}{Y}\left(-\frac{n}{Y}+3m\right)^2 = +n^2$

$-\frac{n}{Y}\left(9m^2-\frac{6mn}{Y}+\frac{n^2}{Y^2}\right) = n^2$

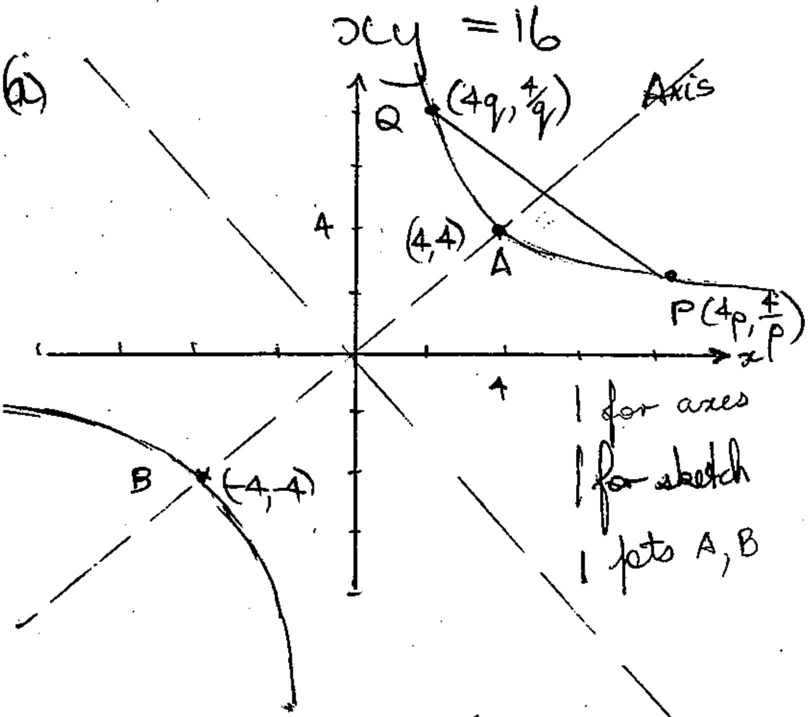
$-9m^2nY^2+6mn^2Y-n^3 = n^2Y^3$

$n^2Y^3+9m^2nY^2-6mnY+n^3=0$

$\therefore x^3+\frac{9m^2}{n}x^2-6mx+n=0$

QUESTION 4

(a)



1 for axes
1 for sketch
1 pts A, B

Intersects @ $A=(4,4)$ $B=(-4,-4)$

(b) $P(4p, \frac{4}{p})$ $Q(4q, \frac{4}{q})$

$$m_{PQ} = \frac{\frac{4}{q}-\frac{4}{p}}{4q-4p} \begin{matrix} \div 4 & \times pq \\ \div 4 & \times pq \end{matrix}$$

$$= \frac{(p-q)}{(q-p)pq}$$

$$= \frac{-1}{pq} \quad 1$$

Chord is $y - \frac{4}{p} = -\frac{1}{pq}(x-4p)$

$$pqy - 4q = -x + 4q$$

$$x + pqy = 4(p+q) \quad 1$$

(c) Tangent $Q \to P$ $\therefore q \to p$ 1

$$x + p^2 y = 8p. \quad 1$$

is equation of tangent

(d)
$$x + p^2 y = 8p$$
$$x + q^2 y = 8q$$
$$\therefore y(p^2 - q^2) = 8(p-q) \quad 1$$

$$\therefore y = \frac{8}{p+q} \quad 1$$

$$\therefore x + \frac{8p^2}{p+q} = 8p$$

$$x = \frac{8p(p+q) - 8p^2}{p+q}$$

$$= \frac{8pq}{p+q} \quad 1$$

T is $\left(\frac{8pq}{p+q}, \frac{8}{p+q}\right)$

(e) Through $(0,8)$

$$\therefore 0 + 8pq = 4(p+q)$$

or $2pq = p+q$ 1

So T is $\left(\frac{8pq}{2pq}, \frac{8}{p+q}\right)$

$\therefore x = 4$ is locus 11

Note $x = pqy$ worth 1

(ii) Note: Given $p>0$ and $q>0$

and $2pq = p+q$

$\therefore q = \frac{p}{2p-1}$

If $p > 1$ $\quad q = \frac{1+k}{2+2k-1}$

$$= \frac{2k+1}{2k+1} - \frac{k}{2k+1}$$

$$< 1.$$

i.e. $4p > 4$ $\therefore$ P to right of A

$4q < 4$ $\therefore$ Q to left of A

$\therefore$ T is between $xy=16$ and axes

$\therefore$ Locus is intercept 1

of $x=4$ from x-axis to A

QUESTION 5

(a) (i) $P(x) = (x-k)^3 \times Q(x)$

$\therefore P'(x) = 3(x-k)^2 Q(x) + (x-k)^3 Q'(x)$

$= (x-k)^2 [3Q(x) + (x-k)Q'(x)]$.

1 $= (x-k)^2 \times R(x)$

$\therefore P''(x) = 2(x-k)R(x) + (x-k)^2 R'(x)$

1 $= (x-k)[2R(x) + (x-k)R'(x)]$.

$\therefore P''(k) = 0$

1 (ii) If root of multiplicity 3 then $P(x)$ has a horizontal point of inflexion on the x-axis.

(iii) $Q(x) = x^4 + 4x^3 - 16x - 16$.

$Q'(x) = 4x^3 + 12x^2 - 16$

$Q''(x) = 12x^2 + 24x$

If $Q''(x) = 0 = 12x(x+2)$

1 $x = 0$ or -2.

Now $Q(0) = -16 \neq 0$

$Q(-2) = 16 - \cancel{32} + 32 - 16$

$= 0$

$\therefore Q(x) = (x+2)^3(x-\alpha)$

$\therefore 8(-\alpha) = -16$

$\alpha = 2$.

1 $\therefore x = -2$ or 2

(iv) $R(x) = 1 + \frac{x}{1} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^5}{5!}$

$\therefore R'(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$

1 $\therefore R''(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!}$

1 But $R(x)$ and $R''(x)$ have no common factor $\therefore R(x) \neq 0$ when $R''(x) = 0$ $\therefore$ No Triple root

(b) $x^3 + 0x^2 + px + q = 0$ has 3 non-zero roots α, β, γ.

$\Sigma\alpha = 0 \quad \Sigma\alpha\beta = p \quad \Sigma\alpha\beta\gamma = -q$

(i) $\therefore \alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2\Sigma\alpha\beta$

$= 0 - 2p$ 1

$= -2p$ 1

(ii) $\alpha^2\beta^2 + \beta^2\gamma^2 + \alpha^2\gamma^2$ see Q3 (

Consider $(\alpha\beta + \beta\gamma + \alpha\gamma)^2$

$= \alpha^2\beta^2 + \alpha\beta^2\gamma + \alpha^2\beta\gamma$

$+ \beta^2\gamma^2 + \alpha\beta^2\gamma + \alpha\beta\gamma^2$

$+ \alpha^2\gamma^2 + \alpha^2\beta\gamma + \alpha\beta\gamma^2$

$= \Sigma(\alpha\beta)^2 - 2\alpha\beta\gamma[\alpha+\beta+\gamma]$ 1

$= p^2 - 2 \times -q \times 0$

$= p^2$ 1

(iii) For Geometric $\frac{\alpha}{\beta} = \frac{\beta}{\gamma}$

$\left.\begin{array}{l}\therefore \alpha\gamma = \beta^2 \\ \text{But } \alpha\gamma + \alpha\beta + \beta\gamma = p\end{array}\right\}$ 1

i.e. $\beta^2 + \beta(\alpha+\gamma) = p$

$\therefore \beta(\alpha+\beta+\gamma) = p$ 1

$\therefore p = 0$.

(iv) For arithmetic $\beta - \alpha = \gamma - \beta$

$\therefore \alpha + \gamma - 2\beta = 0$

i.e. $(\alpha+\beta+\gamma) - 3\beta = 0$ 1

$\therefore 3\beta = 0$

But if α, β, γ are non-zero

$\therefore$ cannot be arithmetic

QUESTION 6.

(a) $i^2 = -1 \quad i^4 = 1$

$\therefore i^{2006} = i^{2004} \times i^2$ 1

$= -1.$

(b)(i) $(18+4i)\overline{(3-i)}$

$= (18+4i)(3+i)$

$= (54-4) + i(18+12)$

$= 50+30i$ 1

(ii) $Z = \frac{18+4i}{3-i} \times \frac{3+i}{3+i}$ 1

$= \frac{50+30i}{10}$

$= 5+3i$ 1

(iii) $|Z| = \sqrt{5^2+3^2}$

$= \sqrt{34}$ 1

$\arg Z = \tan^{-1}\left(\frac{3}{5}\right)$

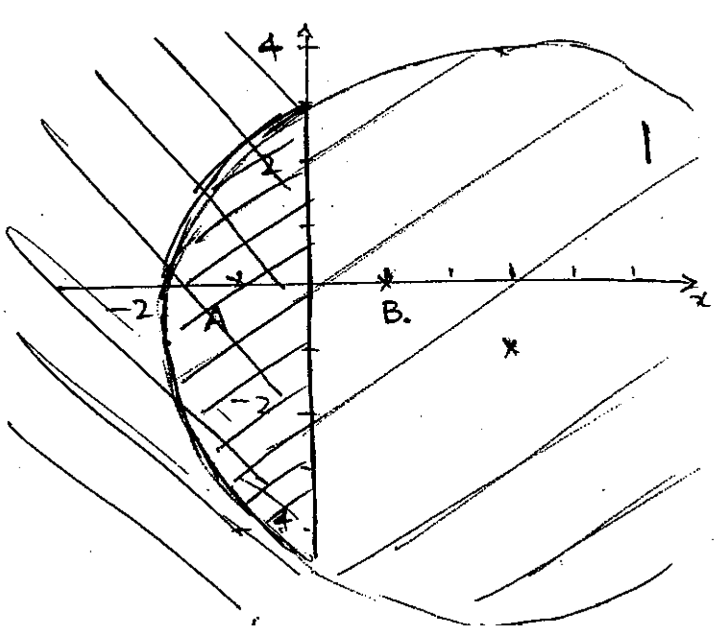
$\doteqdot 0.540$ radians 1

$|z-3+i| \leq 5$ Centre $(3,-1)$ Radius ≤ 5 1

$|z+1| \leq |z-1|$

If $A = (-1,0)$ $B = (1,0)$

Points closer to A than B. 1



(d) $w = \frac{3+4i}{5} \qquad z = \frac{5+12i}{13}$

(i) $|w| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2}$

$= \sqrt{\frac{9+16}{25}}$

$= 1$ 1

(ii) $|z| = \sqrt{\left(5/13\right)^2 + \left(12/13\right)^2}$

$= \sqrt{\frac{25+144}{169}}$

$= 1$ 1

i.e. $|w| = |z| = 1$

(ii) $wz = \frac{(3+4i)}{5} \times \frac{(5+12i)}{13}$

$= \frac{15-48+i(36+20)}{65}$

$= \frac{-33}{65} + \frac{56i}{65}$ 1

$w\bar{z} = \frac{(3+4i)}{5} \times \frac{(5-12i)}{13}$

$= \frac{15+48+i(-36+20)}{65}$

$= \frac{63}{65} + \frac{-16i}{65}$ 1

(iii) $|wz| = |w||z| = 1$

$\therefore 33^2 + 56^2 = 65^2$

also $|w\bar{z}| = |w||\bar{z}| = 1$

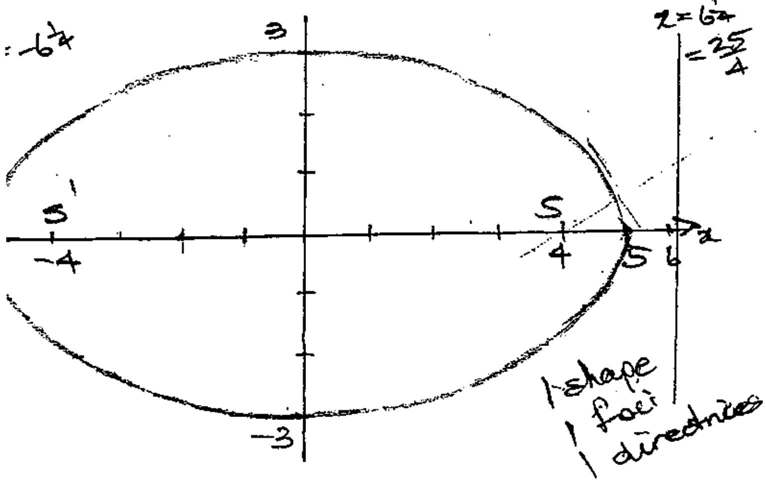
$\therefore 63^2 + 16^2 = 65^2$

i.e. $p = 33 \quad q = 56$ 1

or $p = 16 \quad q = 63.$ 1

QUESTION 7

(i) E: $\left(\frac{x}{5}\right)^2 + \left(\frac{y}{3}\right)^2 = 1.$



$a = 5 \quad b = 3$

$b^2 = a^2(1-e^2)$

$9 = 25(1-e^2)$

$e^2 = 16/25 \quad e = 4/5.$

Directrices are $x = \frac{25}{4} = 6\frac{1}{4}.$

(ii) $\frac{x^2}{25} + \frac{y^2}{9} = 1.$

$\frac{2x}{25} + \frac{2y}{9}\frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\frac{x}{25} \times \frac{9}{y}. \quad 1$

Tangent @ P

$y - y_1 = \frac{-9x_1}{25y_1}(x - x_1) \quad 1$

$25yy_1 - 25y_1^2 = -9x_1x + 9x_1^2$

$9xx_1 + 25yy_1 = 9x_1^2 + 25y_1^2$

i.e. $9xx_1 + 25yy_1 = 225 \quad 1$

(from E: $9x_1^2 + 25y_1^2 = 225$)

(iii) $P(x_1y_1)$ to $Q(x_2y_1)$

$\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}$ through (4,0)

$\therefore \frac{-y_1}{4-x_1} = \frac{y_2-y_1}{x_2-x_1} \quad 1$

Method 2: $m_{PS} = m_{QS}$

$\therefore (4-x_1)(y_2-y_1) = -y_1(x_2-x_1)$

i.e. $4(y_2-y_1) = x_1y_2 - x_1y_1$ 1 $- y_1x_2 + x_1y_1$

$4(y_2-y_1) = x_1y_2 - y_1x_2$

(iv) $Q(x_2y_2)$ on ellipse

Tangent @ P (or use (ii) equation)

$\frac{xx_1}{25} + \frac{yy_1}{9} = 1 \quad$ —① 1

$\frac{xx_2}{25} + \frac{yy_2}{9} = 1 \quad$ —②

From ① $\frac{yy_1}{9} = 1 - \frac{xx_1}{25}$

$y = \frac{9}{y_1}\left(\frac{25 - xx_1}{25}\right)$

Sub in ② $\frac{xx_2}{25} + \frac{y_2}{9} \times \frac{9}{y_1}\left(\frac{25 - xx_1}{25}\right) = 1$

$xx_2 + \frac{y_2}{y_1}(25 - xx_1) = 25$

$xx_2 + \frac{y_2}{y_1}25 - \frac{xx_1y_2}{y_1} = 25$

$x\left(x_2 - \frac{y_2x_1}{y_1}\right) = 25\left(1 - \frac{y_2}{y_1}\right) \quad 1$

$x = 25\left(\frac{y_1-y_2}{\cancel{y_1}}\right) \times \left(\frac{\cancel{y_1}}{x_2y_1 - y_2x_1}\right)$

$= \frac{25(y_1-y_2)}{-4(y_2-y_1)}$ from (iii)

$x = +\frac{25}{4}$ i.e. Directrix 1

(v) Normal $y - y_1 = \frac{25y_1}{9x_1}(x - x_1) \quad 1$

If S (4,0) $-y_19x_1 = 25y_1(4 - x_1)$

$-9x_1y_1 = 100y_1 - 25x_1y_1$

$16x_1\cancel{y_1} = 100\cancel{y_1} \quad y_1 \neq 0$

$\therefore x_1 = \frac{25}{4} \quad 1$

i.e. outside ellipse

$\therefore$ impossible. 1

If S' (−4,0) $\quad x_1 = \frac{100}{34}$ possibl

QUESTION 8.

(a) (i) $T = (a\cos\theta, a\sin\theta)$

as on circle $x^2+y^2=a^2$

The tangent TM is

$$x a\cos\theta + y a\sin\theta = a^2$$

$$\therefore x\cos\theta + y\sin\theta = a \quad 1$$

At M $y=0$

$$\therefore x\cos\theta = a$$

$$x = a\sec\theta.$$

This is also x co-ord of P

$$\therefore \frac{a^2\sec^2\theta}{a^2} - \frac{y^2}{b^2} = 1.$$

$$y^2/b^2 = \sec^2\theta - 1 \quad 1$$

$$y^2 = b^2\tan^2\theta$$

$$y = \pm b\tan\theta$$

$P = (a\sec\theta, b\tan\theta)$ as shown

(ii) $Q = (a\sec\varphi, b\tan\varphi)$

$\therefore$ Chord PQ is

$$\frac{y - b\tan\theta}{x - a\sec\theta} = \frac{b\tan\varphi - b\tan\theta}{a\sec\varphi - a\sec\theta} \quad 1$$

But $\varphi = \frac{\pi}{2} - \theta.$

$$\therefore \frac{y - b\tan\theta}{x - a\sec\theta} = \frac{b\cot\theta - b\tan\theta}{a\operatorname{cosec}\theta - a\sec\theta}$$

$$= \frac{b}{a} \times \frac{\frac{\cos\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta}}{\frac{1}{\sin\theta} - \frac{1}{\cos\theta}} \quad 1$$

$$= \frac{b}{a} \times \frac{\cos^2\theta - \sin^2\theta}{\cos\theta - \sin\theta}$$

$$= \frac{b}{a}(\cos\theta + \sin\theta) \quad 1$$

$$\therefore ay - ab\tan\theta = b(x - a\sec\theta)(\cos\theta + \sin\theta)$$

$$= bx(\cos\theta + \sin\theta) - ba(1 + \tan\theta)$$

$$\therefore ay = bx(\cos\theta + \sin\theta) - ba$$

i.e. $ay = b(\cos\theta + \sin\theta)x - ab.$

(iii) $y = \frac{b}{a}(\cos\theta + \sin\theta)x - b.$

Says for all values of θ

line has y-indpt $-b$ 1

i.e. passes thro' $(0, -b)$ 1

(iv) as $\theta \to \pi/2$ $\varphi \to 0$

Chord $\to y = \frac{b}{a}(0+1)x - b$ 1

i.e. $y = \frac{b}{a}x - b$

which is $\parallel$ to $y = \frac{b}{a}x$ 1

which is an asymptote.

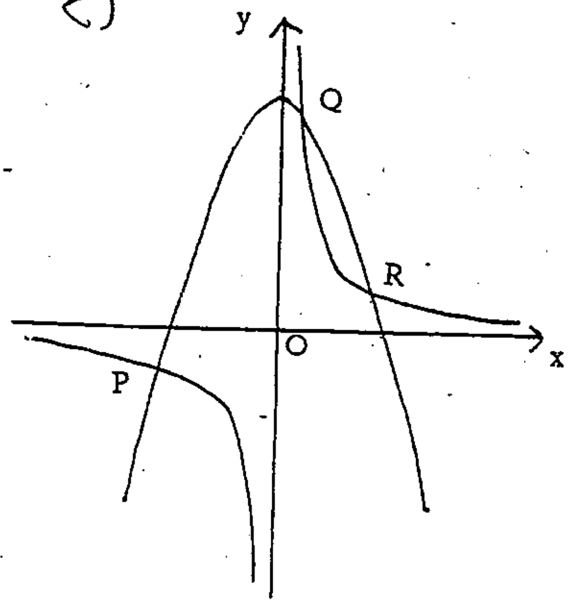
8(a)(i) In ΔOTM $\cos\theta = \frac{a}{OM}$

$\therefore OM = a\sec\theta = P_x$

9

QUESTION 8

(b) $y = k - x^2$ (k real)
$y = \frac{1}{x}$



Intersect @ P, Q, R
$P = (\alpha, \frac{1}{\alpha})$ $Q(\beta, \frac{1}{\beta})$ $R(\gamma, \frac{1}{\gamma})$

(i) α, β, γ are roots of
$k - x^2 = \frac{1}{x}$
i.e. $x^3 - kx + 1 = 0$
For roots $\alpha^2 \beta^2 \gamma^2$ Let $x = \sqrt{x}$
$\therefore (\sqrt{x})^3 - k(\sqrt{x}) + 1 = 0$
$\sqrt{x}(x - k) = -1$
$(x-k)^2 \quad x = 1$
$x^3 - 2kx^2 + k^2x - 1 = 0$

(ii) The above has $\alpha^2 \beta^2 \gamma^2$
To find roots $\frac{1}{\alpha^2}, \frac{1}{\beta^2}, \frac{1}{\gamma^2}$
Let $x = \frac{1}{x}$.
$\therefore \left(\frac{1}{x}\right)^3 - 2k\left(\frac{1}{x}\right)^2 + k^2\left(\frac{1}{x}\right) - 1 = 0$
$1 - 2kx + k^2x^2 - x^3 = 0$
i.e. $x^3 - k^2x^2 + 2kx - 1 = 0$

(iii) $OP^2 + OQ^2 + OR^2$
$= \left(\alpha^2 + \frac{1}{\alpha^2}\right) + \left(\beta^2 + \frac{1}{\beta^2}\right) + \left(\gamma^2 + \frac{1}{\gamma^2}\right)$
$= \left(\alpha^2 + \beta^2 + \gamma^2\right) + \left(\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}\right)$
(Sums of roots previous equ^{n}s
$= +2k \quad + k^2$
$= k^2 + 2k.$

/6