Name:____________________

Teacher:____________________

Killara

CONSERVA PROGREDERE

MATHEMATICS

4 UNIT PAPER

HSC
Assessment N° 1

December 2007

Time Allowed - **70 minutes**

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- Questions are of different value.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Board-approved calculators may be used.

The question paper must be handed to the supervisor at the end of the examination.

1 If $(1+i)z = 3-i$, find

a) z in the form $x+iy$ [2]

b) $|z|$ [1]

c) $\arg z$ [1]

d) $\overline{z}$ [1]

2 If z is a complex number prove that $|z|^2 = z \times \overline{z}$ [3]

3 If the real part of $\dfrac{z-4}{z-2i}$ is zero, prove that the locus of the point representing z in the Argand Diagram is a circle of radius $\sqrt{5}$ units. [5]

4 sketch the region of the Argand Diagram whose points z satisfy the inequalities

a) $\left|z-\overline{z}\right| \le 4$ and $-\dfrac{\pi}{3} \le \arg z \le \dfrac{\pi}{3}$. [3]

b) $1 < \mathrm{Re}(z) < 2$ and $\arg\left(\dfrac{z+1}{z-1}\right) \le \dfrac{\pi}{4}$. [4]

5 a) Write the complex number $1-\sqrt{3}\,i$ in modulus-argument form. [2]

b) Hence use De Moivre's theorem to find $\left(1-\sqrt{3}\,i\right)^{10}$ in the form $a+ib$, where a, b are real numbers. [2]

6 a) If a is a complex cube root of unity, show that ω^2 is the other complex cube root. [2]

b) Evaluate $\dfrac{\omega^2}{2} - \dfrac{\omega+1}{2\omega}$ [2]

7 The complex number z is given by $z = -1 - i\sqrt{3}$

a) Write the values of $\arg z$ and $|z|$. [2]

b) If n is integral, explain why z^n is never purely imaginary. [3]

8 a) Find the six sixth roots of -1. Leave your answers in modulus-argument form. [4]

b) Hence, or otherwise solve $z^4 - z^2 + 1 = 0$. [3]

[total 40 marks]

$(1+i)z = 3-i$

a) $z = \dfrac{3-i}{1+i}$

$= \dfrac{(3-i)(1-i)}{(1+i)(1-i)}$

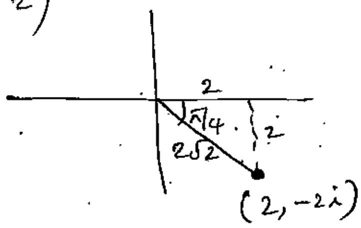
$= \dfrac{3-3i-i+1}{2}$

$= \dfrac{4-4i}{2} = 2-2i$

(b) $|z| = \sqrt{2^2+(-2)^2}$

$= 2\sqrt{2}$.

(c) $\arg(z)$



$= -\dfrac{\pi}{4}$.

(d) $\bar{z} = \overline{2-2i} = 2+2i$

2. Let $z = x+iy$.

$|z|^2 = x^2+y^2$. — ①

$z\bar{z} = (x+iy)(x-iy)$

$= x^2+y^2$ ②.

Using ① & ②,

$|z|^2 = z\bar{z}$.

3. $\dfrac{z-4}{z-2i} = \dfrac{x+iy-4}{x+iy-2i}$

$= \dfrac{x-4+iy}{x+i(y-2)}$

$= \dfrac{(x-4+iy)\,[x-i(y-2)]}{[x+i(y-2)]\,[x-i(y-2)]}$

$= \cancel{x^2-4x+xyi-xyi-4iy-8i+xyi}$

$= \dfrac{x(x-4)+i(x-4)(y-2)+xyi+y(y-2)}{x^2+(y-2)^2}$

$= \dfrac{x^2-4x+xyi-2xi-4yi+8i+xyi+y^2-2y}{x^2+(y-2)^2}$

$= \dfrac{x^2-4x+y^2-2y+i(2xy-2x-4y+8)}{x^2+(y-2)^2}$.

$Re\left(\dfrac{z-4}{z-2i}\right) = 0$.

ie $\dfrac{x^2-4x+y^2-2y}{x^2+(y-2)^2} = 0$.

$\therefore x^2-4x+y^2-2y = 0$.

ie $x^2-4x+4+y^2-2y+1 = 5$

$(x-2)^2+(y-1)^2 = 5$

Circle with centre $(2,1)$ and radius $\sqrt{5}$.

④ a) $|z-\bar{z}| \le 4$

and $-\frac{\pi}{3} \le \arg(z) \le \frac{\pi}{3}$.

$z = x + iy$.

$|z-\bar{z}| = |x+iy - x + iy|$

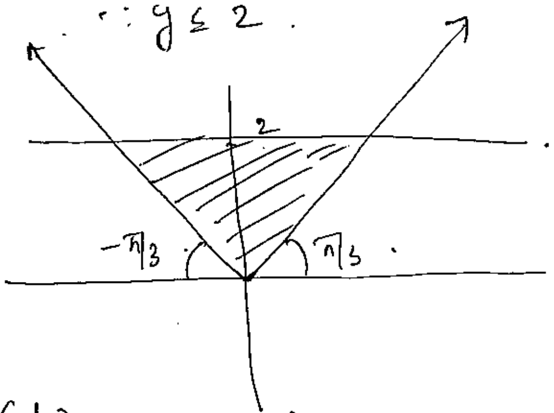
$= |2iy| \le 4$.

$|0+2iy| \le 4$.

$\sqrt{0+(2y)^2} \le 4$.

$2y \le 4$

$\therefore y \le 2$.

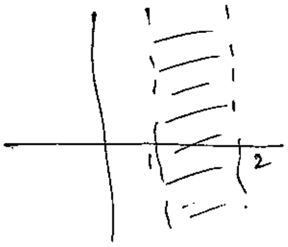


(b) $1 < Re(z) < 2$ and

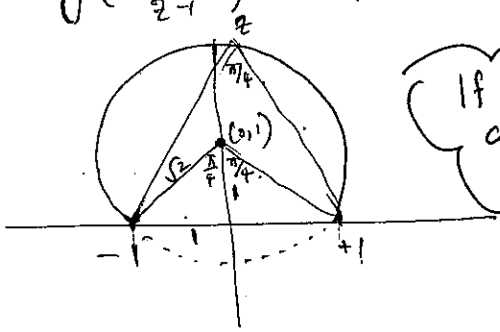
$\arg\left(\frac{z+1}{z-1}\right) \le \pi/4$.

$Re(z) = x$.

$1 < x < 2$



$\arg\left(\frac{z+1}{z-1}\right) \le \pi/4$.

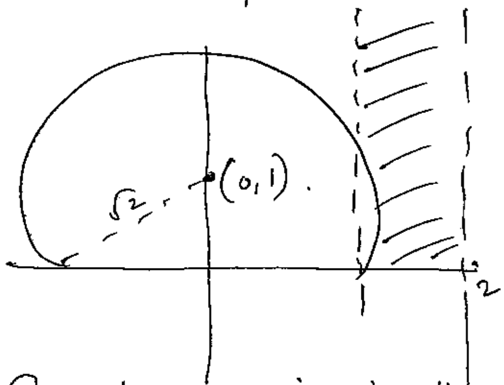


If z was outside the circle, the angle will be less than $\pi/4$

$\therefore$ equation of this circle is

$x^2 + (y-1)^2 = 2$.

When $\arg\left(\frac{z+1}{z+1}\right) \le \pi/4$, it is the outside of this circle.

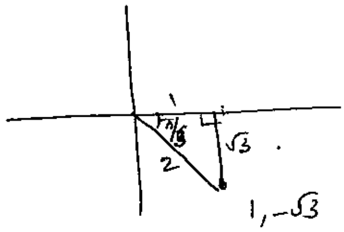


The shaded region is the required area.

⑤ $1-\sqrt{3}i$

a)

$= 2 \operatorname{cis}\left(\frac{\pi}{3}\right)$.



(b) $(1-\sqrt{3}i)^{10}$

$= \left(2 \operatorname{cis}\frac{\pi}{3}\right)^{10} = 2^{10} \operatorname{cis}\frac{10\pi}{3}$.

$= 1024\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$.

$= 512(-1-\sqrt{3}i)$

$= -512(1+\sqrt{3}i)$.

6. (a) Consider the equation $z^3 = 1$.

let $z = r \operatorname{Cis} \theta$

$\therefore r^3 \operatorname{Cis} 3\theta = 1$. $\quad r = 1$.

ie ~~~~ $\cos 3\theta = 1$; $\sin 3\theta = 0$

$\therefore 3\theta = 2k\pi + 0$.

$\therefore \theta = \frac{2k\pi}{3} \quad k = 0, 1, 2$..

$k = 0. \quad z_1 = \operatorname{Cis} 0 = 1$.

$k = 1 \quad z_2 = \operatorname{Cis} \frac{2\pi}{3} = \omega$.

$k = 2 \quad z_3 = \operatorname{Cis} \frac{4\pi}{3} = \left(\operatorname{Cis} \frac{2\pi}{3}\right)^2 = \omega^2$

(using of De-Moivre theorem)

$\therefore 1, \omega, \omega^2$ are the roots of unity.

(b). $\frac{\omega^2}{2} - \frac{\omega + 1}{2\omega}$

$= \frac{1}{2}\left(\omega^2 - \frac{\omega+1}{\omega}\right)$

$= \frac{1}{2}\left(\frac{\omega^3 - \omega - 1}{\omega}\right)$

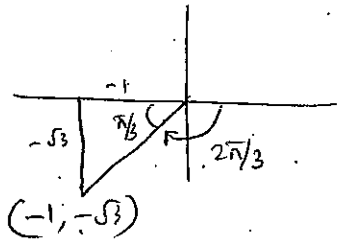
$= \frac{1}{2}\left(\frac{1 - \omega - 1}{\omega}\right)$

$(\omega^3 \text{ being } = 1)$.

$\therefore = \frac{1}{2} \times -1 = \underline{\underline{-\frac{1}{2}}}$.

7. $z = -1 - i\sqrt{3}$

(a)



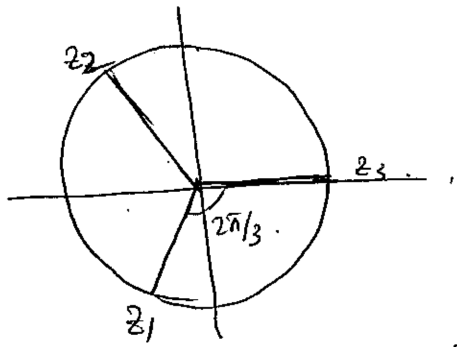
$\arg(z) = -\frac{2\pi}{3}$..

$|z| = \sqrt{(-1)^2 + (-\sqrt{3})^2} = 2$.

(b) $z = 2 \operatorname{cis}\left(-\frac{2\pi}{3}\right)$.

$\therefore z^n = 2^n \operatorname{cis}\left(-\frac{2n\pi}{3}\right)$.

On an Argand diagram the roots ~~of the~~ are as given below for various integral values of n.



This shows that the ~~roots~~ z^n is either complex or real. ie for no integral value of n, ~~the~~ z^n is ~~an~~ purely imaginary.

(8) (a) $z^6 = -1$.

let $z = r\,\text{cis}\,\theta$, $r = 1$.

$\therefore z = \text{cis}\,\theta$.

$z^6 = \text{cis}\,6\theta = -1$.

$\therefore \cos 6\theta = -1 \quad \sin 6\theta = 0$.

$$6\theta = 2k\pi + \pi = (2k+1)\pi$$

~~$\therefore \theta = \frac{(2k+1)\pi}{6}$, $k = 0, 1, 2, 3, 4, 5$.~~

~~$k=0$, $z_1 = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$~~

~~$k=1$, $z_2 = \text{cis}\,\frac{\pi}{2} = i$~~

~~$k=2$ $z_3 = \text{cis}\,\frac{5\pi}{6}$.~~

$$\theta = (2k+1)\frac{\pi}{6}, \quad k = 0, \pm 1, \pm 2, -3$$

$k=0$ $z_1 = \text{cis}\,\frac{\pi}{6}$.

$k=1$ $z_2 = \text{cis}\,\frac{\pi}{2} = i$

$k=-1$, $z_3 = \text{cis}\left(-\frac{\pi}{6}\right)$.

$k=-2$ $z_4 = \text{cis}\left(-\frac{\pi}{2}\right) = -i$

$K=2$ $z_5 = \text{cis}\left(\frac{5\pi}{6}\right)$

$k=-3$ $z_6 = \text{cis}\left(-\frac{5\pi}{6}\right)$.

(b). $z^6 + 1 = (z^2)^3 + 1$

$= (z^2+1)(z^4 - z^2 + 1)$.

z^2+1 have roots $z = \pm i$.

$\therefore$ Roots of $z^4 - z^2 + 1 = 0$ are

$\text{cis}\left(\pm \pi/6\right)$, $\text{cis}\left(\pm \frac{5\pi}{6}\right)$.

$= \frac{\sqrt{3}}{2} \pm \frac{1}{2}i$, $-\frac{\sqrt{3}}{2} \pm \frac{1}{2}i$