Year 12

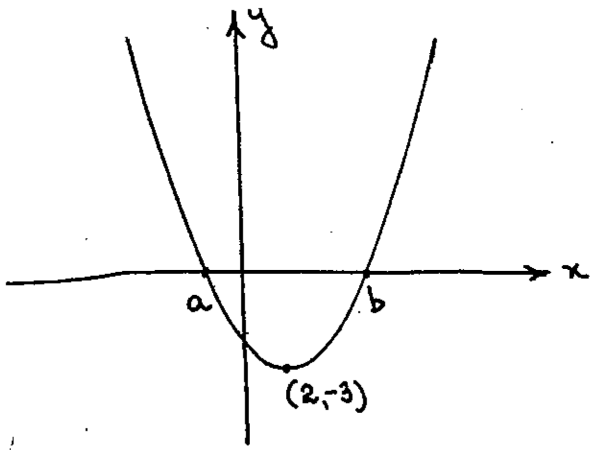
Extension 2 Mathematics

Term 2 Assessment 2007

Time allowed: 70 minutes.
All questions may be attempted.
START EACH QUESTION ON A NEW PAGE

Question 1 (10 marks)

a. The graph of y = f(x) is sketched in the diagram shown.



Copy this graph and draw neat sketches, on separate diagrams, showing all the main features, of

(i) $y = |f(x)|$ 1

(ii) $y = f(-x)$ 1

(iii) $y = \dfrac{1}{f(x)}$ 2

(iv) $y = [f(x)]^2$ 2

(v) $y = \sqrt{f(x)}$ 2

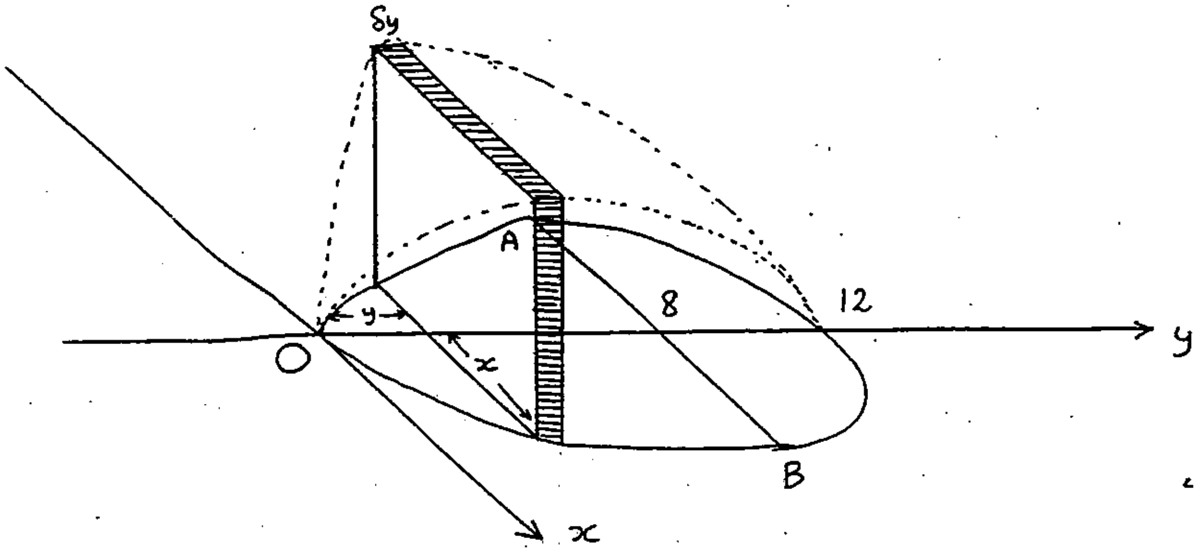
b. Draw a neat sketch, showing the main features, of the graph of $y = \cos(\sin^{-1} x)$. 2

Question 2 (15 marks)

a. The region R in the first quadrant bounded by the curve $y = x(4 - x^2)$ and the x - axis is rotated about the line x = 2 to form a solid of revolution. Use the method of "cylindrical shells" to find the volume of the solid. 4

b. The region bounded by $y = 4x - x^2$ and $0 \le x \le 4$ is rotated about the y-axis. The solid so formed is sliced by planes perpendicular to the y-axis. Express the areas of the cross-sections so formed as a function of y and hence find the volume of the solid. 5

c.



The diagram shows a solid with its base in the x-y plane. Every cross section perpendicular to the y-axis is a square. One part of the base is the segment OAB of the parabola $x^2 = 2y$ cut off by the line y = 8. The other part of the base is a semicircle with diameter AB.
Consider a slice of the solid of width δy and perpendicular to the y-axis as shown.
(i) Find an expression for the volume δV of S in terms of x and δy. 1
(ii) Find the volume of the solid. 5

Question 3 (18 marks)

a. Using the substitution u = x + 2, or otherwise, find $\int \frac{dx}{\sqrt{x^2+4x+3}}$. 2

b. Find $\int \frac{dx}{e^x + e^{-x}}$ using the substitution $u = e^x$. 2

c. Evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{1+\cos x+\sin x}$ using $t = \tan\frac{x}{2}$ 3

d. Using partial fractions, find $\int \frac{6x+4}{(x+2)(4-3x^2)}dx$ 4

e. Find $\int \tan^4 x \ dx$ 3

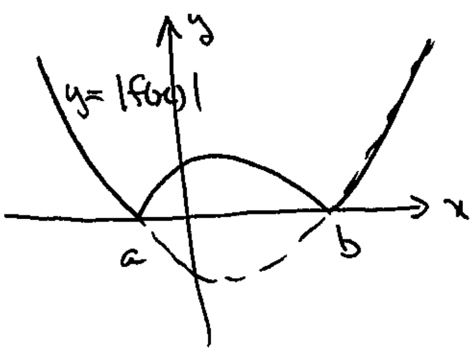
f. (i) If $I_n = \int_1^e x(\ln x)^n \, dx$ for n a positive integer show that

$$I_n = \frac{e^2}{2} - \frac{n}{2} I_{n-1}$$ 3

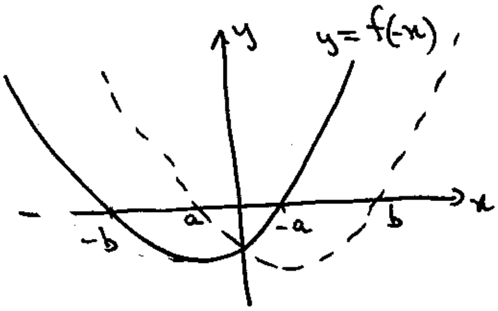
(ii) Hence evaluate $\int_1^e x(\ln x)^3 \, dx$ 1

END OF EXAMINATION

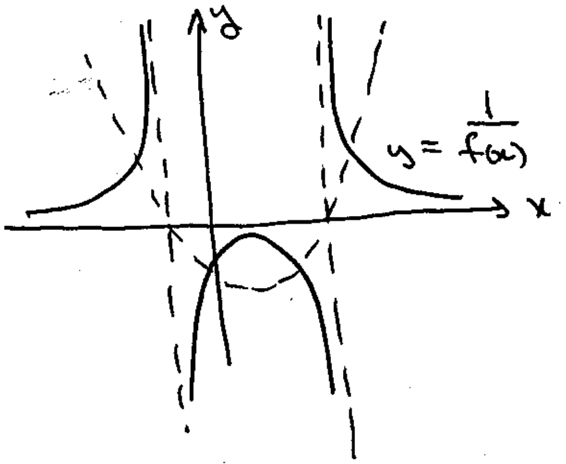
Q.1 a (i)



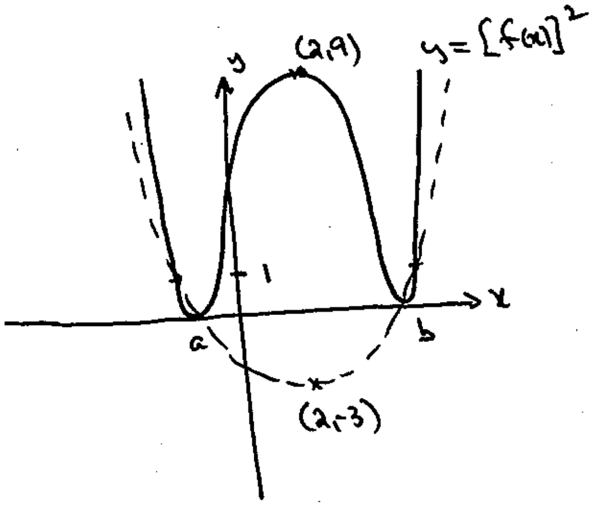
(ii)



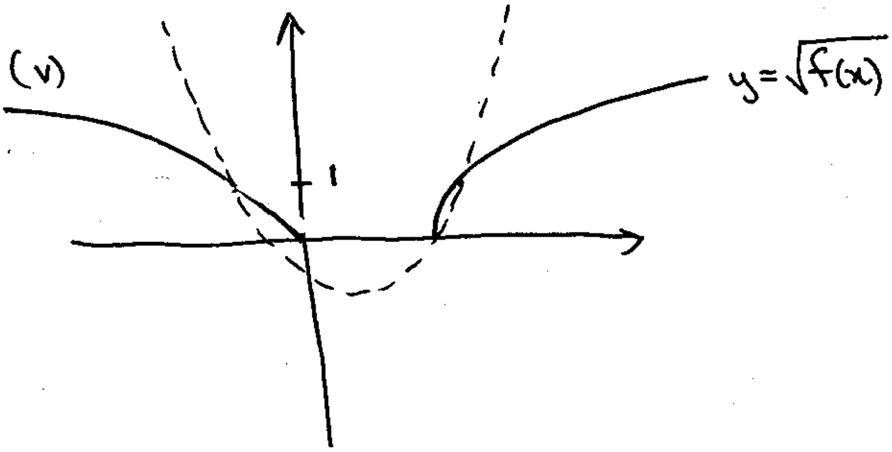
(iii)



(iv)



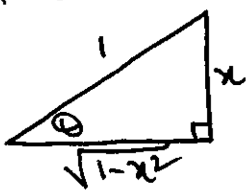
(v)



b) $y=\cos(\sin^{-1}x)$, $-1\leq x\leq 1$

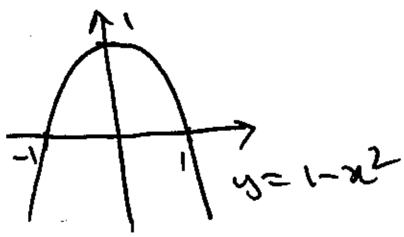
Let $\theta=\sin^{-1}x$, $-\frac{\pi}{2}\leq\theta\leq\frac{\pi}{2}$

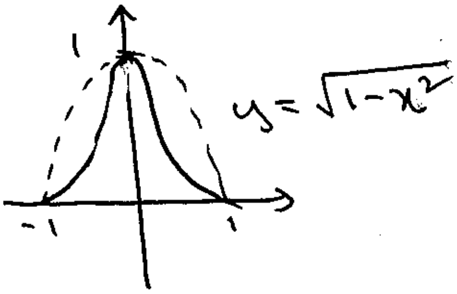
then $\sin\theta=x$



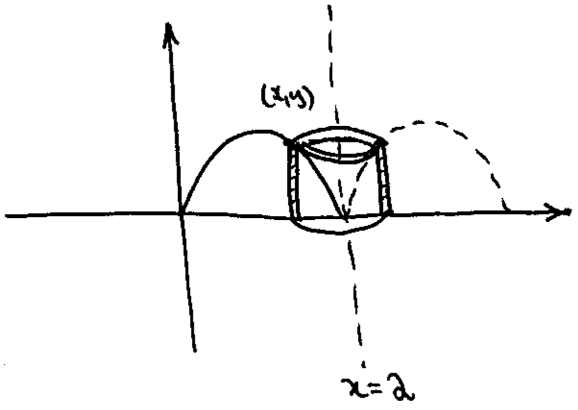
$y=\cos\theta$

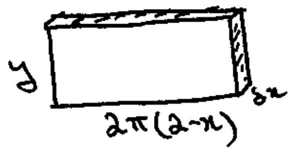
$=\sqrt{1-x^2}$





Q.2 a.





$$\delta V = 2\pi(2-x)y\,\delta x$$

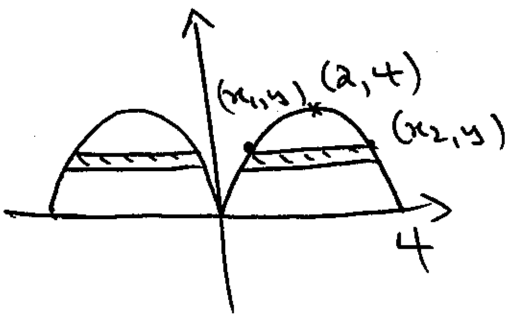
$$V = \int_0^2 2\pi(2-x)(4x-x^3).\,dx$$

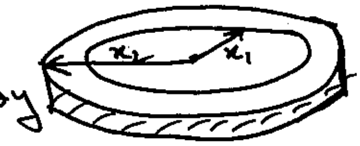
$$= 2\pi\int_0^2 (8x-4x^2-2x^3+x^4).\,dx$$

$$= 2\pi\left[4x^2-\frac{4x^3}{3}-\frac{x^4}{2}+\frac{x^5}{5}\right]_0^2$$

$$= \frac{112\pi}{15}$$

b.





$$\delta V = \pi(x_2^2-x_1^2)\,\delta y \quad 1$$

$$\therefore V = \int_0^4 \pi(x_2+x_1)(x_2-x_1).\,dy$$

$$= \int_0^4 \pi 4\sqrt{16-4y}\,.\,dy$$

$$= -\pi\left[\frac{2}{3}(16-4y)^{3/2}\right]_0^4$$

$$= \frac{128\pi}{3}u^3 \quad 1$$

$y = 4x - x^2$

i.e. $x^2-4x+y=0$

$\therefore\ x_2+x_1 = 4$ ✓

$(x_2-x_1)^2 = (x_2+x_1)^2-4x_2x_1$ 1

$x_2-x_1 = \sqrt{(x_2+x_1)^2-4x_1x_2}$

$= \sqrt{16-4\times y}$

$= \sqrt{16-4y}$ 1

c. (i) $\delta V = 4x^2 \delta y$

(ii) Consider the part with base OAB:

$$V = \int_0^8 4x^2 . dy$$

$$= \int_0^8 8y\,dy \quad \text{since } x^2 = 2y$$

$$= 4\left[y^2\right]_0^8$$

$$= 256 \text{ u}^3$$

Consider the part with semicircular base:

The equ'n of the circle containing this is

$$x^2 + (y-8)^2 = 16 \quad \left(\text{Centre is } (0,8), r = 4\right)$$

ie. $x^2 = 16 - (y-8)^2$

$$\therefore V = \int_8^{12} 4\left[16 - (y-8)^2\right] dy$$

$$= 4\left[16y - \tfrac{1}{3}(y-8)^3\right]_8^{12}$$

$$= 4\left[16\times12 - \tfrac{1}{3}\times4^3 - 16\times8\right]$$

$$= 170\tfrac{2}{3} \text{ u}^3$$

$\therefore$ Volume of solid $= 256 + 170\frac{2}{3}$

$= 426\frac{2}{3}$

Q.3 a. $\displaystyle\int \frac{dx}{\sqrt{x^2+4x+3}} = \int \frac{dx}{\sqrt{(x+2)^2-1}}$

Let $u = x+2$ $\displaystyle = \int \frac{du}{\sqrt{u^2-1}}$

$= \ln\{u + \sqrt{u^2-1}\}$ (table of standard int.'s)

$= \ln\{(x+2) + \sqrt{x^2+4x+3}\}$

b. $u = e^x$

$du = e^x \cdot dx$

or $dx = \frac{du}{u}$

$\displaystyle I = \int \frac{1}{u+\frac{1}{u}} \cdot \frac{du}{u}$

$\displaystyle = \int \frac{du}{u^2+1}$

$= \tan^{-1} u + c$

$= \tan^{-1} e^x + c$

c. $t = \tan\frac{x}{2}$

$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$

$= \frac{1}{2}(1+\tan^2\frac{x}{2})$

$= \frac{1}{2}(1+t^2)$

$dx = \frac{2\,dt}{1+t^2}$

$x=0$, $t=0$

$x=\frac{\pi}{2}$, $t=1$

$\displaystyle I = \int_0^{\frac{\pi}{2}} \frac{1}{1+\frac{1-t^2}{1+t^2}+\frac{2t}{1+t^2}} \times \frac{2dt}{1+t^2}$

$\displaystyle = \int_0^{\frac{\pi}{2}} \frac{2\,dt}{1+t^2+1-t^2+2t}$

$\displaystyle = \int_0^{\frac{\pi}{2}} \frac{2\,dt}{2+2t}$

$\displaystyle = \int_0^{\frac{\pi}{2}} \frac{dt}{1+t}$

$= [\ln(1+t)]_0^1$

$= \ln 2$

d. Let $\dfrac{6x+4}{(x+2)(4-3x^2)} \equiv \dfrac{A}{x+2} + \dfrac{Bx+C}{4-3x^2}$

$$6x+4 \equiv A(4-3x^2) + (Bx+C)(x+2)$$

Put $x=-2 \Rightarrow \quad -8 = A(4-12) \quad \Rightarrow A = 1$

$x=0 \Rightarrow \quad 4 = 4A + 2C \quad \Rightarrow C = 0$

$x=1 \Rightarrow \quad 10 = A + 3B + 3C \Rightarrow B = 3$

$$I = \int \left(\frac{1}{x+2} + \frac{3x}{4-3x^2} \right) dx$$

$$= \ln(x+2) - \frac{1}{2}\ln(4-3x^2) + C$$

e. $\displaystyle\int \tan^4 x \, dx = \int \tan^2 x \,(\sec^2 x - 1).dx$

$$= \int \tan^2 x \sec^2 x . dx - \int \tan^2 x \, dx$$

$$= \frac{1}{3}\tan^3 x - \int (\sec^2 x - 1).dx$$

$$= \frac{1}{3}\tan^3 x - \tan x + x + C$$

f. (i) $\displaystyle I_n = \int_1^e (\ln x)^n . \frac{d}{dx}\left(\frac{x^2}{2}\right) . dx$

$$= \left[\frac{x^2}{2}(\ln x)^n\right]_1^e - \int_1^e \frac{x^2}{2} \times n(\ln x)^{n-1} . \frac{1}{x}$$

$$= \frac{e^2}{2} - \frac{n}{2}\int_1^e x(\ln x)^{n-1} . dx$$

$$= \frac{e^2}{2} - \frac{n}{2} I_{n-1}$$

f. (ii) $$I_3 = \frac{e^2}{2} - \frac{n}{2}\left[I_2\right]$$

$$= \frac{e^2}{2} - \frac{n}{2}\left[\frac{e^2}{2} - \frac{n}{2} I_1\right]$$

$$= \frac{e^2}{2}\left(1 - \frac{n}{2}\right) + \frac{n^2}{4} I_1$$

$$= \frac{e^2}{4}(2-n) + \frac{n^2}{4}\left[\frac{e^2}{2} + I_0\right]$$

$$= \frac{e^2}{4}\left(2 - n + \frac{n^2}{2}\right) + I_0$$

Now $$I_0 = \int_1^e x \cdot dx$$

$$= \left[\frac{x^2}{2}\right]_1^e$$

$$= \frac{e^2}{2} - 1$$

$$\therefore I_3 = \frac{e^2}{8}(4 - 2n + n^2) + \frac{e^2}{2} - 1$$

$$= \frac{e^2}{8}(4 - 2n + n^2 + 4) - 1$$

$$= \frac{e^2}{8}(n^2 - 2n + 8) - 1$$