Question 3 (Begin a new page)

a) (i) Show that the x coordinates of the stationary points on the graph $y = \{f(x)\}^2$ are the same as the x coordinates of the stationary points <u>or</u> intercepts on the x – axis of the graph $y = f(x)$.

(ii) $f(x) = \sin^2 x - \frac{1}{2}$, $0 \leq x \leq \pi$. On the same axes, and without using further calculus, sketch the graphs of $y = f(x)$ and $y = \{f(x)\}^2$.

b) (i) Sketch the graph of $y = x^2 + \frac{2}{x}$ showing any intercepts on the coordinate axes, asymptotes, stationary points or points of inflexion.

(ii) The equation $x^2 + \frac{2}{x} - k = 0$ has exactly two different real solutions. Find the value of k and the real solutions of the equation.

c) (i) Find $\int \sqrt{e^x}\,dx$

(ii) Find $\int (\cos^2 x - \sin^2 x)dx$

Question 4 (Begin a new page)

a) Consider the function $f(x) = \frac{3x}{(x-1)(4-x)}$.

(i) Express $f(x)$ in partial fractions.

(ii) Find the coordinates and nature of any turning points on the graph $y = f(x)$.

(iii) Sketch the graph $y = f(x)$ showing clearly the coordinates of any turning points and the equations of any asymptotes.

(iv) Find the area of the region bounded by the curve $y = f(x)$ and the x-axis between the lines $x = 2$ and $x = 3$.

b) Consider the function $f(x)=\dfrac{x-1}{x}$.

(i) Sketch the graph $y=f(x)$ showing clearly the coordinates of any points of intersection with the axes and the equations of any asymptotes.

(ii) Use the graph $y=f(x)$ to sketch on separate axes the graphs

(α) $y=|f(x)|$

(β) $y=f(|x|)$

(γ) $y=\{f(x)\}^2$

Question 5 (Begin a new page)

a) (i) Show that the tangent to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ at the point $P(a\cos\theta, b\sin\theta)$ in the first quadrant has equation $bx\cos\theta+ay\sin\theta-ab=0$.

(ii) This tangent cuts the x-axis at point A and the y-axis at point B. Find the minimum area of ΔAOB and show that when this occurs P is the midpoint of AB.

b) (i) Show that the tangent to the rectangular hyperbola $xy=4$ at the point $T(2t,\dfrac{2}{t})$ has equation $x+t^2y=4t$.

(ii) This tangent cuts the x-axis at point Q. Show that the line through Q which is perpendicular to the tangent at T has equation $t^2x-y=4t^3$.

(iii) The line through Q cuts the rectangular hyperbola at the points R and S. Show that the midpoint M of RS has coordinates $M(2t,-2t^3)$

Question 6 (Begin a new page)

a) Find $\displaystyle\int\frac{x+1}{\sqrt{x-1}}dx$ using the substitution $u=x-1$

b) Evaluate $\displaystyle\int_1^2\frac{1}{e^x-1}\,dx$

c) Evaluate $\displaystyle\int_0^{\pi/2}\frac{1}{\cos x+2}\,dx$ using the substitution $t=\tan\dfrac{x}{2}$.

d) (i) If $I_n = \int_0^{\pi/4} \tan^n x \, dx$, n = 0,1,2,........

show that $I_n + I_{n-2} = \dfrac{1}{n-1}$, n= 2,3,4, ...

(ii) Hence find the value of $\int_0^{\pi/4} \tan^5 x \, dx$.

Question 7 (Begin a new page)

a) The base of a particular solid is the region bounded by the parabola $y^2 = 4x$ between its vertex (0,0) and its latus rectum. Find the volume of the solid if every cross section perpendicular to the x-axis

(i) is a square with one side in the base of the solid

(ii) is an equilateral triangle with one side in the base of the solid.

b) The equation $x^3 + kx + 2 = 0$ has roots α, β and γ.

(i) Find the value of $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$.

(ii) Find the monic equation with roots α^2, β^2 and γ^2.

(iii) Show that the value of $\alpha^3 + \beta^3 + \gamma^3$ is independent of k.

Question 8 (Begin anew page)

a) An ellipse has equation $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$.

(i) Find the ellipses eccentricity.
(ii) Find the coordinates of the foci S and S^1 and the equations of the directrices.
(iii) Sketch the ellipse showing the above features.
(iv) If $P(x_1, y_1)$ is a point on the ellipse show that $PS + PS^1$ is independent of the position of P.

b) (i) Sketch the graph of $y = x^2 - 4$ showing the intercepts on the axes.

(ii) Without using calculus, use the graph of $y = x^2 - 4$ to sketch on separate axes the graphs of

(α) $y = |x^2 - 4|$

(β) $y = \dfrac{1}{x^2 - 4}$

(γ) $y^2 = x^2 - 4$

In each case show clearly any intercepts on the axes and any asymptotes.

Question 1 (17)

a $\displaystyle\int \frac{x}{\sqrt{x-1}}\,dx$

let $u = x-1$
$\therefore du = dx$
1 and $x = u+1$.

$= \displaystyle\int \frac{u+1}{\sqrt{u}}\,du$ 1

$= \displaystyle\int \frac{u}{u^{\frac{1}{2}}} + \frac{1}{u^{\frac{1}{2}}}\,du$ (84)

$= \displaystyle\int \left(u^{\frac{1}{2}} + u^{-\frac{1}{2}}\right)du$

$= \frac{2}{3}u^{\frac{3}{2}} + 2u^{\frac{1}{2}} + c$ 1 or $\frac{2}{3}u^{\frac{1}{2}}(u+3) + c$

$= \frac{2}{3}\sqrt{(x-1)^3} + 2\sqrt{x-1} + c$ $= \frac{2}{3}\sqrt{x-1}\,(u+2) + c$

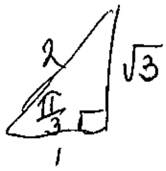
1

b i $\displaystyle\int_{\pi/4}^{\pi/3} (1+\tan^2 x)\tan x\,dx.$

$\tan^2 x + 1 = \sec^2 x.$

$= \displaystyle\int_{\frac{\pi}{4}}^{\pi/3} \sec^2 x \tan x\,dx$

$= \left[\frac{1}{2}\tan^2 x\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$



$= \frac{1}{2}\tan^2\frac{\pi}{3} - \frac{1}{2}\tan^2\frac{\pi}{4}$ (3)

$= \frac{1}{2}\cdot\sqrt{3}^2 - \frac{1}{2}\cdot 1$

$= \frac{1}{2}\cdot 3 - \frac{1}{2}$

$= 1$

Question 1

$\tan^2 + 1 = \sec^2$

b ii $\displaystyle\int_0^{\ln 2} e^x \sec^2(e^x)\,dx.$

$$= \Big[\tan(e^x)\Big]_0^{\ln 2} \checkmark$$

$$= \tan e^{\ln 2} - \tan 1$$

$$= \tan 2 - \tan 1. \quad (\doteqdot 3{\cdot}74) \checkmark$$

(2)

c $\displaystyle\int_0^{\pi/2} \frac{1}{7+5\sin x+5\cos x}\,dx$

let $t = \tan\frac{x}{2}$

$dt = \frac{1}{2}\sec^2\frac{x}{2}\,dx.$

$= \frac{1}{2}\left(\tan^2\frac{x}{2}+1\right)dx$

$= \frac{1}{2}\left(t^2+1\right)dx.$

$\frac{2dt}{t^2+1} = dx \checkmark$

and $\sin x = \frac{2t}{1+t^2}$

$\cos x = \frac{1-t^2}{1+t^2}$

when $x = 0 \Rightarrow t = 0$

$x = \frac{\pi}{2} \Rightarrow t = 1.$

$$= \int_0^1 \left(\frac{1}{7+5\left(\frac{2t}{1+t^2}\right)+5\left(\frac{1-t^2}{1+t^2}\right)}\right)\cdot\frac{2dt}{(1+t^2)} \checkmark$$

$$= \int_0^1 \frac{2dt}{7(1+t^2)+10t+5-5t^2}$$

$$= \int_0^1 \frac{2dt}{7+7t^2+10t+5-5t^2}$$

$$= \int_0^1 \frac{2dt}{2t^2+10t+12.}$$

$$= \int_0^1 \frac{dt}{t^2+5t+6} \checkmark$$

(4)

$$= \int_0^1 \frac{dt}{(t+3)(t+2)}$$

$$= \int_0^1 \frac{1}{t+2} - \frac{1}{t+3}\,dt. \checkmark$$

$$= \Big[\ln(t+2) - \ln(t+3)\Big]_0^1 = \ln\frac{9}{8}.$$

$$= \ln 3 - \ln 4 - \ln 2 + \ln 3$$

Question 1

d $\displaystyle\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{\ln x}{1+x^2}\,dx.$

$u = \frac{1}{x} = x^{-1}$

$du = -x^{-2}\,dx.$

$= -\frac{1}{x^2}\,dx.$

$= -u^2\,dx$

$-\frac{1}{u^2}du = dx.$

when $x = \sqrt{3} \Rightarrow u = \frac{\sqrt{3}}{3}$

when $x = \frac{1}{\sqrt{3}} \Rightarrow u = \sqrt{3}$

$x^2 = \frac{1}{u^2}$

$$= \int_{\sqrt{3}}^{\frac{1}{\sqrt{3}}} \frac{\ln\left(\frac{1}{u}\right)}{1+\frac{1}{u^2}} \cdot \frac{-1}{u^2}\,du. \checkmark$$

$$= \int_{\sqrt{3}}^{\frac{1}{\sqrt{3}}} \frac{\ln 1 - \ln u}{u^2+1} \cdot -1\,du.$$

$$= \int_{\sqrt{3}}^{\frac{1}{\sqrt{3}}} \frac{\ln u}{1+u^2}\,du$$

$$= -\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{\ln u}{1+u^2}\,du.$$

4

$$= -\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{\ln x}{1+x^2}\,dx. \checkmark$$

$$\therefore \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{\ln x}{1+x^2}\,dx = -\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{\ln x}{1+x^2}\,dx$$

$$2\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{\ln x}{1+x^2}\,dx = 0.$$

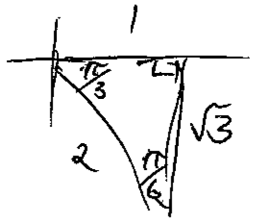
$$\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{\ln x}{1+x^2}\,dx = 0 \checkmark$$

Question 2 16

a i $z = 2 - 2\sqrt{3}i$

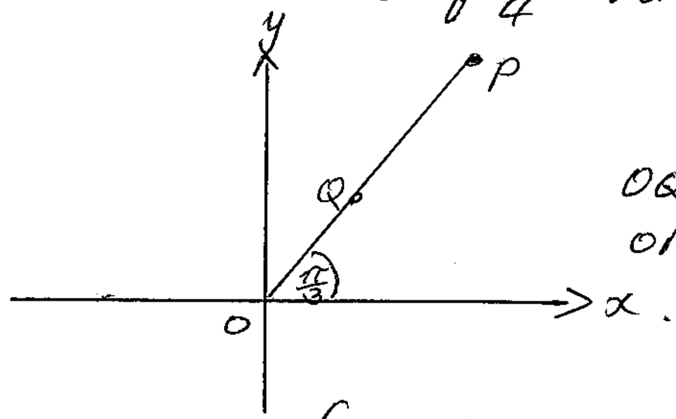
$= 4\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$

$= 4\left\{\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right\}$ 2



ii $\bar{z}$ has modulus of 4 and argument of $\frac{\pi}{3}$

$\frac{1}{z}$ has modulus of $\frac{1}{4}$ and argument of $\frac{\pi}{3}$



$OQ = \frac{1}{4}$

$OP = 4$.

2

OQ represents $\frac{1}{z}$

OP represents $\bar{z}$.

b i) let $z = x + iy$. $x \geq 0$.

$2|z| = 3(z + \bar{z})$

$4|z|^2 = 9(z + \bar{z})^2$

$4(x^2 + y^2) = 9(2x)^2$

$4x^2 + 4y^2 = 36x^2$

$4y^2 = 32x^2$

$y^2 = 8x^2 \quad \therefore \quad y = \pm 2\sqrt{2}x$

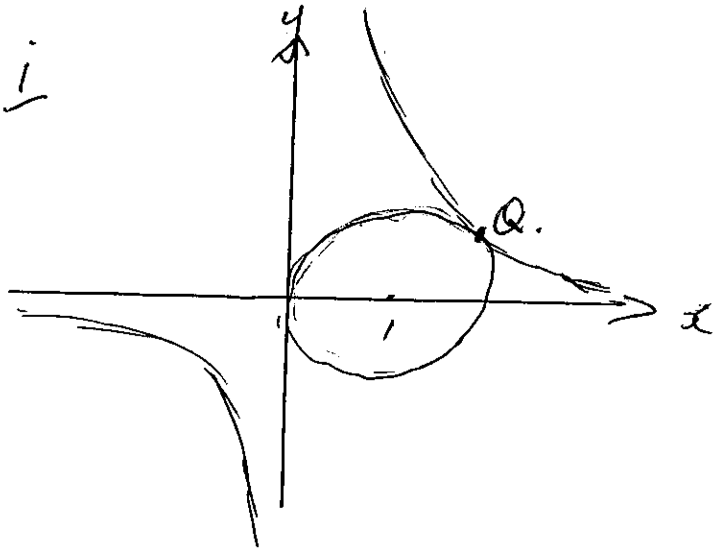
4

Question 2

b ii

1

c i



1

ii if β is a repeated root of $P(x)=0$ then there is a polynomial $Q(x)$ such that

$$P(x) = (x-\beta)^2 Q(x). \checkmark$$

$$\therefore P'(x) = 2(x-\beta)Q(x) + Q'(x)(x-\beta)^2$$

$$\checkmark = (x-\beta)\left[2Q(x) + (x-\beta)Q(x)\right]$$

$$\therefore P'(\beta) = 0. \checkmark$$

3

Question 2

c iii At Q in i

$xy = c^2$

$y = \frac{c^2}{x}$

$y^2 = \frac{c^4}{x^2}$ — ① and $(x-1)^2 + y^2 = 1$ — ②

substitute ① into ②.

$(x-1)^2 + \frac{c^4}{x^2} = 1.$

$x^2(x-1)^2 + c^4 = x^2$

Now, since the circle and hyperbola touch at Q, then the x-coordinate β of Q is a repeated real root of this equation.

There are no other real roots since the circle and hyperbola have no other intersection points.

3

The remaining 2 roots are complex roots that are complex conjugates (α and $\bar{\alpha}$) since the equation has real coefficients.

Question 3 (14)

a i $y = \{f(x)\}^2$

$\frac{dy}{dx} = 2f(x) \cdot f'(x)$. (1)

$\frac{dy}{dx} = 0 \Rightarrow f(x) = 0$ or $f'(x) = 0$

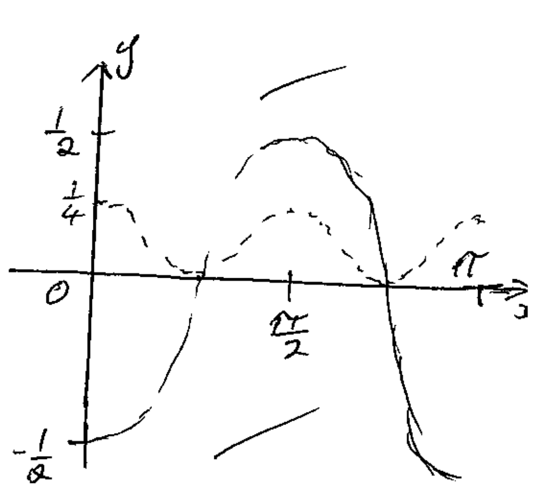
ii $f(x) = \sin^2 x - \frac{1}{2}$.

when $f(x) = 0$

then $\sin^2 x = \frac{1}{2}$

$x = \frac{\pi}{4}, \frac{3\pi}{4}$

(3)



b i

$y = x^2 + \frac{2}{x}$.

* as $x \to 0$, $y \to \infty$ $x \neq 0$
$\therefore$ y is an asymptote.

* $y = 0$ $\therefore x^2 = -\frac{2}{x}$

$x^3 = -2$

* $\frac{dy}{dx} = 2x - \frac{2}{x^2}$

$\frac{dy}{dx} = 0 \Rightarrow \frac{2}{x^2} = 2x$

$x^3 = 1$

$x = 1$.

* $\frac{d^2y}{dx^2} = 2 + \frac{4}{x^3}$

$\frac{d^2y}{dx^2} = 0 \Rightarrow \frac{4}{x^3} = -2$

$x^3 = -2$.

$x = -\sqrt[3]{2}$

* when $x = 1$, $\frac{d^2y}{dx^2}$ is (+)ve

$\therefore$ Min T. Point at $(1, 3)$.

* when $x = -\sqrt[3]{2}$, ~~$\frac{d^2y}{dx^2} =$~~

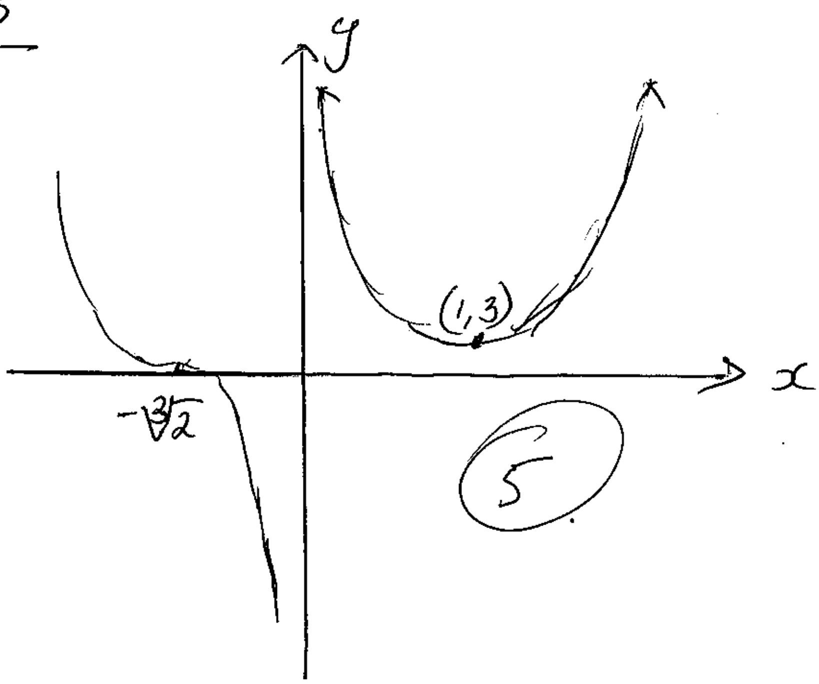
x	<	$-\sqrt[3]{2}$	>
y''	+	0	−

$\therefore$ Point of Inflexion at $(-\sqrt[3]{2}, 0)$.

Question 3

b i

* as $x \to \infty$,
$y \to \infty$.
* as $x \to -\infty$
$y \to \infty$



b ii $y = x^2 + \frac{2}{x} - k$. is the same as the above graph moved down 'k' units

For $x^2 + \frac{2}{x} - k = 0$ to have exactly 2 real roots (different), the graph touches the x-axis at $x = 1$

$\therefore$ the graph is moved down 3 units.

$\therefore k = 3$

$\therefore x^2 + \frac{2}{x} - 3 = 0$ 3

$x^3 - 3x + 2 = 0$.

$x = 1, 1$ are the 2 equal roots

By inspection, the other root is -2.

$(x-1)(x-1)(x+2) = 0$.

$\therefore$ the 2 real solutions are 1 and -2.

Question 3

c i $\int \sqrt{e^x}\, dx = \int e^{\frac{x}{2}}\, dx$

$= 2e^{\frac{x}{2}} + c$

$= 2\sqrt{e^x} + c.$ 1

ii $\int (\cos^2 x - \sin^2 x)\, dx = \int \cos 2x\, dx.$

$= \frac{1}{2}\sin 2x + c$ 1

Question 4 (16)

a
i. $f(x) = \frac{3x}{(x-1)(4-x)}$ let $\frac{3x}{(x-1)(4-x)} = \frac{A}{x-1} + \frac{B}{4-x}$

$\checkmark = \frac{A(4-x)+B(x-1)}{(x-1)(4-x)}$

Equating numerators

$3x = A(4-x) + B(x-1)$

when $x = 4$

$12 = 3B$

$B = 4$

when $x = 1$

$3 = 3A$

$A = 1.$ ✓ (2)

$\therefore f(x) = \frac{1}{x-1} + \frac{4}{4-x}.$

ii $f(x) = \frac{1}{x-1} + \frac{4}{4-x}$

$= (x-1)^{-1} + 4(4-x)^{-1}$

$f'(x) = -1(x-1)^{-2} + 4(4-x)^{-2}$

$= \frac{-1}{(x-1)^2} + \frac{4}{(4-x)^2}$

$= \frac{-(4-x)^2 + 4(x-1)^2}{(x-1)^2(4-x)^2}$

$f'(x) = 0$ when $4(x-1)^2 = (4-x)^2$

ie when $x = \pm 2.$ ✓✓

$f''(x) = 2(x-1)^{-3} + 8(4-x)^{-3}$ (3)

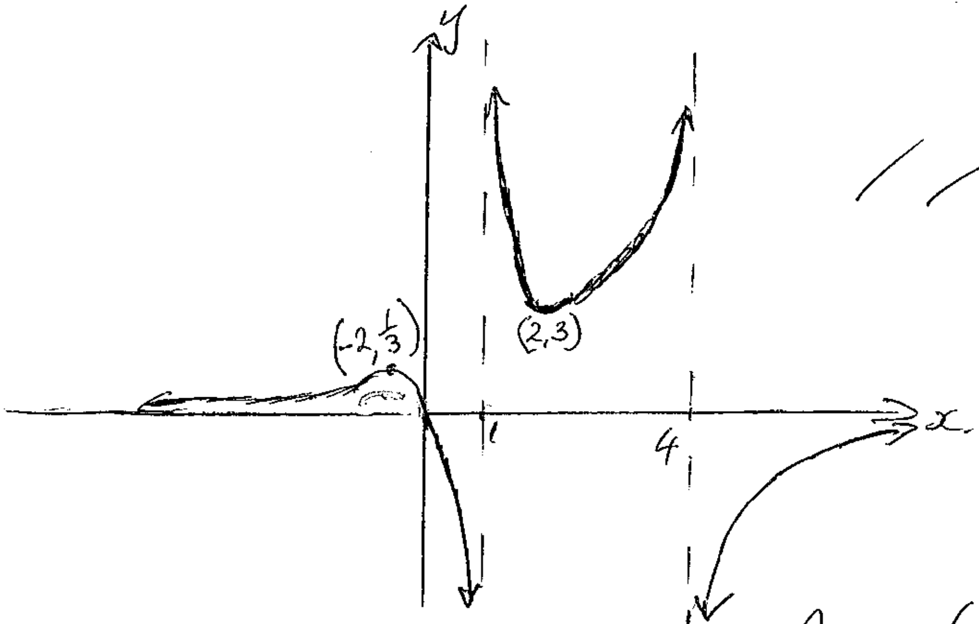
when $x = 2$ $(y = 3)$ $f''(2)$ is positive ✓

$\therefore$ Min T. Point at $(2, 3)$

when $x = -2$ $(y = \frac{1}{3})$ $f''(-2)$ is < 0 $\therefore$ Max T. P at $(-2, \frac{1}{3})$

Question 4

iii



* when $x = 0$
  $f(x) = 0$

* Asymptotes at $x = 1$ and $x = 4$

* as $x \to {}^{+}\infty$
  $f(x) \to {}^{-}0$

* as $x \to {}^{+}4$
  $f(x) \to -\infty$

* as $x \to {}^{-}\infty$
  $f(x) \to {}^{+}0$

②

iv Area $= \int_a^b f(x)\, dx.$

$= \int_2^3 \frac{1}{x-1} + \frac{4}{4-x}\, dx.$

$= \left[\ln(x-1) - 4\ln(4-x)\right]_2^3$

$= (\ln 2 - 4\ln 1) - (\ln 1 - 4\ln 2)$

$= \ln 2 + 4\ln 2$

$= 5\ln 2$ units^2

②

Question 4.

b

i

$f(x) = \frac{x-1}{x} \left(1 - \frac{1}{x}\right)$

* $x \neq 0$
$\therefore$ asymptote at $x = 0$.

* when $x = 1$, $f(x) = 0$.

* $f'(x) = \frac{x - (x-1)}{x^2}$
$= \frac{1}{x^2} \left(x^{-2}\right)$

$f''(x) = -2x^{-3}$
$= \frac{-2}{x^3}$

* $f'(x) \neq 0$.
$\therefore$ there are no turning points

*let $y = \frac{x-1}{x}$

$xy = x-1$

$xy - x = -1$

$x - xy = 1$

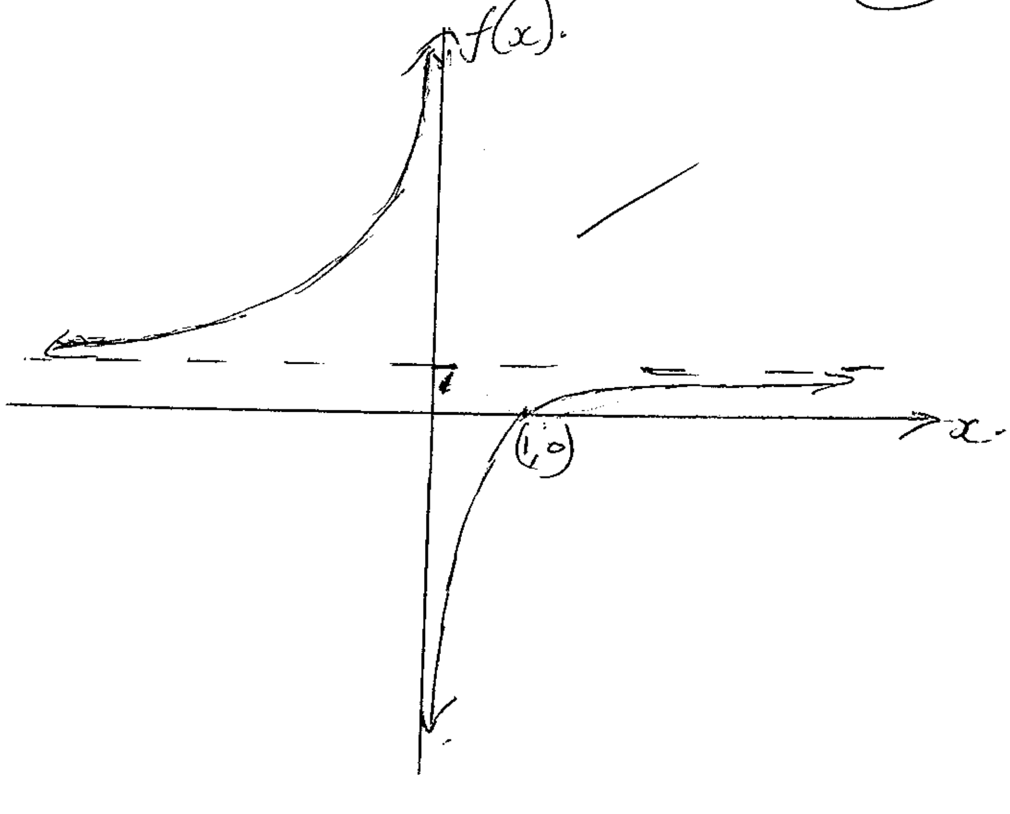
$x(1-y) = 1$

$x = \frac{1}{1-y}$

$y \neq 1$.

$\therefore$ there is a horizontal asymptote at $y = 1$.

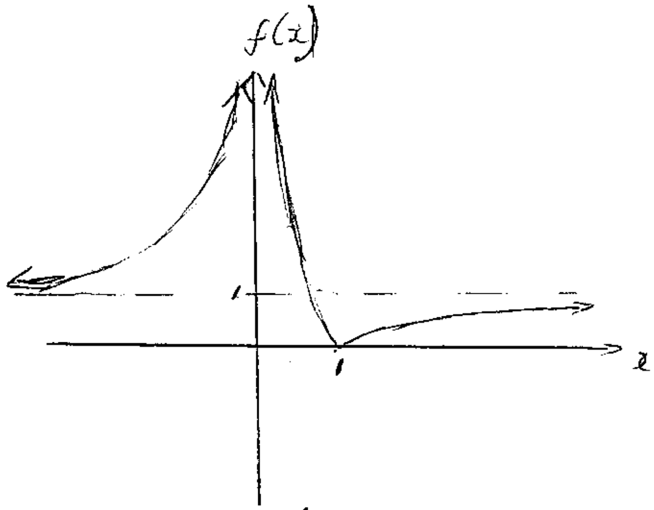
3



Question 4

b ii

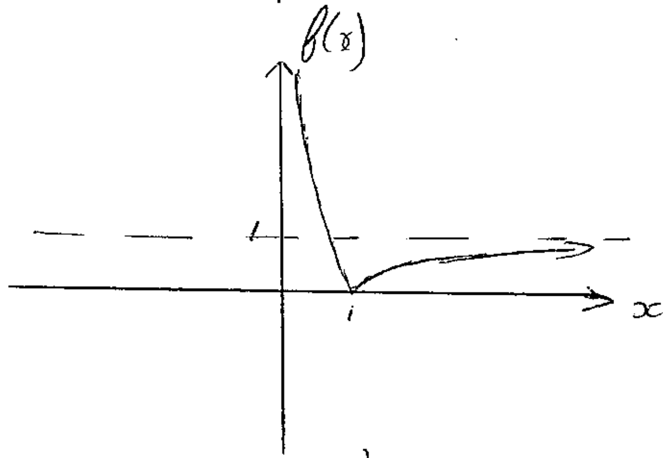
(α)



$|f(x)|$

1

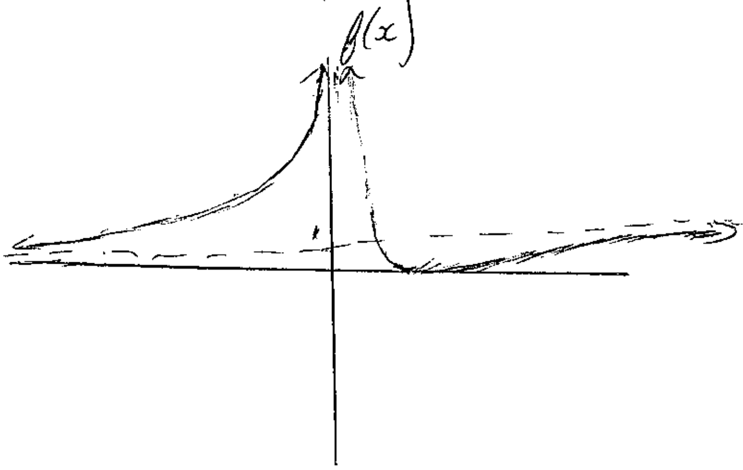
(β)



$f(|x|)$

1

(γ)



$y = \{f(x)\}^2$

1

Question 5. (14)

a i $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$$\frac{2x}{a^2} + \frac{2y}{b^2} \cdot \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{2x}{a^2} \cdot \frac{b^2}{2y}$$

$$= \frac{-b^2}{a^2} \cdot \frac{x}{y}.$$

at P, grad of tangent $m = \dfrac{-b^2}{a^2} \times \dfrac{a\cos\theta}{b\sin\theta}$

$$= \frac{-b\cos\theta}{a\sin\theta}.$$

(3) Equation of Tangent at $P(a\cos\theta, b\sin\theta)$

$$y - y_1 = m(x - x_1)$$

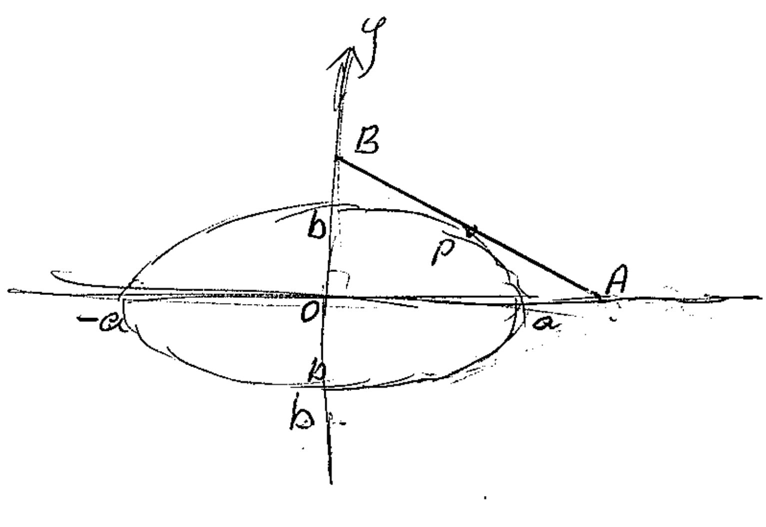
$$y - b\sin\theta = \frac{-b\cos\theta}{a\sin\theta}(x - a\cos\theta).$$

$$ay\sin\theta - ab\sin^2\theta = -bx\cos\theta + ab\cos^2\theta.$$

$$bx\cos\theta + ay\sin\theta - ab(\sin^2\theta + \cos^2\theta) = 0$$

$$bx\cos\theta + ay\sin\theta - ab = 0. \quad — (1)$$

ii



Question 5

a ii From ① when $x = 0$

$$ay \sin\theta - ab = 0$$

$$y = \frac{ab}{a\sin\theta}$$

$$y = \frac{b}{\sin\theta}$$

$\therefore$ B has coordinates $\left(0, \frac{b}{\sin\theta}\right)$ or $(0, b\,\mathrm{cosec}\,\theta)$

From ① when $y = 0$

$$bx\cos\theta - ab = 0$$

$$x = \frac{ab}{b\cos\theta}$$

$$x = \frac{a}{\cos\theta}$$

3

$\therefore$ ~~B~~ A has coordinates $\left(\frac{a}{\cos\theta}, 0\right)$ or $(a\sec\theta, 0)$.

$$\text{Area of } \triangle AOB = \frac{1}{2} \times b \times h$$

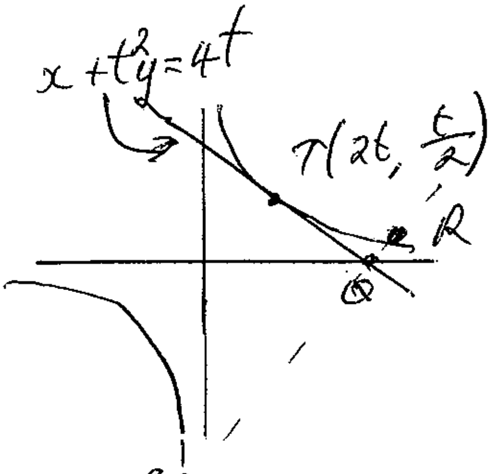
$$= \frac{1}{2} \times \frac{a}{\cos\theta} \times \frac{b}{\sin\theta}$$

$$= \frac{ab}{2\sin\theta\cos\theta}$$

Question 5

b i Calculus 3

ii



for $x + t^2 y = 4t$

when $y = 0$, $x = 4t$.

$\therefore$ Q is $(4t, 0)$

gradient of tangent is $-\frac{1}{t^2}$ 2

$\therefore$ gradient of line through Q is t^2.

Equⁿ of this line is

$$y = t^2(x - 4t)$$

$$y = t^2x - 4t^3$$

$$t^2x - y = 4t^3$$

iii solving $t^2x - y = 4t^3$ and $xy = 4$

$y = \frac{4}{x}$.

substituting gives

$$t^2x^2 - 4xt^3 - 4 = 0.$$

Question 5

b iii Continued.

Now, this is a quadratic

The sum of the roots is; $\alpha + \beta = -\frac{b}{a}$

$= -\frac{-4t^3}{t^2}$

$= 4t$

let R and S have x values α and β.

The midpoint of R and S is given by $\frac{\alpha+\beta}{2}$ (x-values).

$\therefore \frac{\alpha+\beta}{2} = \frac{4t}{2} = 2t$. (x-value).

substitute into $t^2x - y = 4t^3$

gives $2t^3 - y = 4t^3$

$y = -2t^3$

$\therefore$ M is $(2t, -2t^3)$.

3

Question 6 (15)

a $\displaystyle\int \frac{x+1}{\sqrt{x-1}}\,dx.$

$u = x-1$
$du = dx.$
$x = u+1$

$= \displaystyle\int \frac{u+1+1}{\sqrt{u}}\,du$

$= \displaystyle\int \frac{u+2}{\sqrt{u}}\,du$

(3)

$= \displaystyle\int u^{1/2} + 2u^{-1/2}\,du$

$= \frac{2}{3}u^{3/2} + 4u^{1/2} + c$

$= \frac{2}{3}\sqrt{(x-1)^3} + 4\sqrt{x-1} + c$

b $\displaystyle\int_1^2 \frac{1}{e^x - 1}\,dx.$

$= \displaystyle\int_1^2 \frac{e^x - e^x + 1}{e^x - 1}\,dx.$

$= \displaystyle\int_1^2 \frac{e^x - (e^x - 1)}{e^x - 1}\,dx.$

(2)

$= \displaystyle\int_1^2 \frac{e^x}{e^x - 1} - 1\,dx.$

$= \left[\ln(e^x - 1) - x\right]_1^2$

$= \ln(e^2 - 1) - 2 - \ln(e-1) + 1$

$= \ln(e+1) - 1$

Question 6

c $\displaystyle\int_0^{\pi/2} \frac{1}{\cos x + 2}\, dx.$

$t = \tan\frac{x}{2}.$

$\cos x = \dfrac{1-t^2}{1+t^2}$

$dt = \frac{1}{2}\sec^2\frac{x}{2}\, dx.$

$\quad = \frac{1}{2}\left(1+\tan^2\frac{x}{2}\right)dx$

$\quad = \frac{1}{2}\left(1+t^2\right)dx.$

$\dfrac{2dt}{1+t^2} = dx.$

when $x = 0$, $t = 0$

when $x = \frac{\pi}{2}$, $t = 1$

$$= \int_0^1 \frac{1}{\left[\left(\frac{1-t^2}{1+t^2}\right)+2\right]}\cdot\frac{2dt}{(1+t^2)}$$

$$= \int_0^1 \frac{2dt}{1-t^2+2+2t^2}$$

$$= \int_0^1 \frac{2dt}{3+t^2}$$

4

$$= \left[\frac{2}{\sqrt{3}}\tan^{-1}\frac{t}{\sqrt{3}}\right]_0^1$$

$$= \frac{2}{\sqrt{3}}\cdot\frac{\pi}{6}$$

$$= \frac{\pi\sqrt{3}}{9}.$$

d i

$$I_n + I_{n-2} = \int_0^{\frac{\pi}{4}} \tan^n x\, dx + \int_0^{\frac{\pi}{4}} \tan^{n-2} x\, dx.$$

$$= \int_0^{\frac{\pi}{4}} \left(\tan^n x - \tan^{n-2} x\right) dx$$

$$= \int_0^{\frac{\pi}{4}} \tan^{n-2} x\left(\tan^2 x - 1\right) dx.$$

$$= \int_0^{\frac{\pi}{4}} \tan^{n-2} x\, \sec^2 x\, dx.$$

let $u = \tan x$

$du = \sec^2 x\, dx$

$x = \frac{\pi}{4}$, $u = 1$

$x = 0$, $u = 0$

$$= \int_0^{\cancel{\frac{\pi}{4}}\,1} u^{n-2}\, du.$$

Question 6

d i continued.

$$= \left[\frac{u^{n-1}}{n-1}\right]_0^1$$

$$= \cancel{0} \frac{1^{n-1}}{n-1} - \frac{0^{n-1}}{n-1}$$ 1 3

$$= \frac{1}{n-1} - 0$$

$$= \frac{1}{n-1}$$

ii $\int_0^{\frac{\pi}{4}} \tan^5 x \, dx = I_5$

$I_n + I_{n-2} = \frac{1}{n-1}$

$I_n = \frac{1}{n-1} - I_{n-2}$

$$= \frac{1}{4} - I_3$$

$$= \frac{1}{4} - \left(\frac{1}{2} - I_1\right)$$

$$= -\frac{1}{4} + I_1$$ 1 3

$$= I_1 - \frac{1}{4}.$$

Now $I_1 = \int_0^{\frac{\pi}{4}} \tan x \, dx.$

$$= -\int_0^{\frac{\pi}{4}} \frac{-\sin x}{\cos x} \, dx.$$

$$= \left[-\ln(\cos x)\right]_0^{\frac{\pi}{4}}$$

$$= \left[-\ln\left(\tfrac{1}{\sqrt{2}}\right) - -\ln \cancel{1}\right]$$ 1

$$= -\ln\left(\tfrac{1}{\sqrt{2}}\right)$$

$$= \ln \sqrt{2}$$

$$= \frac{1}{2} \ln 2.$$

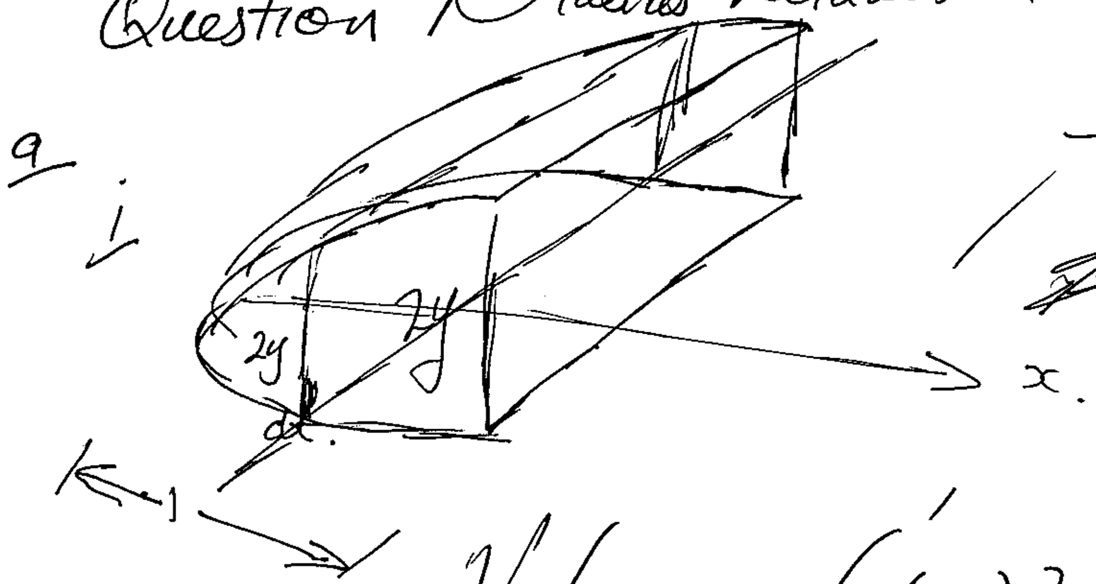
1

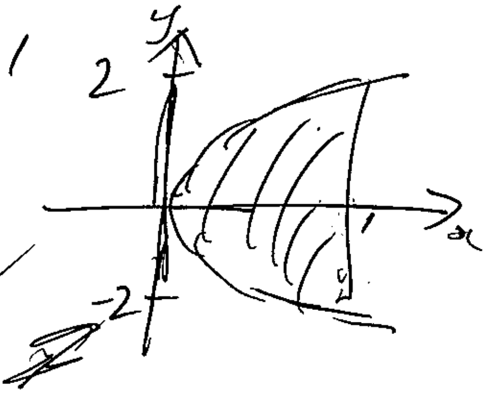
$$= \frac{1}{2} \ln 2 - \frac{1}{4}.$$

$$= \frac{1}{4}(2\ln 2 - 1).$$

Question 7 (15) latus rectum $x = 1$

a i





$$\text{Volume} = \int_0^1 (2y)^2\,dx$$
$$= 4\int_0^1 y^2\,dx$$
$$= 4\int_0^1 4x\,dx$$
$$= 4\left[2x^2\right]_0^1$$
$$= 4 \times 2$$
$$= 8 \text{ units}^3$$

4

Question 7

b $\quad x^3 + kx + 2 = 0$

$x^3 + 0x^2 + kx + 2 = 0$

$\alpha + \beta + \gamma = -\frac{b}{a}$
$= 0$

$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$
$= k.$

$\alpha\beta\gamma = -\frac{d}{a}$
$= -2.$

i $\quad \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\alpha\beta + \beta\gamma + \gamma\alpha}{\alpha\beta\gamma}$

$= \frac{k}{-2}.$

3

ii $\quad \alpha, \beta$ and γ are the roots of
$x^3 + kx + 2 = 0$

$\therefore \alpha^2, \beta^2$ and γ^2 are the roots of.
$(\sqrt{x})^3 + k\sqrt{x} + 2 = 0.$
$\sqrt{x^3} + k\sqrt{x} + 2 = 0.$

2

iii $\quad$ If α a root of $x^3 + kx + 2 = 0$
then $\alpha^3 + k\alpha + 2 = 0$
that is $\alpha^3 = -k\alpha - 2$
Similarly $\beta^3 = -k\beta - 2$
and $\gamma^3 = -k\gamma - 2$

Question 7

b iii Continued

adding gives

$$\alpha^3+\beta^3+\gamma^3 = -k\alpha - k\beta - k\gamma - 6$$
$$= -k(\alpha+\beta+\gamma) - 6$$ ✓ 2
$$= -6 \quad \text{as } \alpha+\beta+\gamma = 0.$$

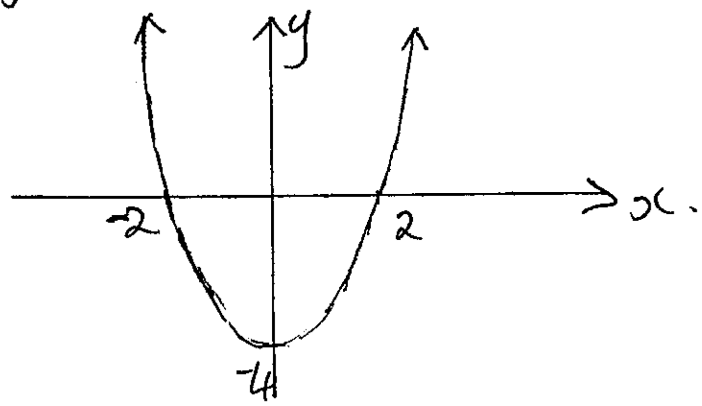
Question 8.

a iv

$$PS + PS' = ePM + ePM'$$
$$= e(PM + PM')$$
$$= e\,MM'$$
$$= \frac{3}{5} \times \frac{50}{3}$$
$$= 10$$

2

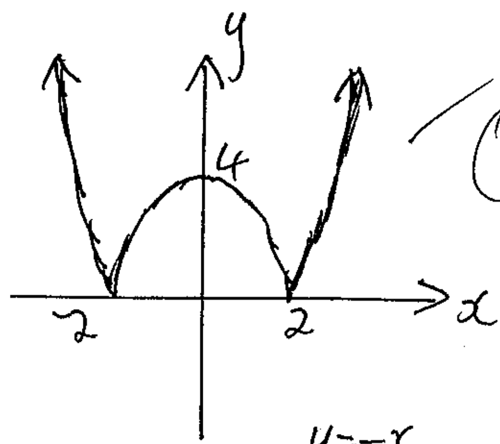
b i) $y = x^2 - 4 = (x-2)(x+2)$



1

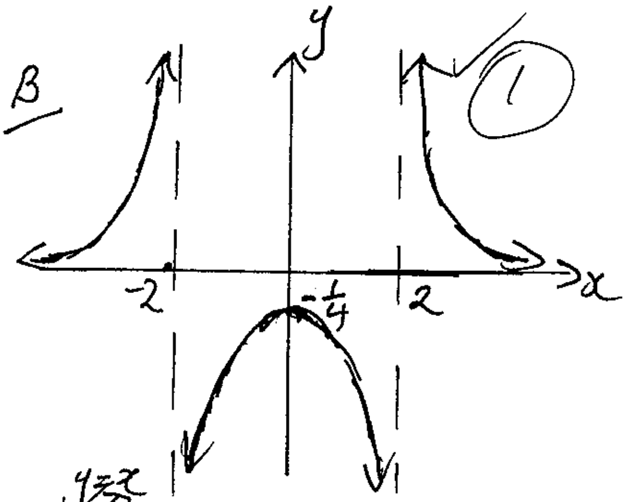
ii

α



1

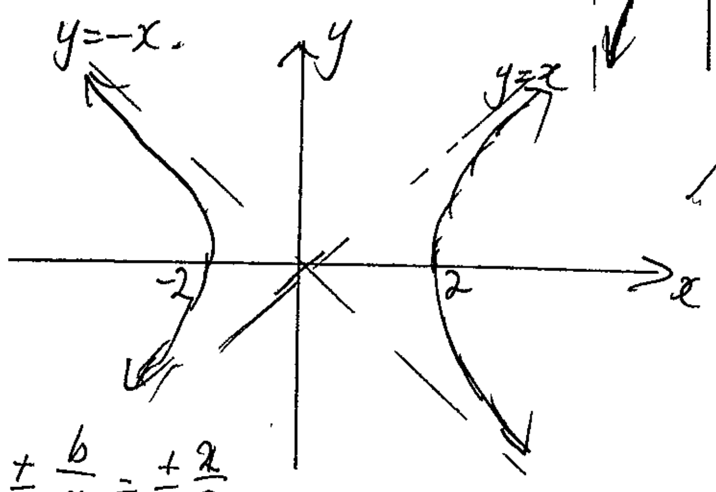
β



1

γ

$$y^2 = x^2 - 4$$
$$x^2 - y^2 = 4$$
$$\frac{x^2}{4} - \frac{y^2}{4} = 1$$



1

Asymptote $y = \pm \frac{b}{a} = \pm \frac{2}{2}$
$= \pm 1$

Question 8 12

a

i $\quad \dfrac{x^2}{5^2} + \dfrac{y^2}{4^2} = 1.$ $\qquad a = 5,\ b = 4 \quad b < a$

eccentricity, $e = \sqrt{1 - \dfrac{16}{25}}$

$= \sqrt{\dfrac{9}{25}}$

$= \dfrac{3}{5}$ $\qquad$ 2

ii $\quad$ Foci's $= (\pm ae, 0) = \left(\pm 5\left(\tfrac{3}{5}\right), 0\right)$

$= (\pm 3, 0).$ $\qquad$ 1

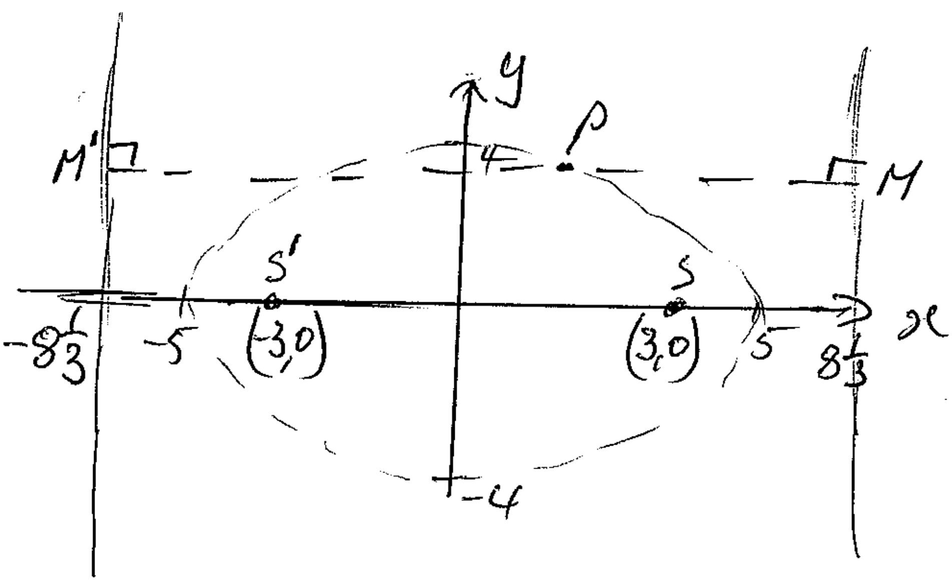
Directrices $\quad x = \pm \dfrac{a}{e}$

$= \pm \dfrac{5}{3/5}$

$= \pm \dfrac{25}{3}$

$= \pm 8\tfrac{1}{3}$ $\qquad$ 1

iii



2