KILLARA HIGH SCHOOL

YEAR 12 MATHEMATICS EXTENSION 2

2008

HSC Assessment 1

Time Allowed: 70 minutes

Instructions:

- Attempt all questions
- Start each question on a new page
- Show ALL necessary working
- Marks may be deducted for untidy or badly arranged work
- Board approved calculators may be used.

Outcomes Assessed:

E1 Appreciates the creativity, power and usefulness of mathematics to solve a broad range of problems

E3 Uses the relationship between algebraic and geometric representations of complex numbers and of conic sections.

E9 Communicates abstract ideas and relationships using appropriate notation and logical argument.

This assessment task constitutes 10% of the final HSC course Assessment.

Question 1 (15 Marks) START A NEW PAGE

(a). If $z = 3 + \sqrt{2}\, i$, find in the form a + ib, **5**

(i) $\bar{z}$ (ii) $z + \bar{z}$ (iii) $z\bar{z}$ (iv) z^{-1}

(b) (i) Find the $\sqrt{16 - 30i}$. **2**

(ii) Hence, or otherwise solve the equation $z^2 - (1 - i)\,z + 7i - 4 = 0$ in the form $a + ib$ **3**

(c) By writing each factor in the modulus-argument form, simplify **5**

$(\sqrt{3} + i)^6 \div (1 - i)^4$

Question 2 (15 Marks) START A NEW PAGE

(a) Sketch and describe the loci defined by the following:

(i) $|z - 4| = |z + 2i|$ **2**

(ii) $z\,\bar{z} = z + \bar{z}$ **4**

(b) 1 – 2i is one of the roots of $x^2 + (1 + i)\,x + k = 0$

(i) State the other root. **2**

(ii) Find the value of k. **2**

(c) Given that 1, ω and ω^2 are the complex roots of unity.
Evaluate:

(i) $1 + \omega + \omega^2$ **2**

(ii) $(1 + \omega^2)^{12}$ **3**

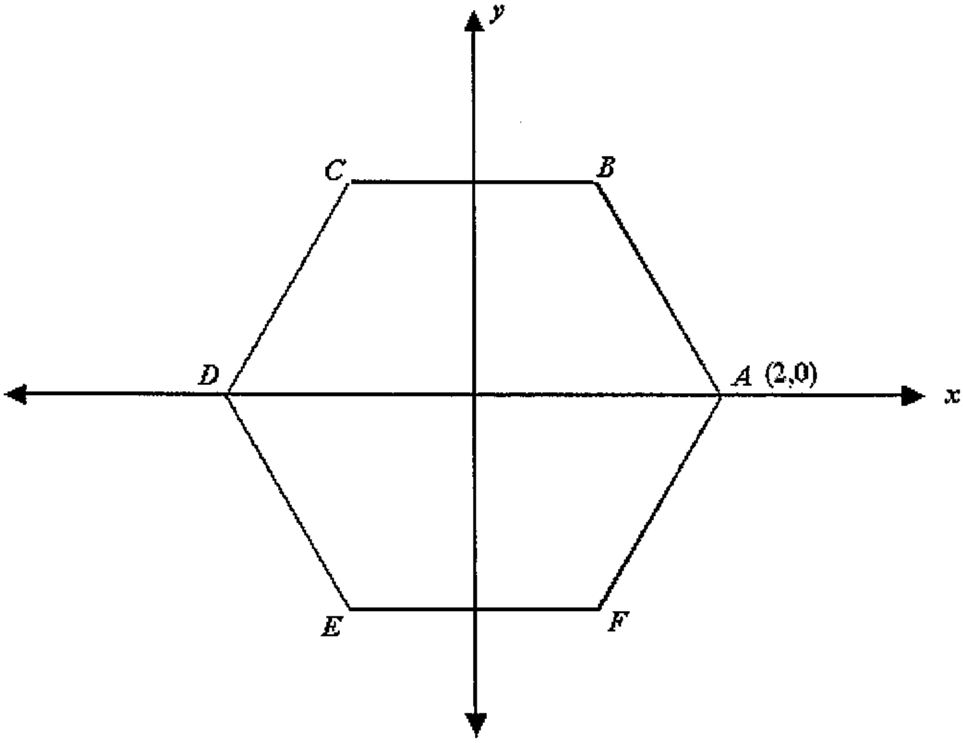
Question 3 (20 Marks) START A NEW PAGE

(a) Find the locus of z, if $W = \dfrac{z-i}{z-2}$ is purely imaginary. **4**

(b) (i) Find the five roots of $Z^5 = 1$. **3**

(ii) Hence, express $Z^4 + Z^3 + Z^2 + Z + 1$ as product of two quadratic factors. **3**

(c)



ABCDEF is a regular hexagon drawn on the Argand diagram with the vertex A at the point (2, 0). O is the centre of the hexagon.

(i) Copy the diagram into your answer book.

(ii) On your diagram show the region within the hexagon in which both the inequalities $|z| \geq 1$ and $-\dfrac{\pi}{3} \leq \arg z \leq \dfrac{\pi}{3}$ are satisfied. **2**

(iii) Find, in the form $|z - c| = R$, the equation of the circle through the points O, B and F. and sketch it on the Argand diagram. **2**

(iv) Find the complex numbers represented by the points C and E **3**

(v) The hexagon is rotated anticlockwise about the origin through an angle of $\dfrac{\pi}{4}$. **3**
Express in the form $r(\cos\theta + i\sin\theta)$ the complex numbers represented by the new positions of C and E.

End of paper

KHS. EXT2 Term 4. Assessment 1 2008.

Question 1

1. $z = 3 + \sqrt{2}i$

(i) $\bar{z} = 3 - \sqrt{2}i$ ✓ 1

(ii) $z + \bar{z}$

$= 3 + \sqrt{2}i + 3 - \sqrt{2}i$

$= 6$ ✓ 1

(iii) $z\bar{z}$

$= (3 + \sqrt{2}i)(3 - \sqrt{2}i)$

$= 9 + 2 = 11$ ✓ 1

(iv) $z^{-1} = \frac{1}{z}$

$= \frac{1}{3 + \sqrt{2}i} = \frac{3 - \sqrt{2}i}{(3 + \sqrt{2}i)(3 - \sqrt{2}i)}$ ✓

$= \frac{3 - \sqrt{2}i}{11}$ ✓ 2.

(b)
(i) $\sqrt{16 - 30i}$

$P = 5 \times 3$
$S = 16$
$F \quad 5 \quad 3i$

$16 - 30i = (5 - 3i)^2$

$\therefore \sqrt{16 - 30i} = \sqrt{(5 - 3i)^2}$ 2

$= \pm(5 - 3i)$ ✓✓

(ii) $z^2 - (1 - i)z + 7i - 4 = 0$

$z = \frac{(1 - i) \pm \sqrt{(1 - i)^2 - 4 \times 1(7i - 4)}}{2}$

$= \frac{(1 - i) \pm \sqrt{1 - 2i - 1 - 28i + 16}}{2}$ ✓

$= \frac{(1 - i) \pm \sqrt{16 - 30i}}{2}$

$= \frac{(1 - i) \pm (5 - 3i)}{2}$ ✓

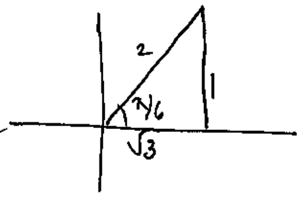
$= \frac{1 - i + 5 - 3i}{2}, \quad \frac{1 - i - 5 + 3i}{2}$

$= \frac{6 - 4i}{2}, \quad \frac{2i - 4}{2}$

$= 3 - 2i, \quad -2 + i$ ✓ 3

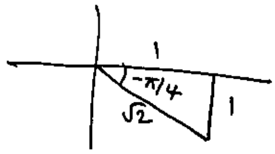
(c) $\sqrt{3} + i =$

$= 2 \operatorname{cis} \frac{\pi}{6}$. ✓



$1 - i =$

$\sqrt{2} \operatorname{cis}\left(-\frac{\pi}{4}\right)$ ✓



$\frac{(\sqrt{3} + i)^6}{(1 - i)^4} = \frac{\left(2 \operatorname{cis} \frac{\pi}{6}\right)^6}{\left(\sqrt{2} \operatorname{cis}\left(-\frac{\pi}{4}\right)\right)^4}$ ✓

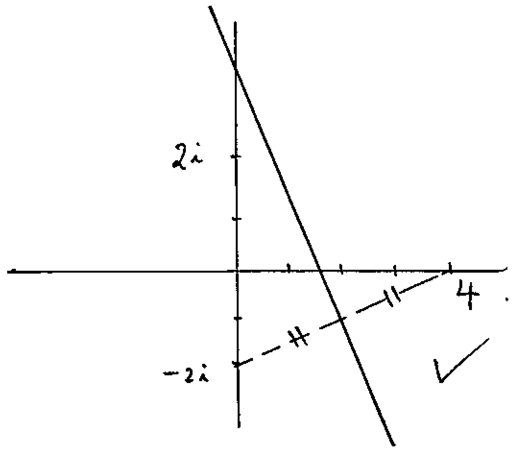
$= \frac{2^6 \operatorname{cis}\left(\frac{\pi}{6} \times 6\right)}{2^2 \operatorname{cis}\left(-\frac{\pi}{4} \times 4\right)}$ ✓ 5

$= \frac{2^6 \operatorname{cis} \pi}{2^2 \operatorname{cis}(-\pi)} = \frac{2^6 \times -1}{2^2(-1 - 0i)}$

$= 2^4 = \underline{16}$ ✓

Question 2

(a) (i) $|z-4| = |z+2i|$



The locus is the perpendicular bisector of the line joining $(4,0)$ and $(0,-2i)$. 2

$$|x+iy-4| = |x+iy+2i|$$

$$|x-4+iy| = |x+i(y+2)|$$

$$(x-4)^2+y^2 = x^2+(y+2)^2$$

$$\cancel{x^2}-8x+16+\cancel{y^2} = \cancel{x^2}+\cancel{y^2}+4y+4$$

$$8x+4y-12=0$$

$$\cancel{4x}\quad 2x+y-3=0$$

(ii) $z\bar{z} = z+\bar{z}$ let $z = x+iy$.

$$(x+iy)(x-iy) = x+iy+x-iy$$

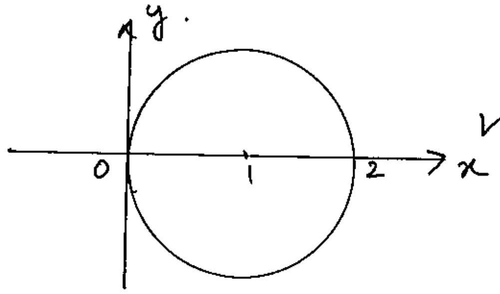
$$x^2+y^2 = 2x$$

$$x^2-2x+y^2 = 0$$

$$x^2-2x+1+y^2 = 1$$

$$(x-1)^2+y^2 = 1$$

The locus is a circle with centre $(1,0)$ and radius 1.



4

(b) $x^2+(1+i)x+K=0$

(i) $1-2i$ is a root.

Let the other root be α.

$\therefore$ Sum of roots $= 1-2i+\alpha = -\dfrac{(1+i)}{1}$

$$\alpha = -1-i-1+2i$$
$$= -2+i$$

2

(ii) Product of roots $= \dfrac{c}{a}$.

$$(1-2i)(-2+i) = \frac{K}{1}$$

$\therefore K = -2+i+4i+2$

$= 5i$

2

(c) (i) Consider the roots of the equation $z^3=1$.

$1, \omega, \omega^2$ are the three roots of $z^3-1=0$

$\therefore$ Sum of roots $= \dfrac{-b}{a}$

$$= \frac{-\text{Coefficient of } z^2}{\text{Coefficient of } z^3} = \frac{0}{1} = 0$$

$\therefore 1+\omega+\omega^2 = 0$.

(ii) $\left(1+\omega^2\right)^{12}$

$1+\omega+\omega^2=0, \quad \omega^3=1$

$\therefore 1+\omega^2=-\omega$ ✓

$\therefore \left(1+\omega^2\right)^{12}=(-\omega)^{12}$

$=\left(\omega^3\right)^4$ ✓

$=1^4$ 3

$=\underline{\underline{1}}$ ✓

<u>Question 3</u>

(a) $\omega=\dfrac{z-i}{z-2}$

Let $z=x+iy$

$\therefore \omega=\dfrac{x+iy-i}{x+iy-2}$

$=\dfrac{x+(y-1)i}{(x-2)+iy}$ ✓

$=\dfrac{x+(y-1)i}{(x-2)+iy}\times\dfrac{(x-2)-iy}{(x-2)-iy}$ ✓

$=\dfrac{x(x-2)-xyi+(y-1)(x-2)i+y(y-1)}{(x-2)^2+y^2}$

$=\dfrac{x^2-2x+y^2-y+i\left(-xy+(y-1)(x-2)\right)}{(x-2)^2+y^2}$ is

purely imaginary.

$\therefore \operatorname{Re}(\omega)=0.$

$(x-2)^2+y^2\neq 0.$

$\therefore x^2-2x+y^2-y=0$ ✓

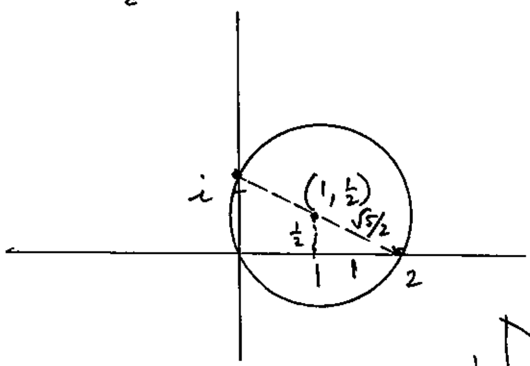
$(x-1)^2+\left(y-\tfrac{1}{2}\right)^2=1+\tfrac{1}{4}$

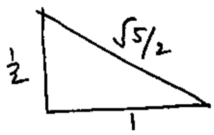
$(x-1)^2+\left(y-\tfrac{1}{2}\right)^2=\tfrac{5}{4}.$ 4

$\therefore$ locus is a circle with ✓ centre $\left(1,\tfrac{1}{2}\right)$ and radius $\tfrac{\sqrt{5}}{2}$.

OR

$\omega=\dfrac{z-i}{z-2}$ is purely imaginary.





Mid-point of $(0,1)$ and $(2,0)$

$=\left(1,\tfrac{1}{2}\right)$ and $r=\tfrac{\sqrt{5}}{2}$.

$\therefore$ locus is a circle with centre $\left(1,\tfrac{1}{2}\right)$ and radius $\underline{\underline{\tfrac{\sqrt{5}}{2}}}$.

$(x-1)^2+\left(y-\tfrac{1}{2}\right)^2=\tfrac{5}{4}.$

(b) (i) $z^5 = 1$

$r = 1$.

Let $z = \text{Cis}\,\theta$.

$z^5 = \text{Cis}\,5\theta = 1 + 0i$

Comparing real and imaginary parts,

$\cos 5\theta = 1 \qquad \sin 5\theta = 0$. ✓

$\therefore 5\theta = 2k\pi + 0$

$\therefore \theta = \frac{2k\pi}{5}, \quad k = 0, \pm 1, \pm 2$ ✓

$k=0, \quad z_1 = \text{Cis}\,0 = 1$

$k=1, \quad z_2 = \text{Cis}\,\frac{2\pi}{5}$

$k=-1, \quad z_3 = \text{Cis}\left(-\frac{2\pi}{5}\right) = \overline{z_2}$

$k=2, \quad z_4 = \text{Cis}\,\frac{4\pi}{5}$

$k=-2, \quad z_5 = \text{Cis}\left(-\frac{4\pi}{5}\right) = \overline{z_4}$

✓ 3

(b)(ii)

$z^5 - 1 = (z-1)(z^4 + z^3 + z^2 + z + 1)$

$\therefore z_2, z_3, z_4, z_5$ are the roots $z^4 + z^3 + z^2 + z + 1 = 0$.

The quadratic with z_2, z_3 as roots.

Sum of roots $z_2 + z_3 = z_2 + \overline{z_2}$

$= 2\cos\frac{2\pi}{5}$

$z_2 z_3$ Product of roots

$= z_2\overline{z_2} = \cos^2\frac{2\pi}{5} + \sin^2\frac{2\pi}{5} = 1$ ✓

$\therefore$ Quadratic is

$z^2 - 2\cos\frac{2\pi}{5}\,z + 1$ ✓

Similarly the other quadratic with z_4, z_5 as roots is

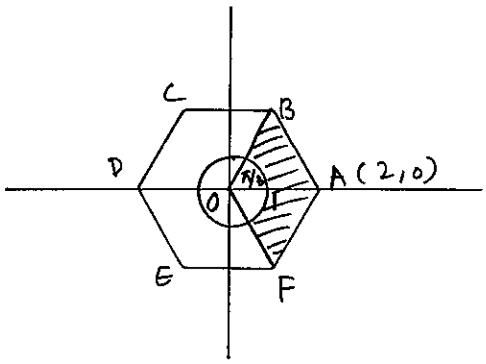
$z^2 - 2\cos\frac{4\pi}{5}\,z + 1$. 3

$\therefore z^4 + z^3 + z^2 + z + 1 =$

$\left(z^2 - 2\cos\frac{2\pi}{5}\,z + 1\right)\left(z^2 - 2\cos\frac{4\pi}{5}\,z + 1\right)$. ✓

(c)

(i)

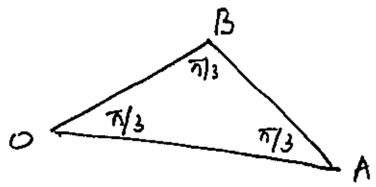


(ii) $|z| \geq 1$ outside the circle

Correct region shown ✓ 2.

$-\pi/3 \leq \arg(z) \leq \pi/3$ Correct region ✓

(iii)



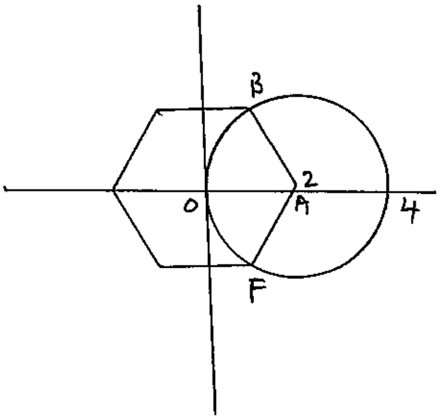
~~OB = OA~~ $\triangle OBA$ is equilateral

$\therefore OB = OA = OF = 2$

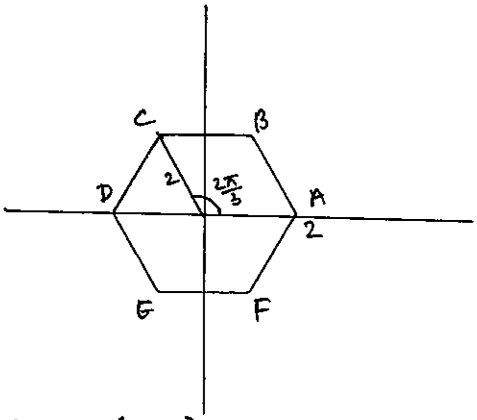
$\therefore$ equation of the circle

$|z - 2| = 2$ ✓ $\boxed{1}$

See diagram below.



(iv)



Point $A(2,0)$

$OA = OC$

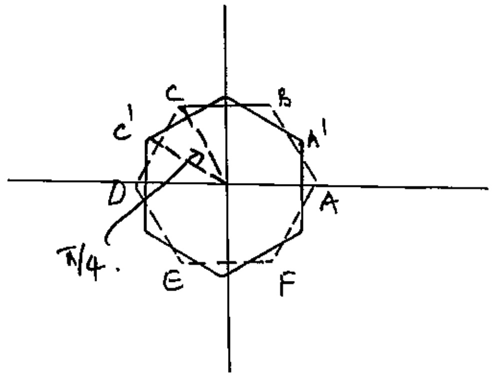
$\overrightarrow{OC}$ is obtained by rotating $\overrightarrow{OA}$ through $\frac{2\pi}{3}$ radians.

$\therefore \overrightarrow{OC} = 2\,\text{cis}\left(\frac{2\pi}{3}\right)$ ✓

$= 2\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = -1+\sqrt{3}i$ ✓

$\therefore E = -1-\sqrt{3}i$ ✓ 3

(v)



$C = 2\,\text{cis}\left(\frac{2\pi}{3}\right)$

$\therefore C' = 2\,\text{cis}\left(\frac{2\pi}{3} + \frac{\pi}{4}\right)$

$= 2\,\text{cis}\left(\frac{11\pi}{12}\right)$ ✓

$= 2\left[\cos\left(\frac{11\pi}{12}\right) + i\sin\left(\frac{11\pi}{12}\right)\right]$

$E = 2\,\text{cis}\left(-\frac{2\pi}{3}\right)$ ✓ 3

$\therefore E' = 2\,\text{cis}\left(-\frac{2\pi}{3} + \frac{\pi}{4}\right)$

$= 2\,\text{cis}\left(-\frac{5\pi}{12}\right)$

$* = \left[2\left[\cos\left(\frac{5\pi}{12}\right) - i\sin\left(\frac{5\pi}{12}\right)\right]\right]$ ✓