KILLARA HIGH SCHOOL
MATHEMATICS
Extension 2
Assessment Task
March 2008

TASK WEIGHTING: 30 %

NAME: ____________________________

Time allowed: 120 short minutes plus 5 minute reading time

The following outcomes will be assessed in this task
H1-9, HE6, HE7, E2, E3, E4, E6, E8

Instructions:

- Show all necessary working
- All questions may be attempted
- Marks may be deducted for careless or poorly arranged work
- Only approved calculators may be used
- Label each question carefully

This is a School Based paper only. The content does not necessarily reflect that of the Higher School Certificate Examination paper.

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \; n \neq 1 : x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \log_e x, \; x > 0.$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \; a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \; a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \; a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \; a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \; a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \; a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \; a > 0, \; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \log_e \left\{ x + \sqrt{x^2 - a^2} \right\}, \; |x| > |a|$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \log_e \left\{ x + \sqrt{x^2 + a^2} \right\}$$

Question 1. (15 marks) Complex Numbers **Start a new page**

(a) Given $z=\sqrt{6}-\sqrt{2}\,i$, find:

(i) $Re\left(z^2\right)$; [1]

(ii) $|\,z\,|$; [1]

(iii) $\arg z$; [2]

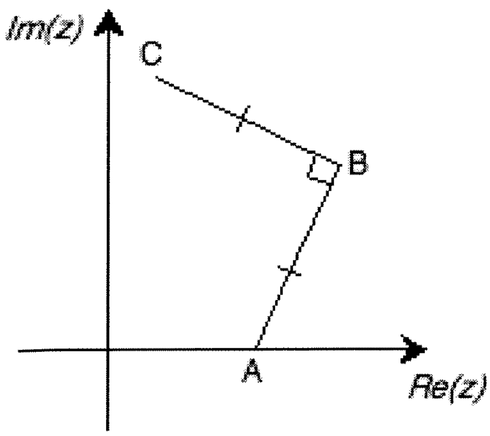
(iv) z^4 in the form $x+iy$. [2]

(b) The equations $|z-8-6i|=2\sqrt{10}$ and $\arg z=\tan^{-1}2$ both represent loci on the Argand plane.

(i) Write down the Cartesian equations of the loci, and hence show that the points of intersection of the loci are $2+4i$ and $6+12i$. [3]

(ii) Sketch both loci on the same diagram, showing their points of intersection. (You need not show the intercepts with the axes.) [2]

(c)



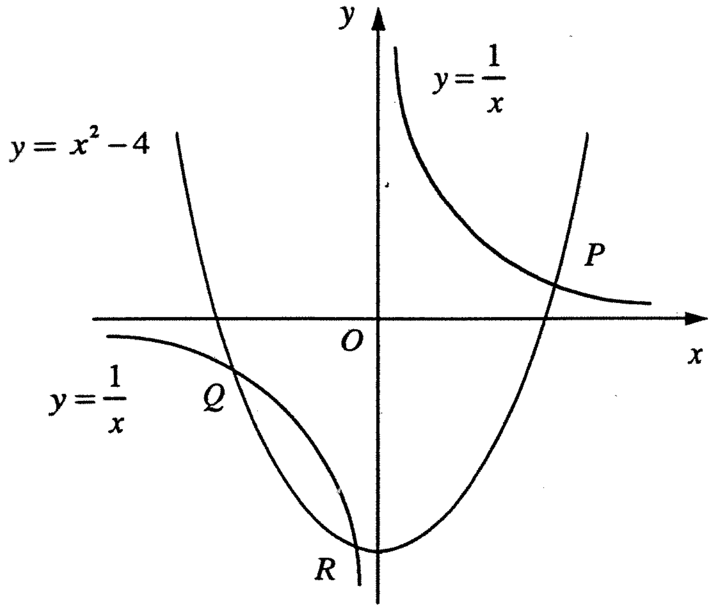
The diagram above shows the fixed points *A*, *B* and *C* in the Argand plane, where $AB = BC$, $\angle ABC=\dfrac{\pi}{2}$, and *A*, *B* and *C* are in anticlockwise order. The point *A* represents the complex number $z_1=2$ and the point *B* represents the complex number $z_2=3+\sqrt{5}\,i$.

(i) Find the complex number z_3 represented by the point *C*. [2]

(ii) *D* is the point on the Argand plane such that *ABCD* is a square. Find the complex number z_4 represented by *D*. [2]

Question 2. (20 marks) Polynomials **Start a new page**

(a)



The curves $y = x^2 - 4$ and $y = \frac{1}{x}$ intersect at the points P, Q and R where $x = \alpha$, $x = \beta$ and $x = \gamma$.

(i) Show that α, β and γ are the roots of the equation $x^3 - 4x - 1 = 0$. [2]

(ii) Find a polynomial equation with numerical coefficients which has roots α^2, β^2 and γ^2. [2]

(iii) Find the polynomial equation with numerical coefficients

which has roots $\frac{1}{\alpha^2}, \frac{1}{\beta^2}$ and $\frac{1}{\gamma^2}$ [2]

(iv) Hence find the numerical value of $OP^2 + OQ^2 + OR^2$ [2]

(b) (i) Use DeMoivre's theorem to show that:

$(1+i\tan\theta)^n + (1-i\tan\theta)^n = \frac{2\cos n\theta}{\cos^n\theta}$ $(\cos\theta \neq 0)$ where n is a positive integer. [3]

(ii) Hence show that the equation $(1+z)^4 + (1-z)^4 = 0$ has roots $\pm i\tan\frac{\pi}{8}, \pm i\tan\frac{3\pi}{8}$. [2]

(iii) Hence show that $\tan^2\frac{\pi}{8} = 3 - 2\sqrt{2}$ [2]

(c) The polynomial function $P(x) = x^4 - 4x^3 - 3x^2 + 50x - 52$ has a zero at $x = 3 - 2i$.

Factorise $P(x)$ over the field of:

(i) rationals. [3]

(ii) complex numbers. [2]

Question 3. (20 marks) Conics **Start a new page**

(a) For the hyperbola $\frac{x^2}{16}-\frac{y^2}{9}=1$, find

(i) the value of the eccentricity, e. [1]

(ii) the co-ordinates of the two foci. S and S' [2]

(iii) the equations of the directrices [2]

(iv) the equation of the tangent to the hyperbola at the point $\left(5,2\frac{1}{4}\right)$ [2]

(v) Make a neat sketch of the hyperbola showing these features [2]

(b) Explain why $\frac{x^2}{k-19}+\frac{y^2}{3-k}=1$ cannot represent the equation of an ellipse. [1]

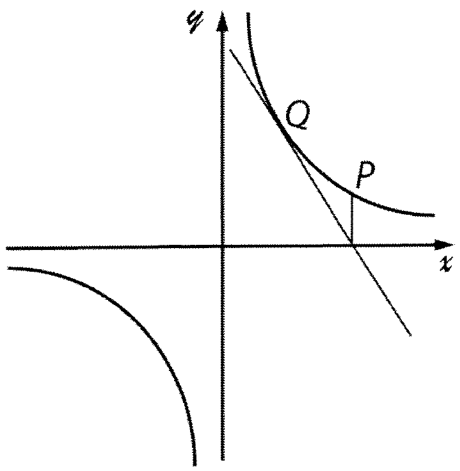
(c) $P\,(a\cos\theta, b\sin\theta)$ lies on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

The normal at P cuts the x-axis at G and N is the foot of the perpendicular from P to the x-axis. Show that

(i) $OG = e^2.ON$ [3]

(ii) $SG = e.SP$ and $S'G = e.S'P$ where S and S' are the foci. [3]

(d) P and Q are variable points on the rectangular hyperbola $xy = c^2$.

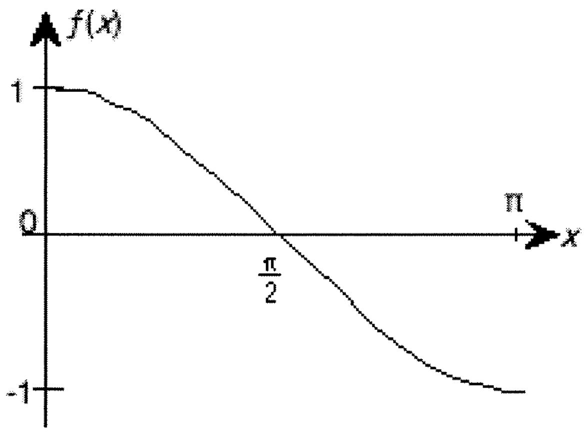


i) The tangent at Q passes through the foot of the ordinate (y-value) of P. If P and Q have parameters p and q, show that $p = 2q$ [2]

ii) Hence prove that the locus of the mid-point of PQ is a rectangular hyperbola. [2]

Question 4. *(20 marks)* *Other Sections* *Start a new page*

(a)



The graph of $f(x) = \cos x$ is shown above for the interval $0 \le x \le \pi$.

Provide separate half-page *sketches* of the graphs of the following

(i) $y = [f(x)]^2$ [2]

(ii) $y^2 = f(x)$ [2]

(iii) $y = \dfrac{1}{|f(x)|}$ [2]

(b) (i) Resolve $\dfrac{5x}{(x+1)(x-2)}$ into partial fractions [2]

(ii) Hence evaluate $\displaystyle\int_3^4 \frac{5x}{(x+1)(x-2)}dx$ [3]

(c) (i) On the same diagram sketch the graphs of $x^2 + y^2 = 1$ and $x^2 - y^2 = 1$, showing clearly the co-ordinates of any points of intersection with the axes and the equations of any asymptotes. [4]

(ii) On a separate sketch shade the region where the inequality:

$(x^2 + y^2 - 1)(x^2 - y^2 - 1) \le 0$ holds. [2]

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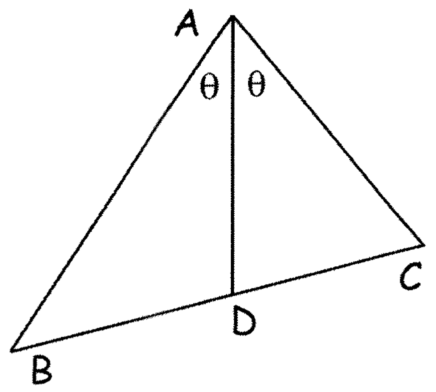
(d)



Not to scale.

In the triangle ABC it is given that AD is the bisector of $\angle BAC$. [3]

(i) Copy the diagram onto your paper

(ii) Prove that $\dfrac{AB}{BD} = \dfrac{AC}{DC}$

End of paper.

Question 1

a) $z = \sqrt{6} - \sqrt{2}.i$

(i) $Re(z^2) = Re(\sqrt{6} - \sqrt{2}.i)^2$

$= Re(6 - 4\sqrt{3}i - 2)$

$= Re(4 - 4\sqrt{3}i)$

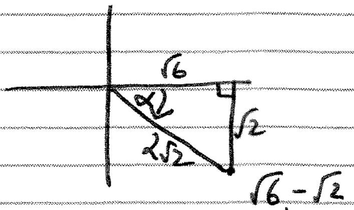
$= 4$ [1]

(ii) $|z| = \sqrt{(\sqrt{6})^2 + (\sqrt{2})^2}$

$= \sqrt{6+2} = \sqrt{8}$

$= 2\sqrt{2}$ [1]

(iii) $\arg z = \frac{\sqrt{2}}{\sqrt{6}} = \frac{1}{\sqrt{3}}$



$\alpha = \pi/6$

$\arg z = \frac{-\pi}{6}$ [2]

(iv) $z^4 = ?$

$z = \sqrt{6} - \sqrt{2}i$

$z^2 = (4 - 4\sqrt{3}i)$ cf (i)

$z^4 = (4 - 4\sqrt{3}i)^2$

$= 16 - 32\sqrt{3}i - 48$

$= -32 - 32\sqrt{3}i$

$= -32(1 + \sqrt{3}i)$

[2]

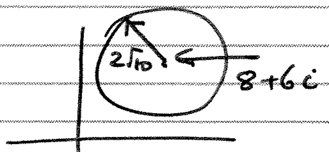
OR.

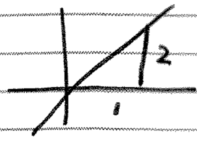
$z = \sqrt{6} - \sqrt{2}i = \sqrt{2}(\sqrt{3} - i)$

$z^4 = [\sqrt{2}(\sqrt{3} - i)]^4$ etc $\left[2\sqrt{2}\operatorname{cis} -\frac{\pi}{6}\right]^4$

$= (2\sqrt{2})^4 \operatorname{cis}\left(-\frac{4\pi}{6}\right) = \ldots$

b) $|z - 8 - 6i| = 2\sqrt{10}$ $\quad \arg z = \tan^{-1} 2$





$y = 2x$

$(x-8)^2 + (y-6)^2 = (2\sqrt{10})^2$

Solve for points of intersection

$(x-8)^2 + (y-6)^2 = 40$

$(x-8)^2 + (2x-6)^2 = 40$

$x^2 - 16x + 64 + 4x^2 - 24x + 36 = 40$

$5x^2 - 40x + 100 = 40$

$5x^2 - 40x + 60 = 0$

$5(x^2 - 8x + 12) = 0$

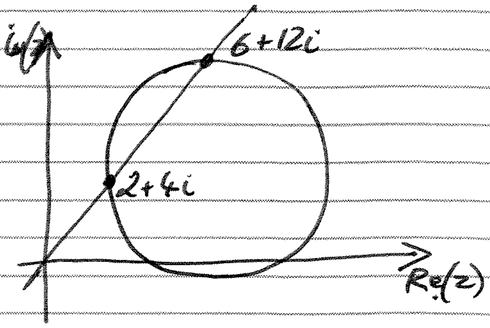
$5(x-6)(x-2) = 0$

$x = 6, 2$

if $x = 6$, $y = 12$ $\quad (6, 12)$ $\quad z = 6 + 12i$

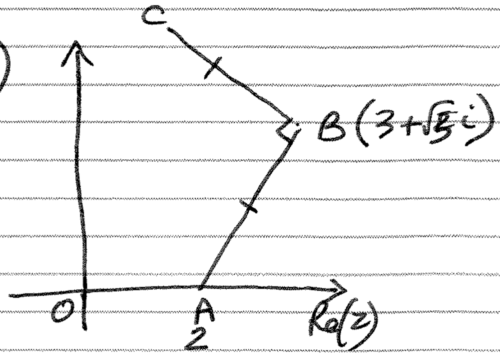
$x = 2$, $y = 4$ $\quad (2, 4)$ $\quad z = 2 + 4i$

[3]



[2]

c)

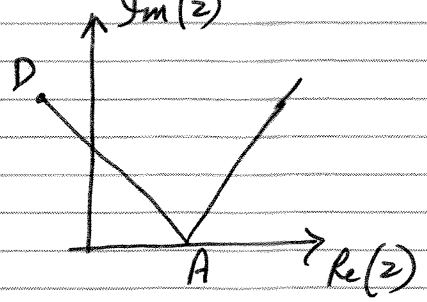


$\vec{OC} = \underbrace{\left[\vec{OA} + \vec{AB}\right]}_{\vec{OB}} + \vec{BC}$

$= 2 + (1 + \sqrt{5}i) + i(1 + \sqrt{5}i)$

$\vec{OC} = 2+1+\sqrt{5}i + i - \sqrt{5}$

$= 3-\sqrt{5} + (1+\sqrt{5})i$ [2]



$\vec{BC} = \vec{AD}$

$\vec{OD} = \vec{OA} + \vec{AD}$

$= 2 + i(1+\sqrt{5}i)$

$= 2 + i - \sqrt{5}$

$= 2-\sqrt{5} + i$ [2]

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Question 2

a) i) $\left.\begin{array}{l} y = x^2 - 4 \\ y = \frac{1}{x}\end{array}\right\}$ intersect at PQR

$x^2 - 4 = \frac{1}{x}$

$x^3 - 4x - 1 = 0$ —— A

Let the roots be α, β, γ [2]

ii) $X = \alpha^2 \quad \therefore \alpha = \sqrt{X}$

this satisfies A

$(\sqrt{X})^3 - 4(\sqrt{X}) - 1 = 0$

$X\sqrt{X} - 4\sqrt{X} - 1 = 0$

$\sqrt{X}(X-4) = 1$

$\left[\sqrt{X}(X-4)\right]^2 = 1^2$

$X(X-4)^2 = 1$

$X(X^2 - 8X + 16) = 1$

$X^3 - 8X^2 + 16X - 1 = 0$ A [2]

(iii) for polynomial with roots $\frac{1}{\alpha^2}, \frac{1}{\beta^2}, \frac{1}{\gamma^2}$

note A has roots $\alpha^2, \beta^2, \gamma^2$

The polynomial we want is

$\left(\frac{1}{X}\right)^3 - 8\left(\frac{1}{X}\right)^2 + 16\left(\frac{1}{X}\right) - 1 = 0$

$\frac{1}{X^3} - \frac{8}{X^2} + \frac{16}{X} - 1 = 0$

$1 - 8X + 16X^2 - X^3 = 0$

$\therefore X^3 - 16X^2 + 8X - 1 = 0$ B

has roots $\frac{1}{\alpha^2}, \frac{1}{\beta^2}, \frac{1}{\gamma^2}$ [2]

(iv) Say $P(\alpha, \frac{1}{\alpha}), Q(\beta, \frac{1}{\beta}), R(\gamma, \frac{1}{\gamma})$

$OP^2 = \alpha^2 + \frac{1}{\alpha^2} \quad OQ^2 = \beta^2 + \frac{1}{\beta^2} \quad OR^2 = \gamma^2 + \frac{1}{\gamma^2}$

$OP^2 + OQ^2 + OR^2 = \alpha^2 + \frac{1}{\alpha^2} + \beta^2 + \frac{1}{\beta^2} + \gamma^2 + \frac{1}{\gamma^2}$

$= \underbrace{\alpha^2 + \beta^2 + \gamma^2}_{\text{from A}} + \underbrace{\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}}_{\text{from B}}$

$= 8 + 16$

$= 24$ [2]

b) (i)

$(1 + i\tan\theta)^n = \left(1 + i\frac{\sin\theta}{\cos\theta}\right)^n$

$= \frac{(\cos\theta + i\sin\theta)^n}{\cos^n\theta}$

$= \frac{1}{\cos^n\theta}\left[\cos n\theta + i\sin n\theta\right]$

by De Moivres Theorem

Similarly

$(1 - i\tan\theta)^n = \frac{1}{\cos^n\theta}\left[\cos\theta - i\sin\theta\right]^n$

Q2bi cont

$$= \frac{1}{\cos^n\theta}\left[\cos(-\theta) + i\sin(-\theta)\right]^n$$

$$= \frac{1}{\cos^n\theta}\left[\cos(-n\theta) + \sin(-n\theta)\right]$$

$$= \frac{1}{\cos^n\theta}\left[\cos n\theta - i\sin n\theta\right]$$

$\therefore (1+i\tan\theta)^n + (1-i\tan\theta)^n$

$$= \frac{1}{\cos^n\theta}\left[\cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta\right]$$

$$= \frac{2\cos n\theta}{\cos^n\theta}$$ [3]

(ii)

if $z = i\tan\theta$

$(1+z)^4 + (1-z)^4 = 0$

i.e. $\dfrac{2\cos 4\theta}{\cos^4\theta} = 0$ note $\cos\theta \neq 0$

$\therefore \cos 4\theta = 0$

$4\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$

$\theta = \frac{\pi}{8}, \frac{3\pi}{8}, \underbrace{\frac{5\pi}{8}, \frac{7\pi}{8}}_{\text{2nd Quad} = -(\text{1st Quad})}$

$\theta = \frac{\pi}{8}, \frac{3\pi}{8}, -\frac{3\pi}{8}, -\frac{\pi}{8}$

$\therefore$ roots of $z = i\tan\theta$

$z = \pm i\tan\frac{\pi}{8}, \pm i\tan\frac{3\pi}{8}$

[2]

(iii)

$$\begin{matrix} & & & 1 & & & \\ & & 1 & 2 & 1 & & \\ & 1 & 3 & 3 & 1 & & \\ 1 & 4 & 6 & 4 & 1 & & \end{matrix}$$

$(1+z)^4 + (1-z)^4 = 0$

$$\begin{aligned} &1 + 4z + 6z^2 + 4z^3 + z^4 \\ &1 - 4z + 6z^2 - 4z^3 + z^4 \end{aligned} = 0$$

$2(1 + 6z^2 + z^4) = 0$

$1 + 6z^2 + z^4 = 0$

Let $m = z^2$

$m^2 + 6m + 1 = 0$

$(m+3)^2 = 8$

$m + 3 = \pm 2\sqrt{2}$

$m = -3 \pm 2\sqrt{2}$

$z^2 = -3 \pm 2\sqrt{2}$

$\left(\pm i\tan\frac{\pi}{8}\right)^2 = -3 \pm 2\sqrt{2}$

or $\left(\pm i\tan\frac{3\pi}{8}\right)^2 = -3 \pm 2\sqrt{2}$.

$-\tan^2\frac{\pi}{8} = -3 \pm 2\sqrt{2}$

$-\tan^2\frac{3\pi}{8} = -3 \pm 2\sqrt{2}$

$\tan^2\frac{\pi}{8} = 3 \pm 2\sqrt{2} = \tan^2\frac{3\pi}{8}$

$\tan\frac{\pi}{8} < \tan\frac{2\pi}{8} < \tan\frac{3\pi}{8}$

$\Downarrow$

$\tan^2\frac{3\pi}{8} = 3 + 2\sqrt{2}$. [2]

2c (i) $P(x) = x^4 - 4x^3 - 3x^2 + 50x - 52$

has a zero at $x = 3 - 2i$

$\therefore$ also at $x = 3 + 2i$

$\therefore (x - (3-2i))(x - (3+2i))$ is a factor.

$(x - 3 + 2i)(x - 3 - 2i)$

$(x-3)^2 - 4i^2 = (x-3)^2 + 4$

$= x^2 - 6x + 13$

$P(x) = (x^2-6x+13)\,Q(x)$

either do synthetic Division

$$\begin{array}{r} x^2+2x-4 \\ x^2-6x+13\overline{)\,x^4-4x^3-3x^2+50x-52} \\ \underline{x^4-6x^3+13x^2} \\ 2x^3-16x^2+50x \\ \underline{2x^3-12x^2+26x} \\ -4x^2+24x-52 \\ \underline{-4x^2+24x-52} \end{array}$$

OR

say $P(x) = (x^2-6x+13)(x^2+ax-4)$

note needed to give -52

LHS of $P(x)$ coeff of $x^3 = -4$

RHS coeff of $x^3 = a-6 \quad \therefore a = 2$

So $P(x) = (x^2-6x+13)(x^2+2x-4)$

[3]

(ii) over the complex field

$x^2+2x-4 = 0$

$x^2+2x = 4$

$x^2+2x+1 = 5$

$(x+1)^2 = 5$

$x+1 = \pm\sqrt{5}$

$x = -1\pm\sqrt{5}$

hence $(x+1+\sqrt{5})(x+1-\sqrt{5})$

$P(x) = (x-3+2i)(x-3-2i)(x+1+\sqrt{5})(x+1-\sqrt{5})$

[2]

/20

Question 3

a) (i) $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$ hyperbola

$a=4,\ b=3$

$e^2 = 1 + \dfrac{b^2}{a^2} = 1 + \dfrac{3^2}{4^2} = \dfrac{25}{16}$

$e>0 \quad e = 5/4$ [1]

(ii) foci $(\pm ae, 0)$

S_1 is $(4.\frac{5}{4}, 0) = (5,0)$

S_2 is $(-4.\frac{5}{4}, 0) = (-5,0)$ [2]

(iii) Directrices $x = \pm\dfrac{a}{e}$

$x = \pm 4/\frac{5}{4} = \pm 16/5 = \pm 3\frac{1}{5}$

[2]

(iv) $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 0$

$\dfrac{2x}{16} - \dfrac{2y}{9}\cdot\dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx} = \dfrac{2x}{16}\times\dfrac{9}{2y} = \dfrac{9x}{16y}$

at $(5, 2\frac{1}{4}) = (5, \frac{9}{4})$

slope of tangent $= \dfrac{9\times 5}{16\times\frac{9}{4}} = \dfrac{5}{4}$

eqn of tangent:

$y - y_1 = m(x-x_1)$

$y - \frac{9}{4} = \frac{5}{4}(x-5)$

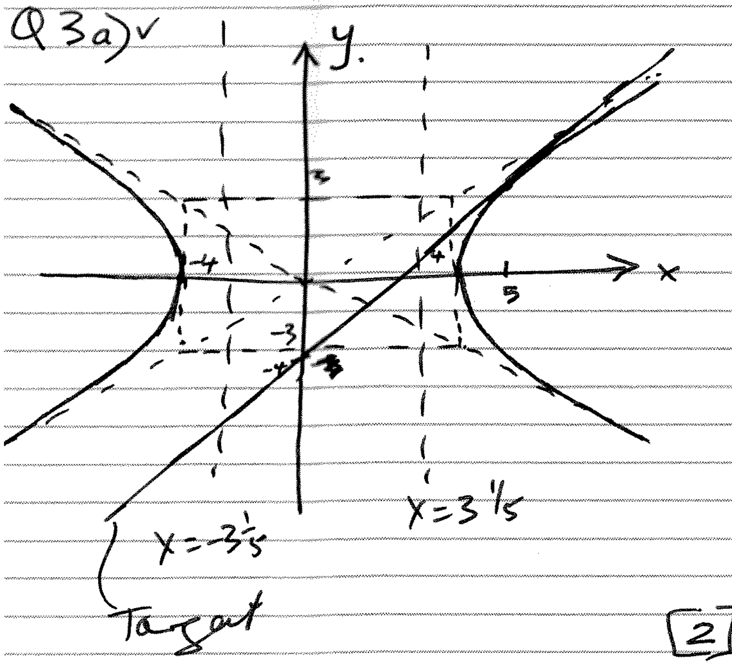
$4y - 9 = 5x - 25$

$4y = 5x - 16$

$5x - 4y - 16 = 0$ or $y = \frac{5}{4}x - 4$

[2]

Q3a) ✓



$\boxed{2}$

b) $\dfrac{x^2}{k-19} + \dfrac{y^2}{3-y} = 1$

note

$k-19 > 0 \Rightarrow k > 19$

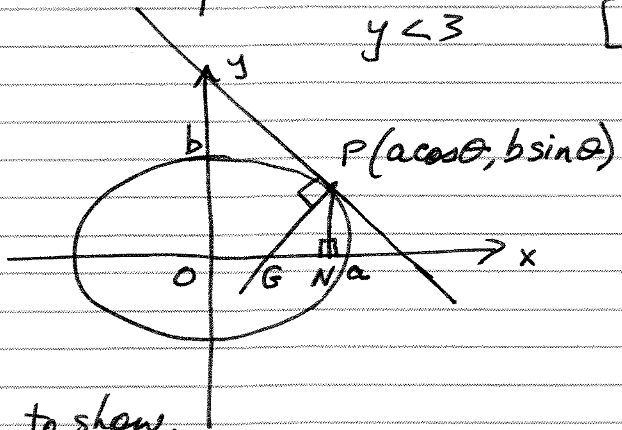
$3-y > 0 \Rightarrow 3 > y$

it is not possible $k > 19$

$y < 3$

$\boxed{1}$

c)



(i) to show.

$OG = e^2 . ON$

$x = a\cos\theta \Rightarrow \dfrac{dx}{d\theta} = -a\sin\theta$

$y = b\sin\theta \Rightarrow \dfrac{dy}{d\theta} = b\cos\theta$

$\dfrac{dy}{dx} = \dfrac{dy}{d\theta} . \dfrac{d\theta}{dx} = b\cos\theta \dfrac{1}{-a\sin\theta}$

$m_{Tangent} = -\dfrac{b}{a}\dfrac{\cos\theta}{\sin\theta}$

$m_{Normal} = \dfrac{a\sin\theta}{b\cos\theta}$

$\therefore$ eqn of PG

$y - b\sin\theta = \dfrac{a\sin\theta}{b\cos\theta}(x - a\cos\theta)$

to find G, let $y = 0$

$-b^2\sin\theta\cos\theta = a\sin\theta x - a^2\sin\theta\cos\theta$

$(a^2-b^2)\cancel{\sin\theta}\cos\theta = a\cancel{\sin\theta}. x$

$x = \left[\dfrac{a^2-b^2}{a}\right]. \cos\theta$

$\therefore OG = \left[\dfrac{a^2-b^2}{a}\right]\cos\theta$

N is $(a\cos\theta, 0)$

$\therefore ON = a\cos\theta$

$OG = \left[\dfrac{a^2-b^2}{a^2}\right]a\cos\theta.$

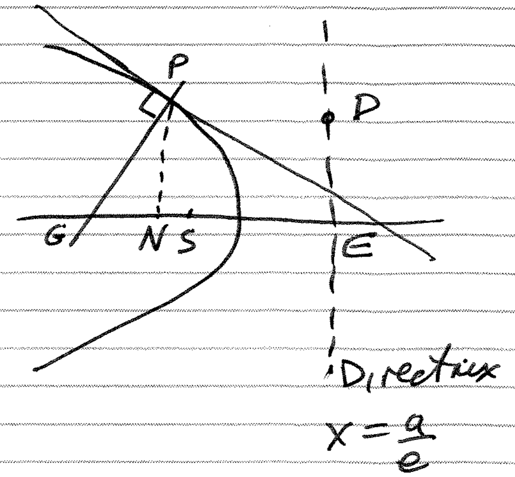
$= \left[1 - \dfrac{b^2}{a^2}\right]a\cos\theta$

$= e^2 . a\cos\theta$

$= e^2 . ON$

$\boxed{3}$

(ii)



Q3cii continue

by definition

$$\frac{PS}{PD} = e \quad \text{——(a)}$$

[To show $SG = eSP$]

$SG = OS - OG$

$= ae - \left(\frac{a^2-b^2}{a}\right)\cos\theta$

$SP = e.PD$ from (a)

$= e.NE$

$= e\left[\frac{a}{e} - a\cos\theta\right]$

$= a - ae\cos\theta$

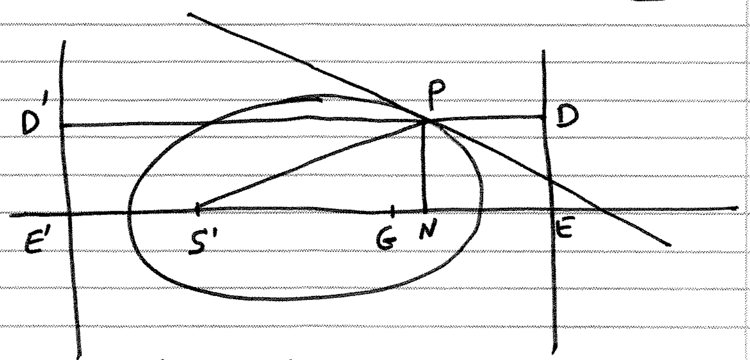
$\therefore e.SP = ae - ae^2\cos\theta$

$= ae - a\left(1-\frac{b^2}{a^2}\right)\cos\theta$

$= ae - \left(\frac{a^2-b^2}{a}\right)\cos\theta$

allow 2

$eSP = SG$



E' is $\left(-\frac{a}{e}, 0\right)$

$\frac{S'P}{D'P} = e \quad \therefore S'P = eD'P$

$e.S'P = e^2D'P = e^2.NE'$

$= e^2\left(a\cos\theta + \frac{a}{e}\right)$

$= ae^2\cos\theta + ae$

$= \frac{a(a^2-b^2)}{a^2}\cos\theta + ae$

$= \left(\frac{a^2-b^2}{a}\right)\cos\theta + ae$

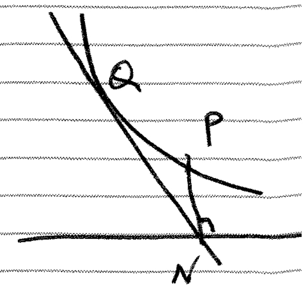
$= S'G$

$\therefore S'G = e.S'P$ allow 1

[3]

Q3d.

(i) $xy = c^2$



$P\left(cp, \frac{c}{p}\right)\ Q\left(cq, \frac{c}{q}\right)$

$y = \frac{c^2}{x} \qquad \frac{dy}{dx} = -\frac{c^2}{x^2}$

at Q $\frac{dy}{dx} = -\frac{c^2}{c^2q^2} = \frac{-1}{q^2}$

eqn of Tangent at Q:

$y - \frac{c}{q} = -\frac{1}{q^2}(x - cq)$

cuts axis at $N(cp, 0)$

$0 - \frac{c}{q} = -\frac{1}{q^2}(cp - cq)$

$cq = \cancel{c}\, cp - cq$

$2cq = cp$

$p = 2q$ [2]

(ii) M is midpoint of PQ.

~~M_x~~ $x = \frac{c}{2}(p+q)$

$x = \frac{c}{2}(3q)$

$\Rightarrow \frac{2x}{3c} = q$

Q3 d(ii)

$y = \frac{c}{2}\left(\frac{1}{p} + \frac{1}{q}\right)$

$= \frac{c}{2}\left(\frac{1}{2q} + \frac{1}{q}\right)$

$= \frac{c}{2}\left(\frac{3}{2q}\right)$

$y = \frac{3c}{4q} \Rightarrow q = \frac{3c}{4y}.$

$\frac{2x}{3c} = \frac{3c}{4y}$

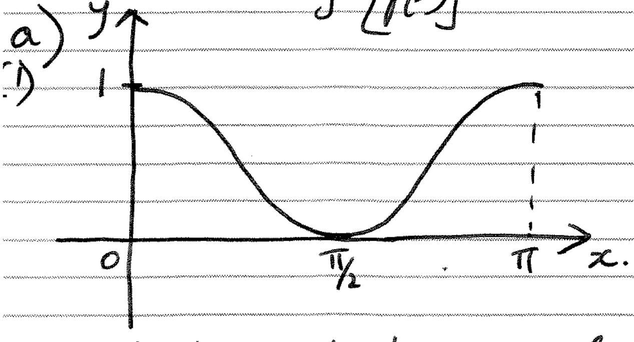
$8xy = 9c^2$

$8xy = (3c)^2$ [2]

which is a rectangular hyperbola

Question 4

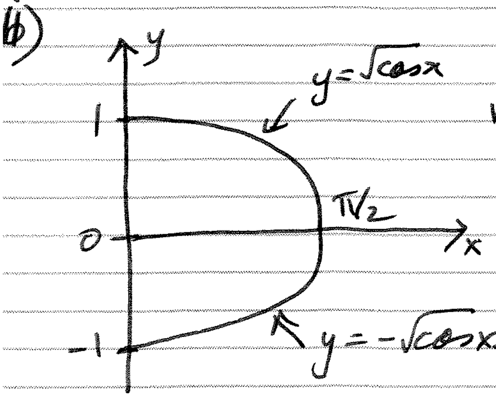
a) (i) $y = [f(x)]^2$



[2]

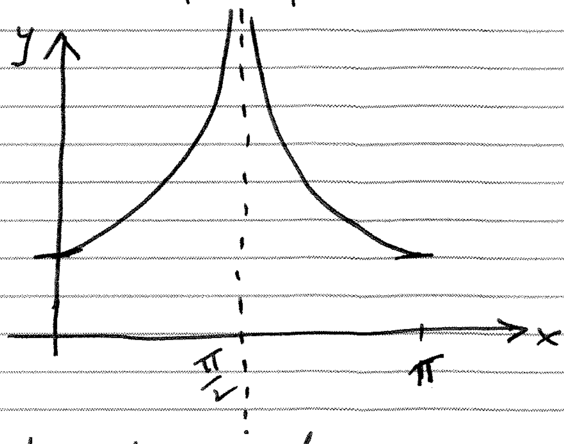
note tangent is horizontal at $x = 0, \pi/2, \pi$

(ii)



vertical tangent at $x = \pi/2$ [2]

(iii) $y = \frac{1}{|\cos x|}$



[2]

discontinuous at $x = \frac{\pi}{2}$

horizontal tangent at $y = 1$

b)(i)

$\frac{5x}{(x+1)(x-2)} = \frac{a}{x+1} + \frac{b}{x-2}$

$= \frac{a(x-2)+b(x+1)}{(x+1)(x-2)}$

$= \frac{ax-2a+bx+b}{(\ )(\ )}$

$= \frac{(a+b)x+(b-2a)}{(\ )(\ )}$

equate coeffecient

$\left.\begin{array}{l} a+b=5 \\ b-2a=0 \end{array}\right\}$ solve

subtract $3a = 5$ $a = \frac{5}{3}$

$b = \frac{10}{3}$

$\frac{5x}{(x+1)(x-2)} = \frac{5}{3(x+1)} + \frac{10}{3(x-2)}$ [2]

(ii)

$I = \int_3^4 \frac{5x}{(x+1)(x-2)}\,dx$

$$I = \int_3^4 \frac{5}{3(x+1)}\,dx + \int_3^4 \frac{10}{3(x-2)}\,dx$$

$$= \frac{5}{3}\Big[\ln(x+1)\Big]_3^4 + \frac{10}{3}\Big[\ln(x-2)\Big]_3^4$$

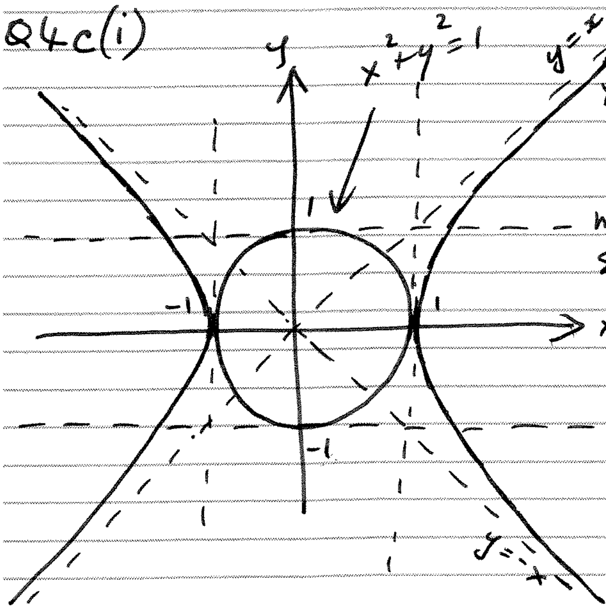
$$= \frac{5}{3}\Big[\ln 5 - \ln 4\Big] + \frac{10}{3}\Big[\ln 2 - \ln 1\Big]$$

$$= \frac{5}{3}\Big[\ln 5 - \ln 4 + 2\ln 2\Big]$$

$$= \frac{5}{3}\ln 5.$$

$[2.682396 \approx 2.68\,(2dp)]$

Q4c(i)

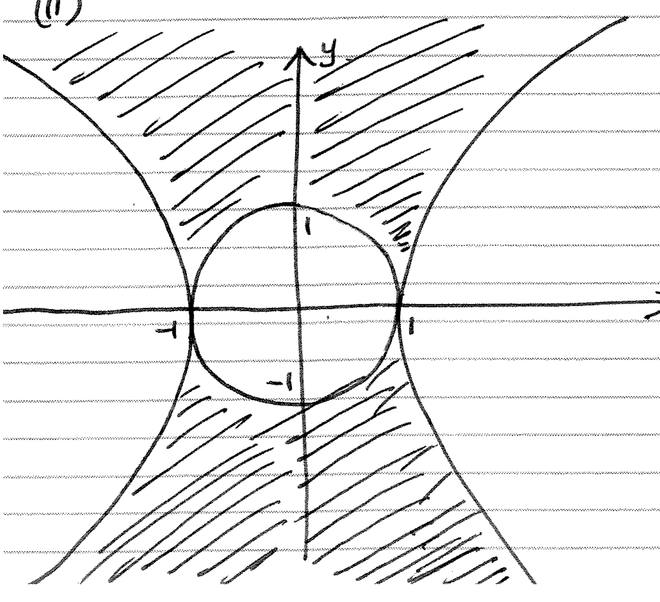


axes •
intersections
shapes ••
asympt •

-1 each missing

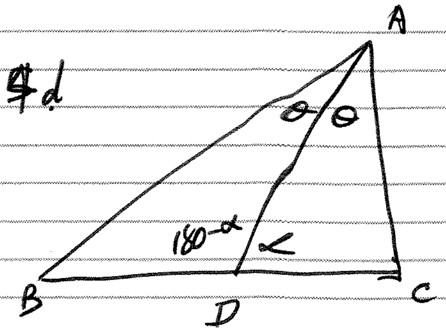
[4]

(ii)



[2]

Q4d



in $\triangle ABD$ $\quad \frac{AB}{\sin D} = \frac{BD}{\sin\theta} \Rightarrow \sin(180-\alpha) = \frac{AB\sin\theta}{BD}$

$\sin D \to 180-\alpha.$

in $\triangle ADC$

$\frac{\sin\alpha}{AC} = \frac{\sin\theta}{DC} \therefore \sin\alpha = \frac{AC\sin\theta}{DC}$

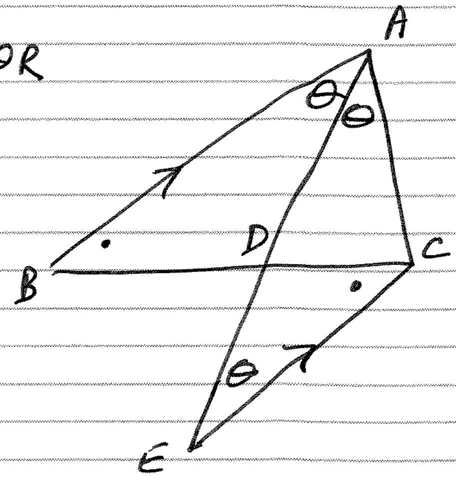
but $\sin(180-\alpha) = \sin\alpha$

$\frac{AB\sin\theta}{BD} = \frac{AC}{DC}\sin\theta.$

[3]

$$\frac{AB}{BD} = \frac{AC}{DC}$$

OR



produce AD TO E such that BA∥CE

Alternate angles equal as marked

$\therefore \triangle ABD \,|||\, \triangle ECD$ (AAA)

$\therefore \frac{AB}{EC} = \frac{BD}{DC}$

but $\triangle ACE$ is isosceles $\therefore CE = AC$

$\frac{AB}{AC} = \frac{BD}{DC}$

$\therefore \frac{AB}{BD} = \frac{AC}{DC}$