KILLARA HIGH SCHOOL

YEAR 12 – 2009

HSC ASSESSMENT 1

MATHEMATICS EXTENSION 2

Time allowed: 70 minutes

Total marks: 40

- **NOT ALL** questions are of equal value.
- Start each question on a new page.
- Show all necessary working.

Start each question on a new page

1. Given that $z = 2 + 3i$ and $w = 3 - i$, find in the form $a + ib$:

a) $z + w$ **1**

b) $z.\overline{w}$ **2**

c) $\dfrac{z}{w}$ **2**

2. a) Find the square root of $5 - 12i$. **2**

b) Hence, or otherwise solve the equation in the form $a + ib$ **3**

$$z^2 - (1 - 4i)z - (5 - i) = 0.$$

3. By expressing z in modulus-argument form simplify

$$(2 - 2i)^{10}.(\sqrt{3} + i)^6$$ **3**

4. a) Find the five fifth roots of $1 + \sqrt{3}i$ **4**

b) Graph these roots on an Argand diagram and find the area of the pentagon. **2**

5. If w is a complex cube root of unity, prove that

a) $1 + w + w^2 = 0$ **2**

b) $\dfrac{a + bw + cw^2}{c + aw + bw^2} = w^2$ **2**

<u>Start each question on a new page</u>

6. a) Draw a neat labelled sketch to indicate the locus of the complex number z

i) If $|z+2-i|=|z-4|$ **3**

i) If both $\arg\left(\frac{z-4}{z}\right)>\frac{\pi}{2}$ *and* $0\le im(z)\le 1$ are satisfied. **3**

7. Complex numbers $z_1=\frac{p}{1+2i}$ *and* $z_2=\frac{q}{1+i}$ where p and q are real. **3**

Find p and q If $z_1-z_2=4i$

8. α and β are two complex numbers.Given that $\alpha - \beta = i(\alpha+\beta)$, on an Argand diagram

a) neatly label and sketch: **2**

i) α

ii) β

iii) $\alpha - \beta$

iv) $\alpha + \beta$

b) Find $\arg\left(\frac{\alpha+\beta}{\beta}\right)$ **2**

c) If $\beta = r(\cos\theta + i\sin\theta)$, find the complex number which represents $\alpha+\beta$. **2**

d) If $\theta=\frac{\pi}{6}$ and r = 1 in part (c). The point Q represents $\alpha+\beta$ is rotated through $\frac{11\pi}{12}$ **1**

clockwise direction to become Q', Find the complex number represented by Q' in the form a + ib.

END OF PAPER

Q1 $z+w = 2+3+3i-i$

$= 5+2i$ ✓

b) $z\,\bar{w} = (2+3i)(3+i)$ ✓

$= 6+2i+9i-3$

$= 3+11i$ ✓

c) $\dfrac{z}{w} = \dfrac{2+3i}{3-i}$

$= \dfrac{(2+3i)(3+i)}{9+1}$ ✓

$= \dfrac{3}{10} + \dfrac{11i}{10}$ ✓

Q2 a) $5-12i = 9-12i+4i^2$ ✓

$= (2i-3)^2$ ✓

$\therefore \sqrt{5-12i} = \pm(2i-3)$

b)

$z = \dfrac{1-4i \pm \sqrt{(1-4i)^2+4(5-i)}}{2}$ ✓

$= \dfrac{1-4i \pm \sqrt{5-12i}}{2}$

$= \dfrac{1-4i+2i-3}{2}, \dfrac{1-4i-2i+3}{2}$

$\therefore$ The 2 roots are

$-(1-i)$ ✓ and $2-3i$ ✓

Q3 $(2-2i)^{10} = \left(\sqrt{8}\, \text{cis} -\frac{\pi}{4}\right)^{10}$ (1)

$= 8^5 \text{cis} \frac{\pi}{2} = 8^5 i$ ✓

$(\sqrt{3}+i)^6 = \left(2 \text{cis} \frac{\pi}{6}\right)^6$ (2)

$= 2^6 \text{cis}\, \pi$

$= -2^6$ ✓

(1)×(2) $= 8^5 i \times -2^6$

$= -2097152\, i$ ✓

Q4

a) $1+\sqrt{3}\, i = 2 \text{cis} \frac{\pi}{3}$

$= r^5 \text{cis} 5\theta$

$\therefore$ The five fifth roots of $1+\sqrt{3}\, i$ are of the form $2^{1/5} \text{cis}\left(\frac{\pi}{15} + \frac{2k\pi}{5}\right)$ ✓ where $k = 0,1,2,3,4$

$\therefore$

$z_1 = 2^{1/5} \text{cis} \frac{\pi}{15}$ ✓

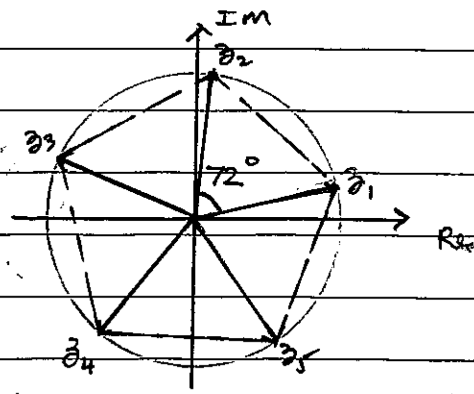
$z_2 = 2^{1/5} \text{cis} \frac{7\pi}{15}$ ✓

(3)

$z_3 = 2^{1/5} \text{cis} \frac{13\pi}{15}$ ✓

$z_5 = 2^{1/5} \text{cis} -\frac{\pi}{3}$ ✓

$z_4 = 2^{1/5} \text{cis} -\frac{11\pi}{15}$ ✓



The roots are equally spaced aroun the circle with $r = 2^{1/5}$, at an angle $\frac{2}{5}$

b) Area of pentagon

$A = 5\left(\frac{1}{2} \times 2^{1/5} \times 2^{1/5} \sin \frac{2\pi}{5}\right)$

$= 3.14$ sq unit. ✓

Q5

(a)

The 3 cube roots of unity are

$-\frac{1}{2}+\frac{\sqrt{3}}{2}i$, $-\frac{1}{2}-\frac{\sqrt{3}}{2}i$ and 1

$\therefore\ 1+\omega+\omega^2 = 1-\frac{1}{2}+\frac{\sqrt{3}}{2}i-\frac{1}{2}-\frac{\sqrt{3}}{2}i$

$= 0$

$= RH.S$

Alternatively

The equation for cube roots of unity is

$z^3-1 = 0$

$(z-1)(z^2+z+1) = 0$

$\omega \neq 1 \ \therefore\ z-1 \neq 0$

hence $z^2+z+1 = 0$

$\therefore\ \omega^2+\omega+1 = 0$

(b)

$L.H.S = \dfrac{(a+b\omega+c\omega^2)\times\omega^3}{(c+a\omega+b\omega^2)\times\omega^3}$

$= \dfrac{\omega^2(a\omega+b\omega^2+c\omega^3)}{c\omega^3+a\omega^4+b\omega^5}$

but $\omega^3=1$, $\omega^4=\omega$ and $\omega^5=\omega^2$

$\therefore\ L.H.S = \dfrac{\omega^2(a\omega+b\omega^2+c)}{c+a\omega+b\omega^2}$

$= \omega^2 \quad = R.H.S$

Q6

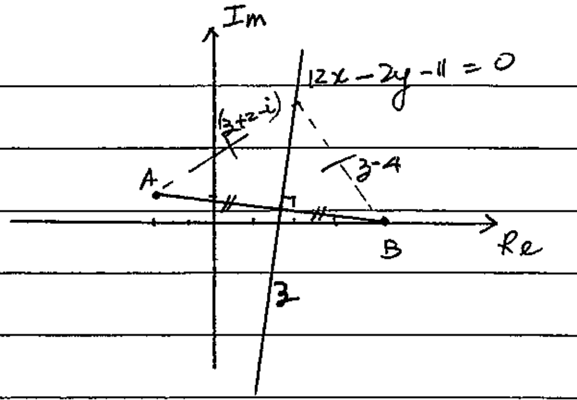
a) $|z+2-i| = |z-4|$

$(x+2)^2+(y-1)^2 = (x-4)^2+y^2$

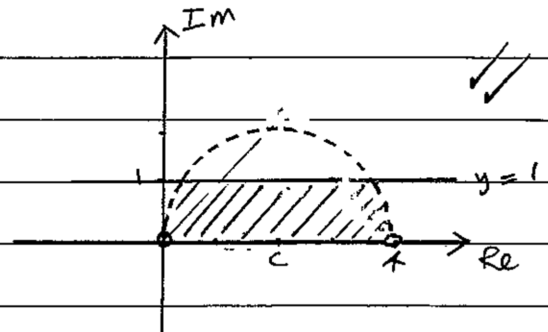
$\cancel{x^2}+4x+4+\cancel{y^2}-2y+1 = \cancel{x^2}-8x+16+\cancel{y^2}$

$12x-2y-11 = 0$

$\therefore$ Locus of z is the bisector to AB with the equation $12x-2y-11=0$



(b)



$Arg\left(\frac{z-4}{z}\right)=\frac{\pi}{2}$ is the semicircle with radius $r=2$, centre $c(2,4)$.

$Arg\left(\frac{z-4}{z}\right) > \frac{\pi}{2}$ is inside the semi-circle.

Q7

$z_1-z_2 = \dfrac{p}{1+2i} - \dfrac{q}{1+i}$

$= \dfrac{p(1-2i)}{5} - \dfrac{q(1-i)}{2}$

$= \dfrac{2p-4pi-5q-5qi}{10} = 4i$

$\therefore\ \dfrac{2p-5q}{10} = 0$ and $\dfrac{4p-5q}{10} = -4$

$\Rightarrow\ 4p-5q = 40$

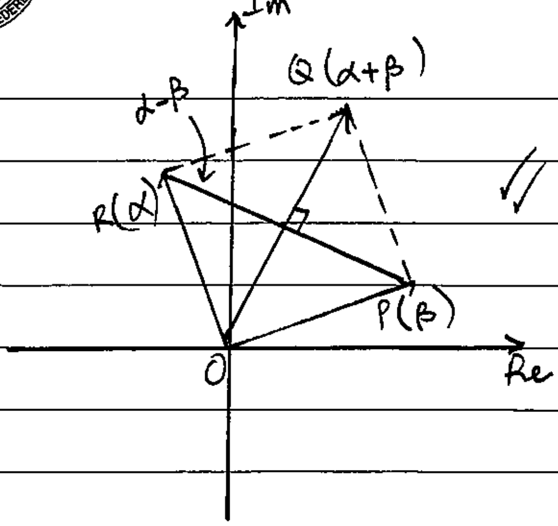
$-\ \ 2p-5q = 0$

$2p = -40 \Rightarrow p = -20$

and

$-40-5q = 0 \Rightarrow q = -8$

(a)



✓✓

(b)

Since $\alpha-\beta = i(\alpha+\beta)$, from the diagram

$\vec{PR}$ is the rotation of $\vec{OQ}$, anticlockwise, by $\frac{\pi}{2}$, about O.

$\therefore$ PR = OQ

$\therefore$ OPQR is a square (diags' are equal & bisect at ⊥)

hence $\angle QOP = \pi/4$ ✓

$\arg\left(\frac{\alpha+\beta}{\beta}\right) = \arg(\alpha+\beta) - \arg\beta$

$= \angle QOP$

$= \pi/4$ ✓

(diag. of a square bisects the angle which it passes.)

(c)

$|\alpha+\beta| = \sqrt{r^2+r^2}$

$= r\sqrt{2}$ ✓

$\arg(\alpha+\beta) = \arg\beta + \angle QOP$

$= \theta + \pi/4$

$\therefore \alpha+\beta = r\sqrt{2}\,\text{cis}\,(\theta+\pi/4)$ ✓

(d) if $\theta = \pi/6$, $r=1$

then $\alpha+\beta = \sqrt{2}\,\text{cis}\,\frac{5\pi}{12}$

$\therefore$ The complex number represents Q' is

$\sqrt{2}\,\text{cis}\left(\frac{5\pi}{12} - \frac{11\pi}{12}\right) = \sqrt{2}\,\text{cis}\left(-\pi/2\right)$

$= \sqrt{2}\left(\cos -\pi/2 + i\sin -\pi/2\right)$

$= -\sqrt{2}\,i$ ✓