KILLARA HIGH SCHOOL

Year 12 Term 2 Assessment 2009

Extension 2 Mathematics

Time Allowed: 70 minutes

Total marks: 45

Instructions

1. Start each question in a new booklet.
2. Write on one side of the paper only.
3. Show all necessary working.

Question 1

Evaluate:

a. $\int \frac{e^{\tan x}}{\cos^2 x} dx$ 1 mark

b. $\int \frac{dx}{4+3\cos 2x}$ 4 marks

c. $\int \frac{3x+1}{(x-1)^2(2x+1)} dx$ 3 marks

Question 2

a. Evaluate $\int_1^2 \frac{dx}{x^2\sqrt{x-1}}$ 3 marks

b. Evaluate $\int_0^1 \sqrt{\frac{3-x}{4+x}} dx$ 4 marks

c. i) Use the substitution $u=-x$ to show that
$\int_{-a}^{0} f(x)dx = \int_0^a f(-x)dx$ 2 marks

ii) Deduce that $\int_{-a}^{a} f(x)dx = \int_0^a [f(x)+f(-x)]dx$ 2 marks

iii) Hence evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^5 \cos x dx$ 2 marks

Question 3

a. Obtain a reduction formula for $I_n = \int_0^{\frac{\pi}{2}} \cos^n x \; dx \; (n \geq 2)$. 3 marks

b. Use the result from part 'a' to evaluate $\int_0^{\frac{\pi}{2}} \cos^6 x \; dx$ 3 marks

Question 4

a. The base of a solid is an ellipse whose major and minor axes are 16 cm and 12 cm respectively. Vertical cross sections taken through this solid in a direction parallel to the minor axis is an equilateral triangle with one side in the base of the solid. Find the volume of the solid.

5 marks

b. The region bounded by the curve $y = (x-2)^2$, the x and y axes is rotated about the line $y = -\frac{3}{2}$. Calculate the exact volume of the solid of revolution by taking slices parallel to the y-axis.

5 marks

c. Evaluate the exact volume obtained when the area bounded by $y = \frac{2}{x+2}$, the x-axis, the y-axis and the line $x = 3$ is rotated about the y-axis. 4 marks

d. i) Sketch the graph of $y = e^{-t^2}$ 1 mark

ii) The region bounded by the graph the t-axis, in the domain $-a \leq t \leq a$, that is rotated about the y-axis forms a solid. Use the method of cylindrical shells, where a shell has radius 't' to show that the shell has an approximate volume of "$\delta V = 2\pi t e^{-t^2} \delta t$". 1 marks

iii) Calculate the volume of the solid. 2 marks

Question 1

a $\int \frac{e^{\tan x}}{\cos^2 x}\,dx$

$= \int \sec^2 x \; e^{\tan x}\,dx$

$= e^{\tan x} + c$ ✓

b. $\int \frac{dx}{4+3\cos 2x}$

let $t = \tan x$

where $\cos 2x = \frac{1-t^2}{1+t^2}$

$dt = \sec^2 x \; dx$

$= 1+\tan^2 x \; dx$

$= (1+t^2)\,dx$

$\frac{dt}{1+t^2} = dx.$

$= \int \frac{\frac{dt}{1+t^2}}{4+3\left(\frac{1-t^2}{1+t^2}\right)}$ 1

1

$= \int \frac{dt}{4(1+t^2)+3(1-t^2)}$

$= \int \frac{dt}{t^2+7}$ 1

$= \frac{1}{\sqrt{7}} \tan^{-1} \frac{t}{\sqrt{7}} + c$

$= \frac{1}{\sqrt{7}} \tan^{-1}\left(\frac{\tan x}{\sqrt{7}}\right) + c$ 1

Question 1

c.

$$\int \frac{3x+1}{(x-1)^2(2x+1)}\,dx \qquad \text{let } \frac{3x+1}{(x-1)^2(2x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{2x+}$$

$$\therefore\ 3x+1 = A(x-1)(2x+1) + B(2x+1) + C(x-1)^2$$

✓ let $x=1 \Rightarrow B = \frac{4}{3}$

let $x = -\frac{1}{2} \Rightarrow C = \frac{-2}{9}$

Equating coefficients of x^2

$A = \frac{1}{9}$.

$$\therefore \int \frac{3x+1}{(x-1)^2(2x+1)}\,dx$$

$$= \int \frac{1}{9(x-1)}\,dx + \frac{4}{3}\int \frac{dx}{(x-1)^2} - \frac{2}{9}\int \frac{dx}{2x+1} \quad ✓$$

$$= \frac{1}{9}\ln(x-1) + \frac{4}{3}\left(\frac{-1}{x-1}\right) - \frac{2}{18}\ln(2x+1) + c$$

$$= \frac{1}{9}\ln\left|\frac{x-1}{2x+1}\right| - \frac{4}{3}\left(\frac{1}{x-1}\right) + C \quad ✓$$

Question 2

<u>a</u> $\int_1^2 \frac{dx}{x^2\sqrt{x-1}}$

let $x = \sec^2\theta$
$= 1 + \tan^2\theta$
✓ $dx = 2\tan\theta\sec^2\theta\, d\theta$
$x = 1 \Rightarrow \theta = 0$
$x = 2 \Rightarrow \theta = \frac{\pi}{4}$.

$$= \int_0^{\frac{\pi}{4}} \frac{2\tan\theta\sec^2\theta\, d\theta}{\sec^4\theta\,\tan\theta} \quad ✓$$

$$= 2\int_0^{\frac{\pi}{4}} \cos^2\theta\, d\theta.$$

$$= \int_0^{\frac{\pi}{4}} [1 + \cos 2\theta]\, d\theta.$$

$$= \frac{\pi}{4} + \frac{1}{2} \quad ✓$$

<u>b</u> $\int_0^1 \sqrt{\frac{3-x}{4+x}}\, dx$

$$= \int_0^1 \frac{3-x}{\sqrt{12-x-x^2}}\, dx \quad ✓$$

$$= \frac{1}{2}\int_0^1 \frac{6-2x}{\sqrt{12-x-x^2}}\, dx \quad ✓$$

$$= \frac{1}{2}\int_0^1 \frac{-1-2x}{\sqrt{12-x-x^2}}\, dx + \frac{1}{2}\int_0^1 \frac{7\, dx}{\sqrt{(\frac{7}{2})^2 - (\frac{1}{2}+x)^2}} \quad ✓$$

$$= \left[(12-x-x^2)^{\frac{1}{2}}\right]_0^1 + \frac{7}{2}\left[\sin^{-1} \frac{2(x+\frac{1}{2})}{7}\right]_0^1$$

$$= \sqrt{10} - \sqrt{12} + \frac{7}{2}\left(\sin^{-1}\frac{3}{7} - \sin^{-1}\frac{1}{7}\right) \quad ✓$$

Question 2

c i) $LHS = \int_{-a}^{0} f(x)\,dx$ let $u = -x$

$\therefore du = -dx$

when $x = 0 \Rightarrow u = 0$

when $x = -a \Rightarrow u = a$

$= -\int_{a}^{0} f(-u)\,du$ ✓

✓ $= \int_{0}^{a} f(-u)\,du$

$= RHS$

ii) $\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$ ✓

$= \int_{0}^{a} f(-x)\,dx + \int_{0}^{a} f(x)\,dx$ ✓

$= \int_{0}^{a} \left(f(x) + f(-x) \right) dx.$

iii) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (x^5 \cos x)\,dx = \int_{0}^{\frac{\pi}{2}} \left((x^5 \cos x) + (-x)^5 \cos(-x) \right) dx$ ✓

$= \int_{0}^{\frac{\pi}{2}} \left(x^5 \cos x - x^5 \cos x \right) dx$

$= 0$ ✓

Question 3

a $I_n = \int_0^{\frac{\pi}{2}} \cos^n x \, dx$

$= \int_0^{\frac{\pi}{2}} \cos x \cos^{n-1} x \, dx.$

$= \int_0^{\pi/2} \cos^{n-1} x \, \frac{d}{dx}(\sin x) \, dx.$

$= \left[\sin x \cos^{n-1} x\right]_0^{\pi/2} - \int_0^{\frac{\pi}{2}} \sin x (n-1)\cos^{n-2} x(-\sin x) \, dx$

$= 0 + (n-1)\int_0^{\frac{\pi}{2}} (1-\cos^2 x)\cos^{n-2} x \, dx.$

$= (n-1) I_{n-2} - (n-1) I_n$

b. $I_n(1+n-1) = (n-1) I_{n-2} \quad \therefore \; I_n = \frac{n-1}{n} I_{n-2}.$

let $I_6 = \cos^6 x \, dx$

$I_6 = 5I_4 - 5I_6$

$6I_6 = 5I_4$

$I_6 = \frac{5}{6} I_4 = \frac{5}{6} \cdot \frac{3}{4} \cdot I_2$

Now, $I_2 = \int_0^{\pi/2} \cos^2 x \, dx = \frac{1}{2}\int_0^{\pi/2} 1 + \cos 2x \, dx = \frac{[illegible]}{4}$

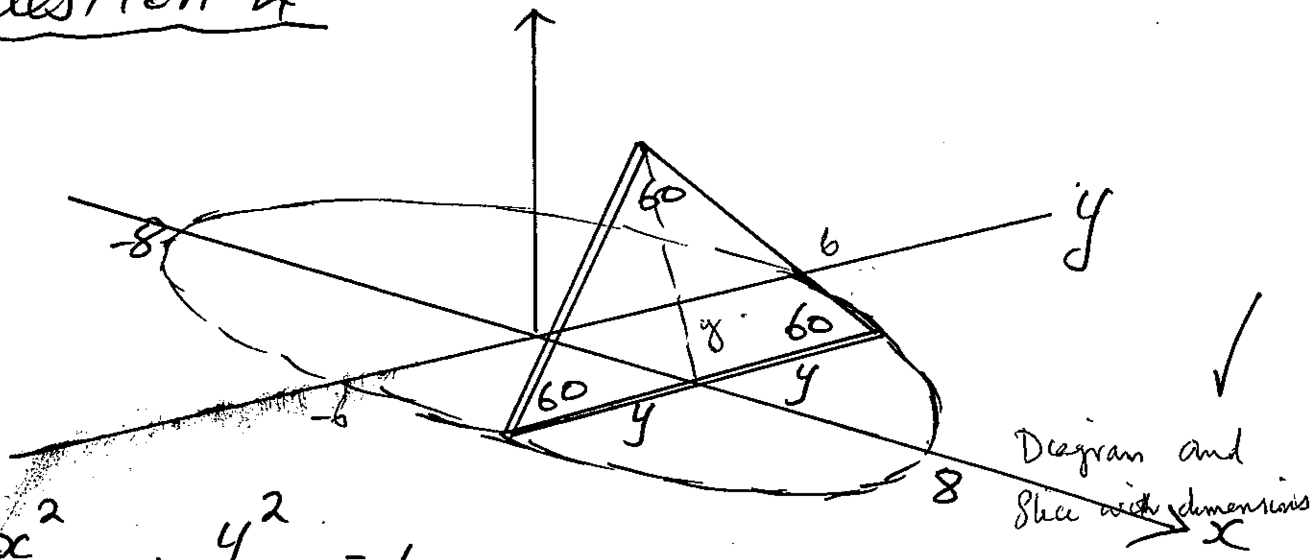
$I_6 = \frac{5}{6} \times \frac{3}{4} \times \frac{\pi}{4}$

$= \frac{15\pi}{96}$

$= \frac{5\pi}{32}$

Question 4

a



$$\frac{x^2}{64} + \frac{y^2}{36} = 1$$

Let the surface area of the equilateral triangle, A, have side length $2y$.

Area of $A = \frac{1}{2} \times 2y \times 2y \times \sin 60 = \sqrt{3}\, y^2$ ✓

Let Volume of cross sectional slice be dV

$$\therefore dV = \lim_{dx \to 0} \sum_{x=-8}^{8} \sqrt{3}\, y^2\, dx .$$ ✓

$$\text{Volume} = \sqrt{3} \int_{-8}^{8} 36 - \frac{9x^2}{16}\, dx .$$ ✓

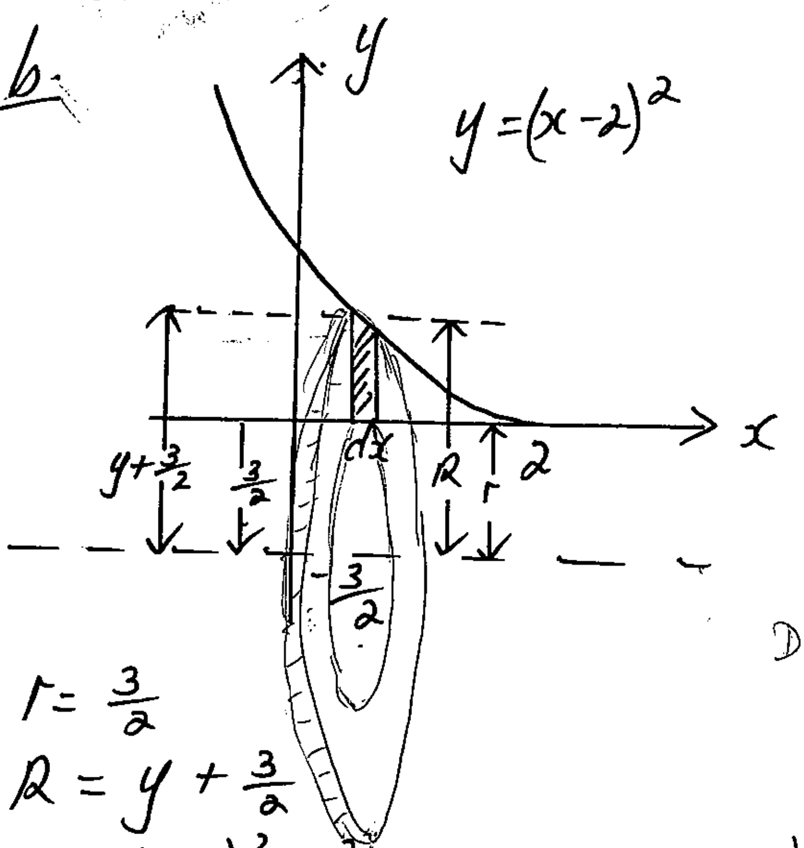
$$= 2\sqrt{3} \int_{0}^{8} 36 - \frac{9}{16}x^2\, dx$$

$$= 2\sqrt{3} \left[36x - \frac{9x^3}{16 \times 3} \right]_0^8 = 2\sqrt{3}(288 - 96 - 0)$$

$$= 384\sqrt{3} \text{ units}^3$$ ✓

Question 4

b.



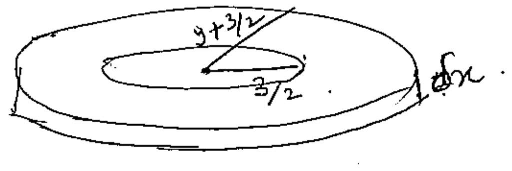


Diagram with Slice

$r = \frac{3}{2}$

$R = y + \frac{3}{2}$

$= (x-2)^2 + \frac{3}{2}$

Let dV be the volume of the annulus with inner radius r and outer radius R of thickness dx.

$\therefore \; dV = \pi (R^2 - r^2)\, dx.$

$= \pi (R-r)(R+r)\, dx$

$= \pi \; y(y+3)\, dx.$

$= \pi (x-2)^2 \left((x-2)^2 + 3\right) dx$

$= \pi \left[(x-2)^4 + 3(x-2)^2\right] dx$

Volume

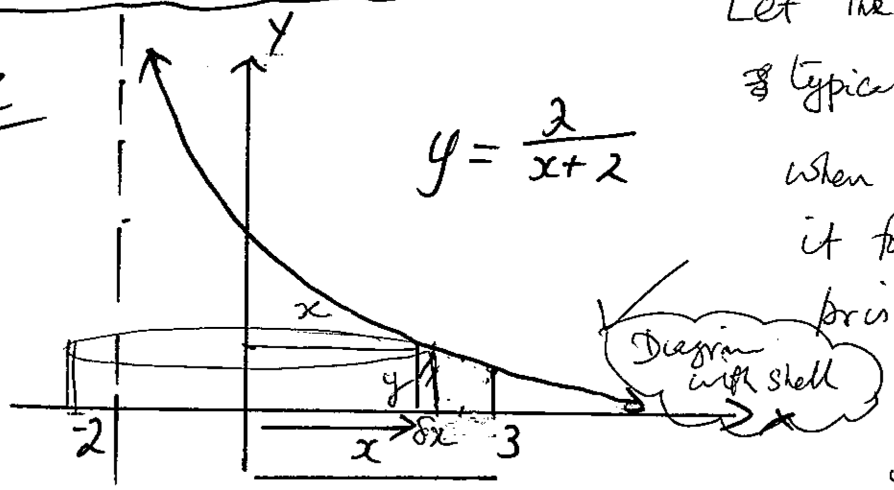
$= \pi \int_0^2 (x-2)^4 + 3(x-2)^2 \, d$

$= \pi \left[\frac{(x-2)^5}{5} + (x-2)^3\right]_0^2$

$= 14\frac{2}{5}\pi$ units^3

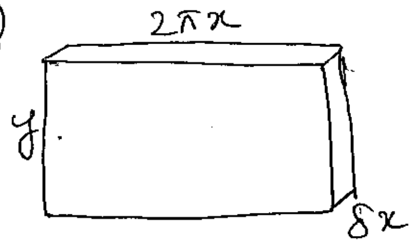
Question 4

c



Let the volume of the typical shell $= \delta V$.

When the shell is opened, it forms a rectangular prism as shown below.



$$\therefore dV = 2\pi x y \, dx.$$

$$\text{Volume} = \lim_{dx \to 0} \sum_{x=0}^{3} 2\pi x y \, dx.$$

$$= \int_0^3 2\pi x \left(\frac{2}{x+2}\right) dx$$

$$= 4\pi \int_0^3 \frac{x}{x+2} \, dx.$$

$$= 4\pi \int_0^3 \frac{x+2-2}{x+2} \, dx.$$

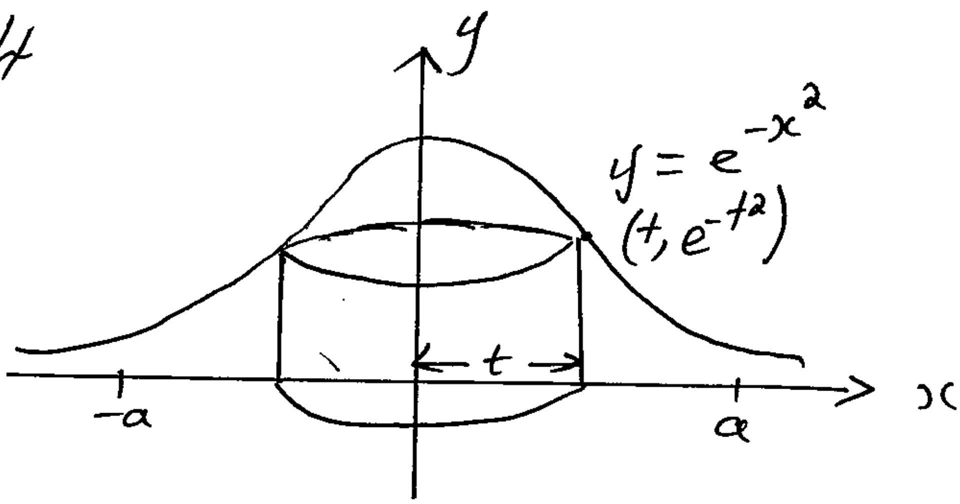
$$= 4\pi \int_0^3 \left(1 - \frac{2}{x+2}\right) dx.$$

$$= 4\pi \Big[x - 2\ln(x+2)\Big]_0^3$$

$$= 4\pi \left[3 + 2\ln\left(\frac{2}{5}\right)\right] \text{ units}^3$$

Question 4

d. i



ii Let the width of a shell be dt.

$$\therefore \quad dV = 2\pi x y \, dt$$
$$= 2\pi \cancel{t} e^{-\cancel{t}^2} \, dt.$$

iii
$$\text{Volume} = \lim_{d\cancel{t} \to 0} \sum_{x=0}^{a} 2\pi t e^{-t^2} dt$$
$$= -\pi \int_0^a -2te^{-t^2} dt$$
$$= -\pi \left[e^{-t^2} \right]_0^a$$
$$= \pi \left(1 - e^{-a^2}\right) \text{ units}^3$$