KILLARA HIGH SCHOOL

2009

HALF-YEARLY HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics
Extension 2

General Instructions

- Reading Time- 5 minutes
- Working Time – 3 hours
- Write using a blue or black pen
- Approved calculators may be used
- A table of standard integrals is provided at the back of this paper.
- All necessary working should be shown for every question.
- Untidy work or work with poor logic may be penalised.

Total marks (**120**)

- Attempt Questions 1-8
- All questions are of equal value
- Start a fresh booklet of paper for each question.
- Put your name and the question number at the top of booklet.

Total marks - 120
Attempt questions 1 – 8
All questions are of equal value

Answer each question on a separate writing booklet.

Question 1 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Sketch the graph of $y = (2x + 1)(x + 1)$ showing clearly the intercepts on the coordinate axes and the coordinates of any turning points. **2**

(b) Use the graph in part (a) to sketch the following graphs:
(equations of asymptotes, turning points and intercepts must be shown)

(i) $y = \dfrac{1}{(2x+1)(x+1)}$ **2**

(ii) $y = |(2x+1)(x+1)|$ **2**

(iii) $y = \sqrt{(2x+1)(x+1)}$ **2**

(iv) $y = \ln\{(2x+1)(x+1)\}$ **3**

(c) Find the equation of the tangent to the curve $y = \ln\{(2x+1)(x+1)\}$ at the origin. **4**

Hence find the values of the real number k such that exactly one of the solutions of the equation $\ln\{(2x+1)(x+1)\} = kx$ is a positive number.

Question 2 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) $-3+4i$ has two square roots z_1 and z_2. Find z_1 and z_2 in the form $a+ib$. **2**

Show the points representing $-3+4i$, z_1 and z_2 on an Argand diagram. Show that These three points are vertices of a right-angled triangle. **2**

(b) The complex number z is represented by the point P on an Argand diagram. Indicate clearly on a single diagram the locus of P in each of the following cases. **4**

(i) $|z-4| = |z+2i|$

(ii) $\arg(z-3) = \dfrac{\pi}{4}$

Show that there is a point representing the complex number of the form ib, where b is real, which lies on both loci.

(c) The equation $z^2+(1-2i)z-(7+i)=0$ has roots α and β. **1**
Find the monic equation with numerical coefficients $\alpha - i$ and $\beta - i$.
Hence find the values of α and β.

(d) On an Argand diagram the point A represents the real number 1. O is the origin and the point P represents the complex number z which satisfies the condition $\arg(z-1) = 2\arg(z)$.

(i) Show this information on a diagram and deduce that the triangle OAP is isosceles. **3**

(ii) Deduce that the locus of P is a circle and show this circle on your diagram. **1**

(iii) Find z in modulus/argument form if z also satisfies the condition $|z| = |z-1|$. **2**

Question 3 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) The equation $x^3 + 2x - 1 = 0$ has roots α, β and γ. In each of the following cases, find the equations with the roots as stated.

(i) $-\alpha, -\beta, -\gamma$. **1**

(ii) $\alpha, -\alpha, \beta, -\beta, \gamma, -\gamma$. **2**

(iii) $\alpha^2,\ \beta^2,\ \gamma^2$ **2**

(b) (i) Prove that for any polynomial $P(x)$, if k is a zero multiplicity 2, then k is also a zero of $P'(x)$. **2**

(ii) Show that $x = 1$ is a double root of the equation: **2**

$$x^{2n} - nx^{n+1} + nx^{n-1} - 1 = 0.$$

(iii) Find all the roots of $3x^3 - 26x^2 + 52x - 24 = 0$, given that the roots are in geometric progression. **3**

(c) On the same set of axes, sketch the graphs of **3**

$$y = \frac{1}{2}x + 1 \text{ and } y = |1 - x|.$$

Hence, or otherwise determine the values of x for which $\frac{1}{2}x + 1 \geq |1 - x|$.

Question 4 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Find:

(i) $\displaystyle\int \frac{e^x}{\left(1-e^x\right)^2}dx$ **2**

(ii) $\displaystyle\int_1^6 x\sqrt{6-x}\,dx$ **3**

(iii) Use the substitution $x = \sin^2\theta$ to evaluate $\displaystyle\int_0^{\frac{1}{2}} \frac{\sqrt{x}}{(1-x)^{3/2}}\,dx$ **4**

(b) (i) By considering the sum of the terms of an arithmetic series, show that **2**

$(1+2+3+....+n)^2 = \frac{1}{4}n^2(n+1)^2$

(ii) By principle of mathematical induction, prove that **4**

$$1^3+2^3+3^3+....+n^3 = (1+2+3+....+n)^2$$

Question 5 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) (i) Solve $\tan 4\theta = 1, \quad 0 \le \theta \le \pi.$ **2**

(ii) Prove that $\tan 4\theta = \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$. **3**

(iii) Using (i) and (ii) solve $z^4 + 4z^3 - 6z^2 - 4z + 1 = 0$ **3**

(iv) Hence, find:

(α) $\tan\dfrac{\pi}{16} + \tan\dfrac{5\pi}{16} + \tan\dfrac{9\pi}{16} + \tan\dfrac{13\pi}{16}$ **1**

(β) $\tan^2\dfrac{\pi}{16} + \tan^2\dfrac{5\pi}{16} + \tan^2\dfrac{9\pi}{16} + \tan^2\dfrac{13\pi}{16}$ **2**

(d) (i) If $2\sin 2x + \cos 2x = k$, show that: **2**

$(1+k)\tan^2 x - 4\tan x - 1 + k = 0.$

(ii) Also, show that if $\tan x_1$ and $\tan x_2$ are the roots of this quadratic equation in $\tan x$, then $\tan(x_1 + x_2) = 2$. **2**

Question 6 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) (i) Sketch the curve $y = 2x - \dfrac{1}{x-2}$ **2**

(ii) Evaluate $\displaystyle\int_{-1}^{1} 2x - \frac{1}{x-2}\, dx.$ **3**

Express your answer in the form *ln k*, where *k* is a real constant.

(b) (i) Explain why the parametric coordinates of the ellipse $4x^2 + 9y^2 = 36$ is $(3\cos\theta, 2\sin\theta)$ **1**

(ii) What are the parametric coordinates of the ellipse of the equation **2**

$$\frac{(x+1)^2}{4} + \frac{(y-3)^2}{2} = 1?$$

(c) The hyperbola $\mathcal{H}$ has Cartesian equation $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ and $P(x_1, y_1)$ is a point on $\mathcal{H}$.

The tangent to the hyperbola through *P* cuts the directrices at *T* and *T′*.

(i) Find the equation of the tangent to the hyperbola at the point *P*. **2**

(ii) Find the coordinates of *T*. **2**

(iii) Show that *PT* subtends a right angle at the focus *S*. **3**

Question 7 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) For the ellipse $9x^2 + 16y^2 = 144$, find the:

(i) eccentricity **2**

(ii) co-ordinates of the foci **1**

(iii) equations of the directrices. **1**

Hence,

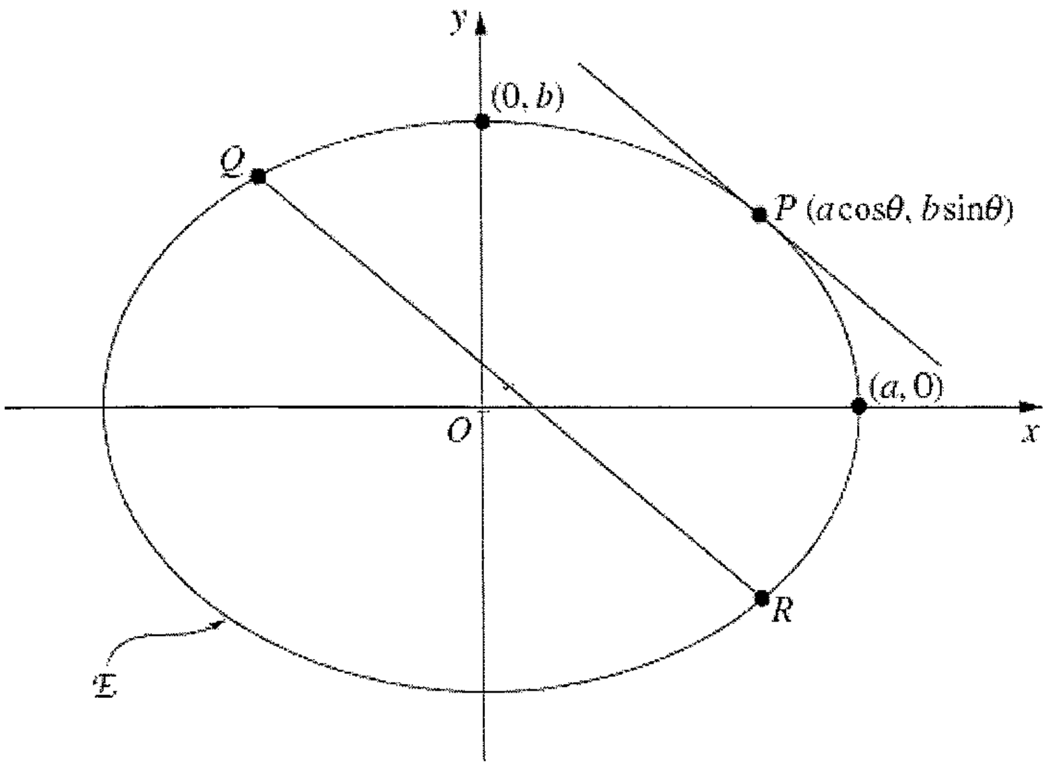
(iv) sketch the ellipse showing the vertices, foci and directrices. **2**

(v) Find the length of the latus rectum of the ellipse given above. **2**

Question 7 continues on page 9

Marks

(b)



Consider the ellipse $\mathcal{E}$ with equation $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, and the points $P(a\cos\theta, b\sin\theta)$ $Q(a\cos(\theta+\varphi), b\sin(\theta+\varphi))$ and $R(a\cos(\theta-\varphi), b\sin(\theta-\varphi))$ on $\mathcal{E}$.

(i) Show that the equation of the tangent to $\mathcal{E}$ at the point P is $\dfrac{x\cos\theta}{a}+\dfrac{y\sin\theta}{b}=1$. **2**

(ii) Show that the chord QR is parallel to the tangent at P. **2**

(iii) Deduce that OP bisects the chord QR. **3**

Question 8 (15 marks) Use a SEPARATE writing booklet **Marks**

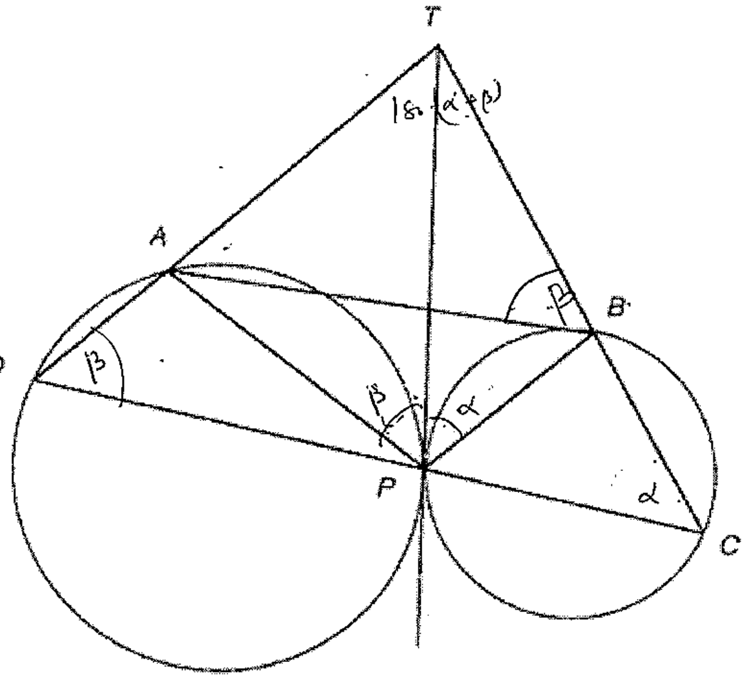
(a) The quadratic equation $x^2 - x + k = 0$, where k is a real number, has two distinct positive real roots α and β.

(i) Show that $0 < k < \frac{1}{4}$ **2**

(ii) Show that $\alpha^2 + \beta^2 = 1 - 2k$ and deduce that $\alpha^2 + \beta^2 > \frac{1}{2}$. **2**

(iii) Show that $\dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2} > 8$. **2**

(b)



Circles *APD* and *BPC* touch at *P*. *D*, *P* and *C* are collinear.
TP is the common tangent at *P*. *TC* cuts circle *BPC* in *B* while *TD* cuts circle *APD* in *A*.

(i) Copy the diagram.

(ii) Show that *ATBP* is a cyclic quadrilateral. **4**

(iii) Show that *ABCD* is a cyclic quadrilateral. **3**

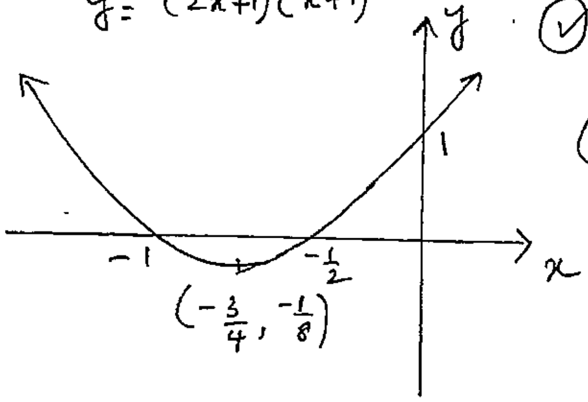
End of paper

KILLARA HIGH SCHOOL

Mathematics Extension 2 – Half-Yearly 2009

Question 1

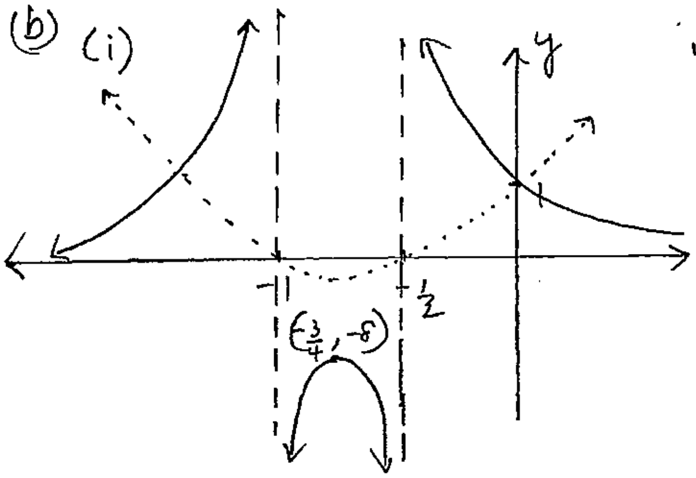
(a) $y = (2x+1)(x+1)$



✓ Shape & intercepts

✓ turning point

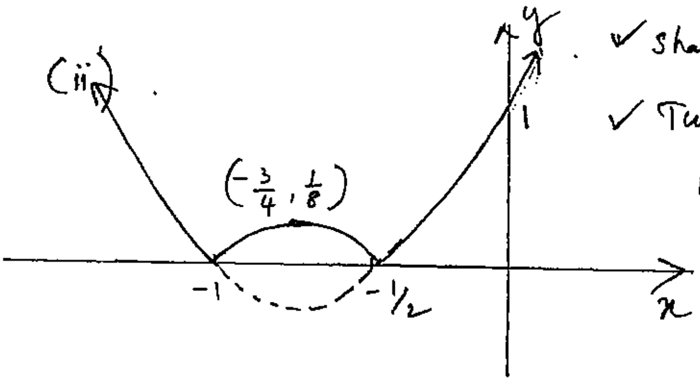
(b) (i)



✓ Asymptotes & shape.

✓ Turning point.

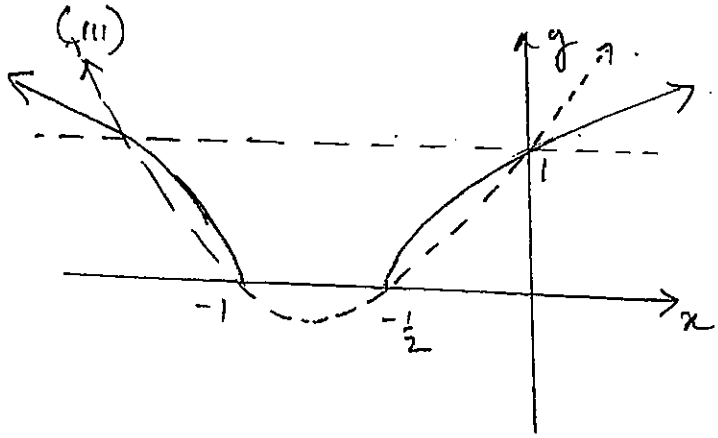
(ii)



✓ shape

✓ Turning point.

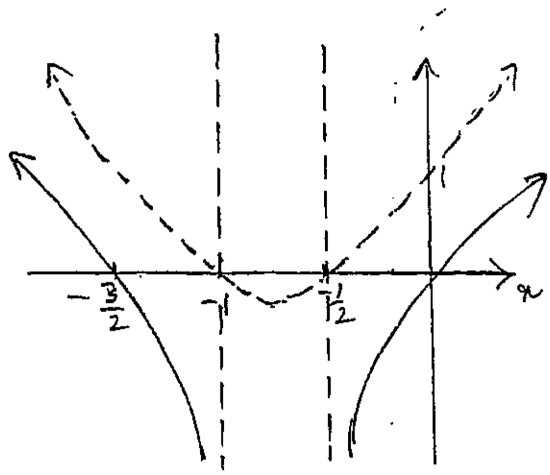
(iii)



✓ Shape above x-axis

✓ region below x-axis is excluded

(iv) $y = \ln\{(2x+1)(x+1)\}$



Asymptotes ✓

Intercepts, ✓ $-1 < x < -\frac{1}{2}$ ignored

shape ✓

(c) $y = \ln(2x+1)(x+1)$

$$= \ln(2x+1) + \ln(x+1) \checkmark$$

$$\frac{dy}{dx} = \frac{2}{2x+1} + \frac{1}{x+1}$$

$$= \frac{2(x+1) + (2x+1)}{(2x+1)(x+1)}$$

$$= \frac{4x+3}{(2x+1)(x+1)} \checkmark$$

$\frac{dy}{dx}$ at $x = 0$

is $\frac{3}{1} = 3$.

Equation of tangent at $(0,0)$ is $y = 3x$. ✓

Exactly one positive solution for $0 < k < 3$. ✓

Question 2.

(a) $-3+4i = (a+ib)^2$.

$a^2-b^2=-3$

$2ab=4$

$\therefore ab=2$ ✓

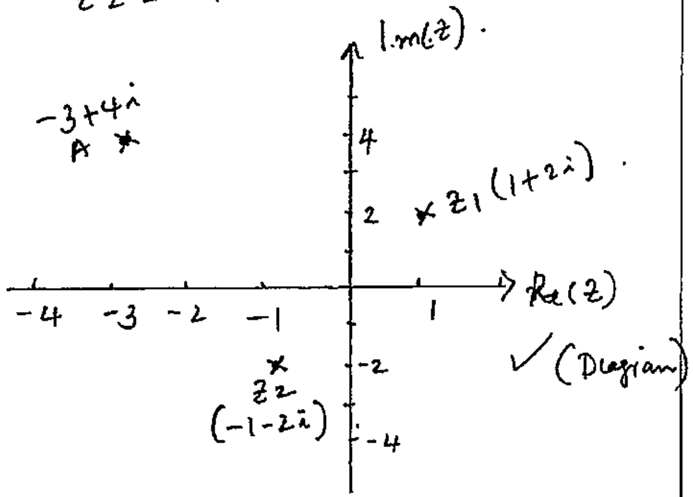
factors $2i, 1$.

$\therefore -3+4i = (1+2i)^2$.

$\therefore \sqrt{-3+4i} = \pm(1+2i)$.

$\therefore z_1 = 1+2i$

$z_2 = -1-2i$ ✓ (2)



gradient of $Az_2 = \frac{6}{-2} = -3$

grad. of $z_1z_2 = \frac{4}{2} = 2$.

grad. of $Az_1 = \frac{2}{-4} = -\frac{1}{2}$. (2)

$m_{z_1z_2} \times m_{Az_1} = 2 \times -\frac{1}{2} = -1$.

$\therefore \Delta Az_1z_2$ is right-angled. ✓ Proof

(b)(i) $|z-4| = |z+2i|$.

$|z-(4+0i)| = |z-(0-2i)|$.

The locus is the perpendicular bisector of the line joining $A(4,0)$ and $B(0,-2)$

$m_{AB} = \frac{2}{4} = \frac{1}{2}$. Mid-point $(2,-1)$.

equation of the perpendicular bisector is

$y+1 = -2(x+2)$. ✓ (1)

$2x+y-3=0$ is the locus of z.

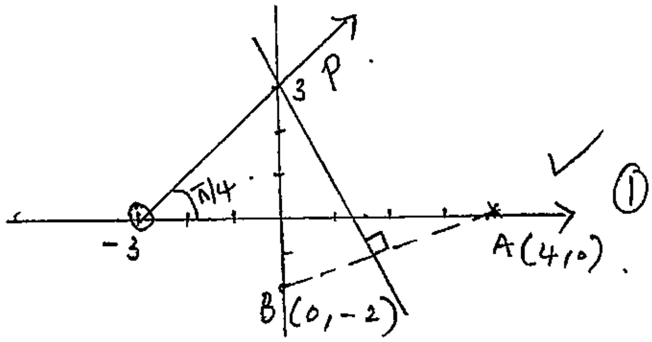
(ii) $\arg(z-(-3+0i)) = \frac{\pi}{4}$.

The line from P to $(-3,0)$ makes an angle of $\frac{\pi}{4}$. $\therefore m=1$.

$\therefore$ equation of the line is

$y = 1(x+3)$. (1)

$x-y+3=0$ ✓



✓ (1)

P lies both on $2x+y-3=0$ and $x-y+3=0$.

$\therefore P(0,3)$. ✓ (1)

P is complex number $3i$

(c). $P(z) = z^2 + (1-2i)z - (7+i) = 0$ has roots α and β.

$\therefore P(z+i)$ has roots $\alpha - i, \beta - i$.

$P(z+i) = (z+i)^2 + (1-2i)(z+i) - (7+i) = 0$.

$= z^2 + \cancel{2iz} - 1 + z \cancel{+i} - \cancel{2iz} + 2 - 7 - \cancel{i} = 0$.

$z^2 + z - 6 = 0$. 1

$\therefore P(z+i) = 0$ has roots $-3, 2$.

α and β are $-3+i$, $2+i$.

(d)

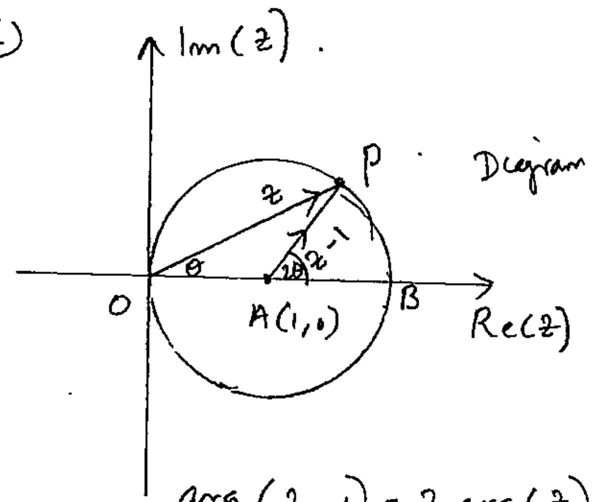


Diagram 1

$\arg(z-1) = 2\arg(z)$

Let $\arg(z) = \theta$ $\therefore \angle POA = \theta$

$\therefore \arg(z-1) = 2\arg(z) = 2\theta$.

ie $\angle PAB = 2\theta$.

$\therefore \angle OPA = \theta$. (exterior angle = sum of the opposite interior angles.

$\therefore \angle OPA = \angle POA = \theta$.

$\therefore AO = AP$ (Sides opposite equal angles are equal)

$\therefore \triangle OAP$ is isosceles. 2

(ii) $PA = 1$.

ie $|z-1| = 1$

$\therefore |z - (1+0i)| = 1$.

$\therefore P$ moves on a circle with centre $(1,0)$ and radius 1. 1

(iii) $|z| = |z-1|$

$\therefore \triangle OAP$ is equilateral with side 1 unit.

$\therefore |z| = 1$ and $\arg(z) = \frac{\pi}{3}$. 2

ie $z = 1\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$.

Question 3.

(a) (i) $x^3+2x-1=0$ has roots α, β, γ.

To find equation with roots $-\alpha, -\beta$ and $-\gamma$, replace x by $-x$.

$\therefore (-x)^3+2(-x)-1=0$ ✓ (1)

ie $x^3+2x+1=0$ has roots $-\alpha, -\beta, -\gamma$.

(ii) $x^3+2x-1=0$ has roots α, β, γ.
$x^3+2x+1=0$ has roots $-\alpha, -\beta, -\gamma$ ✓

$\therefore (x^3+2x-1)(x^3+2x+1)=0$ has roots $\pm\alpha, \pm\beta, \pm\gamma$.

$(x^3+2x)^2-1=0$. ✓ (2)

$x^6+4x^4+\underline{\underline{4x^2}}-1=0$.

(iii) To find equation with roots $\alpha^2, \beta^2, \gamma^2$, replace x by $\sqrt{x}$.

$(\sqrt{x})^3+2\sqrt{x}-1=0$.

$x^{3/2}+2x^{1/2}=1$.

$x^{\frac{1}{2}}(x+2)=1$.

$x+2=x^{-\frac{1}{2}}$.

Squaring,

$x^2+4x+4=\frac{1}{x}$.

$x^3+4x^2+4x-1=0$ is the required equation.

(b) (i) $P(x)=(x-k)^2Q(x)$. ✓

$P'(x)=2(x-k)Q(x)+(x-k)^2Q'(x)$

$P'(k)=2(k-k)Q(k)+(k-k)^2Q'(k)$

$=0$. ✓

ie. k is also a root of $P'(x)=0$.

(ii) Let $P(x)=x^{2n}-nx^{n+1}+nx^{n-1}-1$

$P(1)=1-n+n-1=0$.

$\Rightarrow x-1$ is a root of $P(x)=0$. ✓

$P'(x)=2nx^{2n-1}-n(n+1)x^n+n(n-1)x^{n-2}$

$P'(1)=2n-n(n+1)+n(n-1)$

$=2n-n^2-n+n^2-n$

$=0$. ✓

$\Rightarrow x=1$ is a root of $P'(x)=0$ and $x=1$ is a double root.

(iii) Let the roots be α, β, γ.

The roots are in G.P.

$\therefore \beta^2=\alpha\gamma$ (1)

$3x^3-26x^2+52x-24=0$.

$\alpha+\beta+\gamma=\frac{26}{3}$ (2).

$\alpha\beta+\beta\gamma+\alpha\gamma=\frac{52}{3}$ (3).

$\alpha\beta\gamma=8$ (4) ✓

From (i) and (4),

$\beta^3 = 8 \quad \therefore \beta = 2$ ✓

From (2) and (4)

$\alpha + \gamma = \frac{26}{3} - 2 = \frac{20}{3}$.

$\alpha\gamma = 4$.

$\therefore \alpha + \frac{4}{\alpha} = \frac{20}{3}$.

$3\alpha^2 - 20\alpha + 12 = 0$.

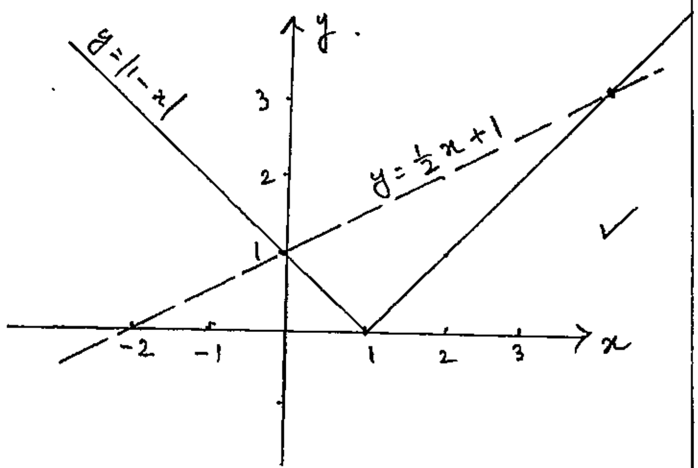
$(3\alpha - 2)(\alpha - 6) = 0$. $\boxed{3}$

$\alpha = \frac{2}{3}$ or $\alpha = 6$ ✓

$\gamma = 6 \qquad \gamma = \frac{2}{3}$.

$\therefore$ Roots are $\frac{2}{3}$, 2, and 6.

(c)



The graphs intersect when

$x - 1 = \frac{1}{2}x + 1$.

$\therefore x = 4$ ✓

$\boxed{3}$

From graph,

$\frac{1}{2}x + 1 \geq |1 - x|$ when

$0 \leq x \leq 4$. ✓

Question 4

(a) (i) $\int \frac{e^x}{(1-e^x)^2}\,dx$.

let $1 - e^x = u$.

$-e^x\,dx = du$ ✓

$\therefore -\int \frac{1}{u^2}\,du$.

$= -\frac{u^{-1}}{-1} + C$

$= \frac{1}{1-e^x} + C$ ✓ ②

(ii) $\int_1^6 x\sqrt{6-x}\,dx$.

$\left.\begin{array}{l} \text{let } 6 - x = u. \quad \therefore x = 6 - u \\ \therefore dx = -du. \\ x = 1, \quad u = 5 \\ x = 6, \quad u = 0. \quad ✓ \; ① \end{array}\right\}$

$\therefore -\int_5^0 (6-u)\sqrt{u}\,du$.

$\int_0^5 (6-u)\sqrt{u}\,du$.

$\int_0^5 6u^{\frac{1}{2}} - u^{\frac{3}{2}}\,du$. ✓ ①

$= \left[6 \cdot \frac{2}{3} u^{3/2} - \frac{2}{5} u^{5/2}\right]_0^5$

$= \left[4u\sqrt{u} - \frac{2}{5}u^2\sqrt{u}\right]_0^5$

$= \left(4 \times 5\sqrt{5} - \frac{2}{5} \times 25\sqrt{5}\right) - 0$

$= \underline{\underline{10\sqrt{5}}}$ ✓ ①

6

(iii) $\int_0^{1/2} \frac{\sqrt{x}}{(1-x)^{3/2}} dx$.

Let $x = \sin^2\theta$. $\therefore dx = 2\sin\theta\cos\theta\, d\theta$

$x=0,\ \theta = 0$.

$x = \frac{1}{2},\ \theta = \pi/4$. ✓ (1)

$\therefore \int_0^{\pi/4} \frac{\sin\theta}{(1-\sin^2\theta)^{\frac{3}{2}}} \cdot 2\sin\theta\cos\theta\, d\theta$.

$= \int_0^{\pi/4} \frac{\sin\theta}{\cos^3\theta} \cdot 2\sin\theta\cos\theta\, d\theta$.

$= 2\int_0^{\pi/4} \tan^2\theta \cdot d\theta$. ✓ (1)

$= 2\int_0^{\pi/4} (\sec^2\theta - 1)\, d\theta$. ✓ (1) $\boxed{4}$

$= 2\left[\tan\theta - \theta\right]_0^{\pi/4}$

$= 2\left(1 - \frac{\pi}{4}\right) - 0$. (1)

$= 2\left(1 - \frac{\pi}{4}\right)$ ✓

(b) (i) $1+2+3+\cdots+n$. A.P with

$a=1,\ d=1\ \ l=n$.

$\therefore$ Sum of series $= \frac{n}{2}(1+n)$ ✓

$\therefore (1+2+3+\cdots+n)^2$

$= \left[\frac{n}{2}(1+n)\right]^2$ (2)

$= \frac{1}{4}(n^2)(n+1)^2$ ✓

(ii) Let S_n be the sum of n terms in the series

$1^3+2^3+3^3+\cdots+n^3 = \frac{1}{4}(n^2)(1+n)^2$

Step 1 When $n=1$.

$S_1 = 1^3 = \frac{1}{4}\times 1^2(1+1)^2$

$= 1$

LHS = RHS. ✓

$\therefore S_1$ is true.

Step 2 Assume S_K is true.

ie $1^3+2^3+3^3+\cdots+K^3 = \frac{1}{4}K^2(K+1)^2$ ✓

Step 3 Prove S_{K+1} is true.

$1^3+2^3+3^3+\cdots+K^3+(K+1)^3 =$

$\frac{1}{4}K^2(K+1)^2 + (K+1)^3$

$= \frac{1}{4}(K+1)^2\left[K^2+4(K+1)\right]$

$= \frac{1}{4}(K+1)^2\left[K^2+4K+4\right]$ $\boxed{4}$

$= \frac{1}{4}(K+1)^2(K+2)^2$ ✓

$\therefore S_{K+1}$ is true, if S_K is true.

In step 1, S_1 is true.

$\therefore$ By principle of mathematical induction, S_n is true for $n = 2, 3, \ldots$ all integral values of n. Using result (i). ✓

$1^3+2^3+3^3+\cdots+n^3 = (1+2+3+\cdots+n)^2$

Question 5

(a) (i) $\tan 4\theta = 1$.

S	A ✓
✓ T	C

$\therefore 4\theta = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}$ ✓ $\boxed{2}$

$\therefore \theta = \frac{\pi}{16}, \frac{5\pi}{16}, \frac{9\pi}{16}, \frac{13\pi}{16}$ ✓

(ii) Consider $(\cos\theta + i\sin\theta)^4$

Let $c = \cos\theta$, $s = \sin\theta$.

By De-Moivre theorem,

$\cos 4\theta + i\sin 4\theta = (\cos\theta + i\sin\theta)^4$

1 4 −6 −4 1

$\cos 4\theta + i\sin 4\theta = (c + is)^4$

$= c^4 + 4c^3si - 6c^2s^2 - 4cs^3i + s^4$ ✓

$\therefore \cos 4\theta = c^4 - 6c^2s^2 + s^4$ ✓

$\sin 4\theta = 4c^3s - 4cs^3$

$\therefore \tan 4\theta = \frac{\sin 4\theta}{\cos 4\theta}$

$= \frac{4c^3s - 4cs^3}{c^4 - 6c^2s^2 + s^4}$ $\boxed{3}$

Dividing by c^4 on RHS

$\tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$ ✓

(iii) $z^4 + 4z^3 - 6z^2 - 4z + 1 = 0$.

from (i), $\tan 4\theta = 1$.

Using (ii),

$\therefore \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta} = 1$. ✓

Let $z = \tan\theta$.

$\therefore 4z - 4z^3 = 1 - 6z^2 + z^4$

$\therefore z^4 - 4z^3 - 6z^2 - 4z + 1 = 0$.

Using (i),

$\theta = \frac{\pi}{16}, \frac{5\pi}{6}, \frac{9\pi}{16}, \frac{13\pi}{16}$ ✓ $\boxed{3}$

$\therefore z = \tan\theta =$ ✓

$\tan\frac{\pi}{16}, \tan\frac{5\pi}{16}, \tan\frac{9\pi}{16}, \tan\frac{13\pi}{16}$.

(iv) ~~Sum of~~

(α) $z^4 + 4z^3 - 6z^2 - 4z + 1 = 0$.

Sum of roots $= \frac{-b}{a} = -4$

$\therefore \tan\frac{\pi}{16} + \tan\frac{5\pi}{16} + \tan\frac{9\pi}{16} + \tan\frac{13\pi}{16}$ ✓

$= \underline{\underline{-4}}$. $\boxed{1}$

(β) $(\alpha + \beta + \gamma + \delta)^2 = \Sigma\alpha^2 + 2\Sigma\alpha\beta$ ✓

$(-4)^2 = \Sigma\alpha^2 + 2 \times -6$. $\boxed{2}$

$\therefore \Sigma\alpha^2 = 16 + 12 = 28$ ✓

$\therefore \tan^2\frac{\pi}{16} + \tan^2\frac{5\pi}{16} + \tan^2\frac{9\pi}{16} + \tan^2\frac{13\pi}{16}$

$= \underline{\underline{2}}$.

(b) Let $t = \tan x$.

(i) Then $2\sin 2x + \cos 2x = k$

$$2 \times \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2} = k$$ ✓

$$4t + 1 - t^2 = k(1+t^2)$$

$$4t + 1 - t^2 = k + kt^2$$

$$t^2(k+1) - 4t + k - 1 = 0$$ ✓

ie $(k+1)\tan^2 x - 4\tan x + k - 1 = 0$

(ii) $\tan x_1$ and $\tan x_2$ are roots of this equation.

$\therefore$ Sum of roots, product of roots $\boxed{4}$

$$\left.\begin{aligned} \tan x_1 + \tan x_2 &= \frac{4}{k+1} \\ \tan x_1 \cdot \tan x_2 &= \frac{k-1}{k+1} \end{aligned}\right\}$$ ✓

$$\tan(x_1 + x_2) = \frac{\tan x_1 + \tan x_2}{1 - \tan x_1 \cdot \tan x_2}$$

$$= \frac{\frac{4}{k+1}}{1 - \frac{k-1}{k+1}} = \frac{4}{\cancel{k}+1-\cancel{k}+1}$$

$$= \frac{4}{2} = \underline{\underline{2}}.$$ ✓

Question 6.

(a) (i) $y = 2x - \frac{1}{x-2}$

$\therefore$ asymptotes $x = 2$.

Oblique asymptote $y = 2x$.

y-intercept

Let $x = 0$. $\therefore y = \frac{1}{2}$.

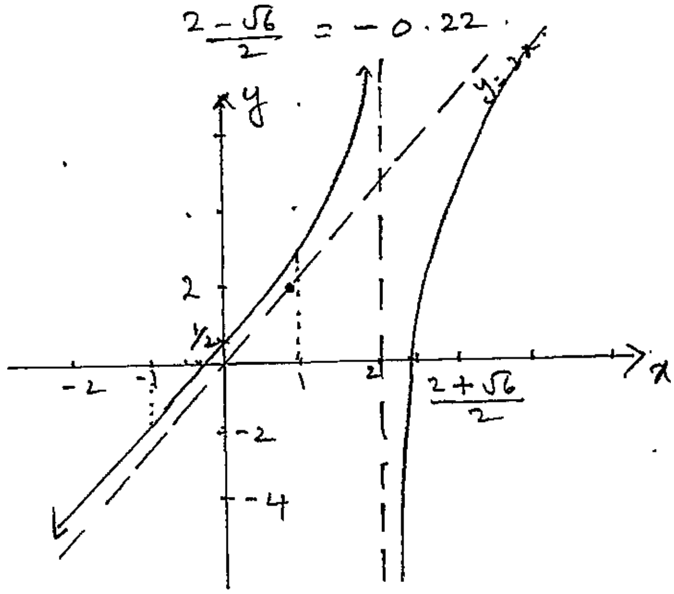
x-intercepts Let $y = 0$.

$$\therefore 2x = \frac{1}{x-2}$$

$$2x^2 - 4x - 1 = 0.$$

$$x = \frac{4 \pm \sqrt{24}}{4} = \frac{2+\sqrt{6}}{2} \doteqdot 2.22$$

$$\frac{2-\sqrt{6}}{2} = -0.22.$$

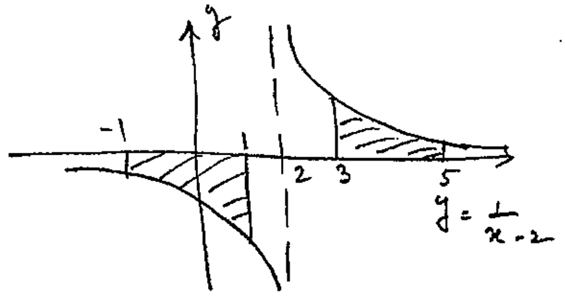


✓ Intercepts, asymptotes.

✓ Shape. $\boxed{2}$

(ii) $$\int_{-1}^{1} 2x - \frac{1}{x-2}\,dx$$

$$= \int_{-1}^{1} 2x\,dx - \int_{-1}^{1} \frac{1}{x-2}\,dx.$$ ✓



$$= \left[x^2\right]_{-1}^{1} - \left[\ln(x-2)\right]_{-1}^{1}$$

When $x = 1$ or -1,

$x - 2$ is negative.

$\therefore \log_e (x-2)$ doesn't exist.

But from the graph of $y = \frac{1}{x-2}$, it can be observed that

$\int_{-1}^{1} \frac{1}{x-2}\,dx$ is symmetrical to $\int_{3}^{5} \frac{1}{x-2}\,dx$.

$\therefore$

$$\left[x^2\right]_{-1}^{1} - \left[\ln(x-2)\right]_{3}^{5}$$

$$= 0 - \left[\ln(3) - \ln(1)\right]$$

$$= 0 - \ln 3 + 0 = -\ln 3.$$

But $\int_{-1}^{1} \frac{1}{x-2}\,dx$ is below x-axis. $\therefore -(-\ln 3) = \ln 3$.

$$\therefore \int_{-1}^{1} 2x - \frac{1}{x-2}\,dx = \underline{\underline{\ln 3}}$$

$\therefore \underline{\underline{K = 3}}$. $\boxed{3}$

OR

$$\int_{-1}^{1} 2x - \frac{1}{x-2}\,dx$$

$$= \left[x^2 - \ln|x-2|\right]_{-1}^{1}$$

$$= \left(1 - \ln|-1|\right) - \left(1 - \ln|-3|\right)$$

$$= \cancel{1} - 0 - \cancel{1} + \ln 3.$$

$$= \ln 3.$$

$\therefore K = 3$

(b) (i) $4x^2 + 9y^2 = 36$.

$x = 3\cos\theta$, $y = 2\sin\theta$.

$$4(3\cos\theta)^2 + 9(2\sin\theta)^2$$

$$= 4 \times 9\cos^2\theta + 9 \times 4\sin^2\theta$$

$$= 36(\cos^2\theta + \sin^2\theta) = 36 = RHS.$$

ie $(3\cos\theta, 2\sin\theta)$ satisfies $4x^2 + 9y^2 = 36$.

$\therefore (3\cos\theta, 2\sin\theta)$ is the parametric representation of any point on $4x^2 + 9y^2 = 36$ $\boxed{1}$

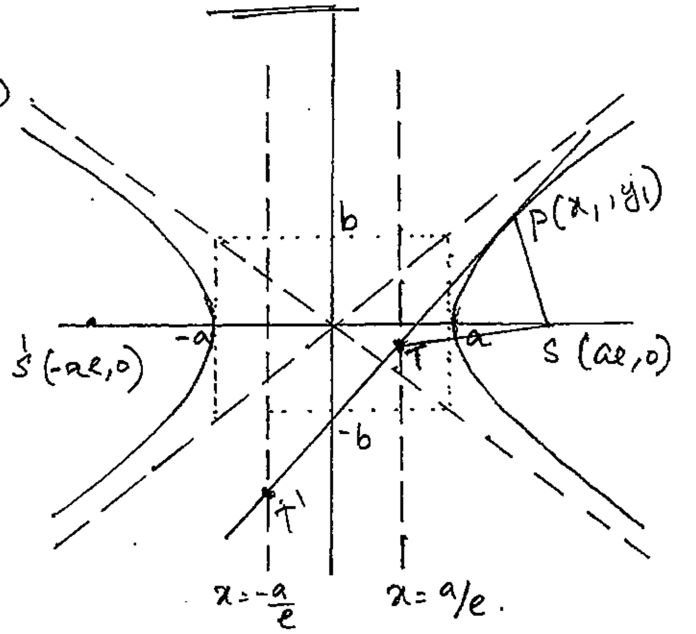
(ii) $\frac{(x+1)^2}{4} + \frac{(y-3)^2}{2} = 1$.

$$\left.\begin{array}{l} a = 2 \quad b = \sqrt{2} \\ x + 1 = 2\cos\theta \\ y - 3 = \sqrt{2}\sin\theta \end{array}\right\} \begin{array}{l} \therefore x = 2\cos\theta - 1. \\ \therefore y = 3 + \sqrt{2}\sin\theta. \end{array}$$

$\therefore$ Parametric coordinates are $\boxed{2}$

$$\left(2\cos\theta - 1,\ 3 + \sqrt{2}\sin\theta\right).$$

(c)



(i) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$\frac{2x}{a^2} - \frac{2yy'}{b^2} = 0.$

$\frac{2yy'}{b^2} = \frac{2x}{a^2}$

$\therefore y' = \frac{2x}{a^2} \cdot \frac{b^2}{2y}$

$= \frac{b^2}{a^2} \frac{x}{y}.$

y' at $P(x_1, y_1)$ is

$\frac{b^2}{a^2} \frac{x_1}{y_1}$ ✓

Equation of the tangent is

$y - y_1 = \frac{b^2}{a^2} \frac{x_1}{y_1} (x - x_1)$

$a^2 y y_1 - a^2 y_1^2 = b^2 x x_1 - b^2 x_1^2$

$b^2 x x_1 - a^2 y y_1 = b^2 x_1^2 - a^2 y_1^2$

Divide by $a^2 b^2$

$\therefore \frac{x x_1}{a^2} - \frac{y y_1}{b^2} = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2}$

(x_1, y_1) is a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$

$\therefore \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1.$

$\therefore$ Equation of the tangent at $P(x_1, y_1)$ is ✓ $\boxed{2}$

$\frac{x x_1}{a^2} - \frac{y y_1}{b^2} = 1.$

(ii) Solve simultaneously equation of tangent and the directrix $x = \frac{a}{e}$ to get coordinates of T.

$\frac{x x_1}{a^2} - \frac{y y_1}{b^2} = 1$

$x = \frac{a}{e}.$

$\frac{a x_1}{e a^2} - \frac{y y_1}{b^2} = 1.$

$\frac{y y_1}{b^2} = \frac{x_1}{ae} - 1 = \frac{x_1 - ae}{ae}$ ✓

$\therefore y = \frac{b^2 (x_1 - ae)}{y_1 ae}.$ ✓

$\therefore T \left[\frac{a}{e}, \frac{b^2 (x_1 - ae)}{a e y_1} \right].$

(iii) $m_{TS} = \frac{\frac{b^2(x_1 - ae)}{a e y_1} - 0}{\frac{a}{e} - ae}.$

$= \frac{\frac{b^2 (x_1 - ae)}{\cancel{a} e y_1}}{\frac{a - ae^2}{\cancel{e}}}$

$= \frac{b^2 (x_1 - ae)}{a y_1 \times a(1 - e^2)}$

$= \frac{b^2 (x_1 - ae)}{a^2 y_1 (1 - e^2)}$ ✓

$$m_{PS} = \frac{y_1 - 0}{x_1 - ae}$$

$$m_{TS} \times m_{PS} = \frac{b^2 \cancel{(x_1 - ae)}}{a^2 \cancel{y_1}(1-e^2)} \times \frac{\cancel{y_1}}{\cancel{(x_1-ae)}}$$

$$= \frac{b^2}{a^2(1-e^2)} = \frac{b^2}{-a^2(e^2-1)} \quad ✓$$

But $a^2(e^2-1) = b^2$, $e > 1$

for a hyperbola.

$$\therefore m_{TS} \times m_{PS} = \frac{b^2}{-b^2} = -1 .$$

$\boxed{3}$

ie $TS \perp PS$. ✓

ie PT subtends a right-angle at the focus.

Question 7

(a) $9x^2 + 16y^2 = 144$.

(i) $\frac{x^2}{16} + \frac{y^2}{9} = 1$.

$a = 4 \quad b = 3$. ✓

$\therefore \; b^2 = a^2(1-e^2)$.

$9 = 16(1-e^2)$.

$1 - e^2 = \frac{9}{16}$. (2)

$e^2 = 1 - \frac{9}{16} = \frac{7}{16}$

$\therefore e = \frac{\sqrt{7}}{4}$. ✓

(ii) foci are $(\pm ae, 0)$.

$(\pm 4 \times \frac{\sqrt{7}}{4}, 0)$ ✓ (1)

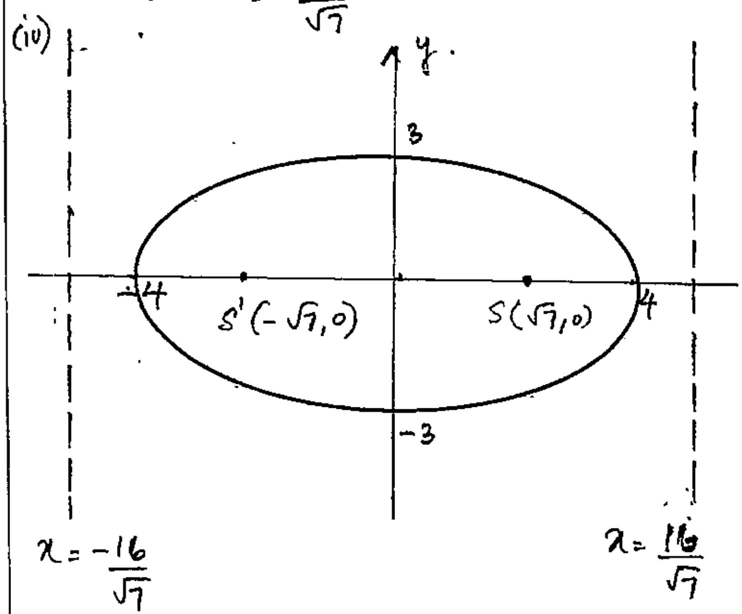
$S(\sqrt{7}, 0) \quad S'(-\sqrt{7}, 0)$.

(iii) Directrices are

$x = \pm \frac{a}{e} = \pm \frac{4}{\sqrt{7}/4}$

$x = \pm \frac{16}{\sqrt{7}}$. ✓ (1)

(iv)

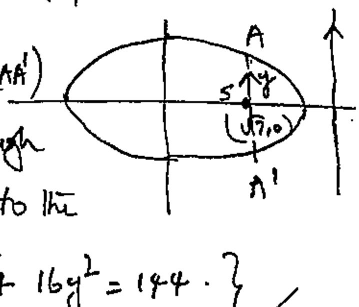


(2) ✓ foci and vertices

✓ ... directrices shown and labelled.

(v) $9x^2 + 16y^2 = 144$

The latus rectum (AA') is the chord through the focus parallel to the directrix.



$\left.\begin{array}{l} 9x^2 + 16y^2 = 144. \\ x = \sqrt{7} \end{array}\right\}$ ✓.

$\boxed{2}$.

$63 + 16y^2 = 144$.

$16y^2 = 81$.

$y = \frac{9}{4}$

$\therefore AA' = 2 \times \frac{9}{4}$ ✓

$= \underline{\underline{4.5 \text{ units}}}$.

(b)

(i) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$

$x = a\cos\theta \qquad \frac{dx}{d\theta} = -a\sin\theta.$

$y = b\sin\theta. \qquad \frac{dy}{d\theta} = b\cos\theta.$

$\therefore \frac{dy}{dx} = -\frac{b}{a}\frac{\cos\theta}{\sin\theta}$ —— (1)

$\frac{dy}{dx}$ at

$\therefore$ Equation of the tangent is

$y - b\sin\theta = \frac{-b}{a}\frac{\cos\theta}{\sin\theta}(x - a\cos\theta).$

$ay\sin\theta - ab\sin^2\theta = -bx\cos\theta + ab\cos^2\theta.$

$bx\cos\theta + ay\sin\theta = ab(\sin^2\theta + \cos^2\theta)$

$bx\cos\theta + ay\sin\theta = ab \times 1.$

Divide by ab, $\boxed{2}$

$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$

(ii) gradient of QR

$= \frac{b\sin(\theta+\phi) - b\sin(\theta-\phi)}{a\cos(\theta+\phi) - a\cos(\theta-\phi)}.$

$= \frac{b\,[\sin(\theta+\phi) - \sin(\theta-\phi)]}{a[\cos(\theta+\phi) - \cos(\theta-\phi)]}$

$= \frac{b \cdot 2\cos\frac{\theta+\phi+\theta-\phi}{2}\sin\frac{\theta+\phi-\theta+\phi}{2}}{a \cdot -2\sin\frac{\theta+\phi+\theta-\phi}{2}\sin\frac{\theta+\phi-\theta+\phi}{2}}$

$= -\frac{b}{a} \cdot \frac{\cos\theta\sin\phi}{\sin\theta\sin\phi}$

$= -\frac{b}{a}\frac{\cos\theta}{\sin\theta}.$ $\boxed{2}$

Using (1) from (b)(i),

$m_{QR} = m_{\text{tangent at } P}.$

$\therefore$ $QR \parallel$ tangent at P.

(iii) Let M be the mid-point of QR.

$M\left\{ \frac{a[\cos(\theta+\phi) + \cos(\theta-\phi)]}{2}, \frac{b[\sin(\theta+\phi) + \sin(\theta-\phi)]}{2} \right\}$

$= M\left\{ \frac{a \cdot 2\cos\theta \cdot \cos\phi}{2}, \frac{b \cdot 2\sin\theta\cos\phi}{2} \right\}$

$= M\{a\cos\theta\cos\phi,\ b\sin\theta\cos\phi\}$

equation of OP is

$y = \frac{b}{a}\frac{\sin\theta}{\cos\theta}x.$

prove that M satisfies equation of OP.

$b\sin\theta\cos\phi = \frac{b}{\cancel{a}}\frac{\sin\theta}{\cancel{\cos\theta}} \cdot \cancel{a\cos\theta}\cos\phi$

$= b\sin\theta\cos\phi$ TRUE.

ie OP bisects the chord QR. $\boxed{3}$

Question 8.

(a) $x^2 - x + k = 0$.

Distinct roots α, β. both positive

(i) $\left.\begin{array}{l}\alpha\beta > 0 \\ \alpha\beta = k\end{array}\right\} \therefore k > 0$ — ①

$\Delta > 0$.

$(-1)^2 - 4k > 0$

$1 - 4k > 0 \quad \therefore 4k < 1 \Rightarrow k < \frac{1}{4}$ — ②

from ① & ②,

$0 < k < \frac{1}{4}$. $\boxed{2}$

(ii) $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$.

$= 1^2 - 2 \times k = 1 - 2k$.

Using ② from part (i),

$\alpha^2 + \beta^2 > 1 - 2 \times \frac{1}{4}$

$\therefore \alpha^2 + \beta^2 > \frac{1}{2}$. $\boxed{2}$

(iii) $\frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{\alpha^2+\beta^2}{(\alpha\beta)^2} = \frac{1-2k}{k^2}$

$> \frac{1 - 2 \times \frac{1}{4}}{\left(\frac{1}{4}\right)^2} = 8$.

$\therefore \frac{1}{\alpha^2} + \frac{1}{\beta^2} > 8$.

(iv) $\alpha^3 + \beta^3 = (\alpha+\beta)(\alpha^2 - \alpha\beta + \beta^2)$

$= 1 \times (1 - 2k - k)$

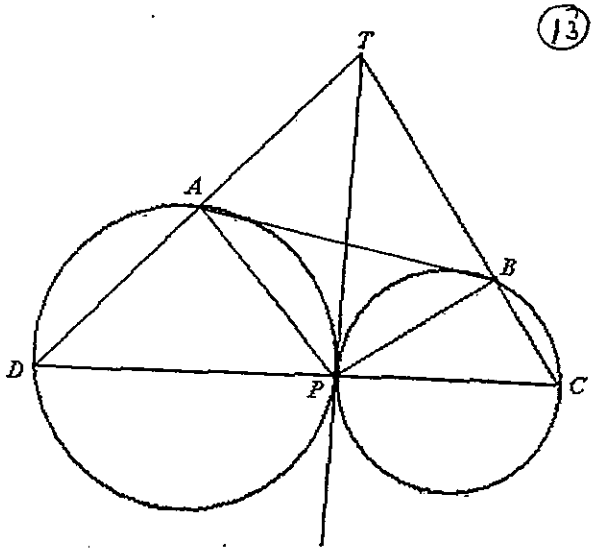
$= 1 - 3k$.

$> 1 - 3 \times \frac{1}{4}$

$> \frac{1}{4}$.

(b)

(i)



(ii) $\left.\begin{array}{l}\angle TPB = \angle BCP \\ \angle TPA = \angle ADP\end{array}\right\}$ Angle between tangent and chord equals angle in the alternate segment. — ①

$\angle ATB + \angle ADP + \angle BCP = 180°$ ($\angle$ sum of ΔDTC)

$\therefore \angle ATB + \angle TPA + \angle TPB = 180°$ using ①

$\therefore \angle ATB + \angle APB = 180°$ ($\angle APB = \angle TPA + \angle TPB$)

$\therefore$ ATBP is a cyclic quadrilateral.

(one pair of opposite $\angle$'s are supplementary). $\boxed{4}$

(iii) $\angle TBA = \angle TPA$ ($\angle$'s at circumference standing on the $\overset{\frown}{AT}$ of circle ATBP.)

$\therefore \angle TBA = \angle ADP$ ($\angle TPA = \angle ADP$ proved in (ii))

$\therefore$ ABCD is cyclic (exterior angle equal to opposite interior angle). $\boxed{3}$

—— END ——