Killara HIGH SCHOOL

STUDENT NUMBER

MATHEMATICS EXTENSION 2

ASSESSMENT TASK 1

Monday 6th December, 2010

TIME ALLOWED: *Reading time: 5 minutes*
Working time: 60 minutes

INSTRUCTIONS: This paper consists of 2 questions
ANSWER all two questions
Start each question in a new booklet
Marks may not be given for untidy and poorly arranged work.
All relevant working should be shown.

THIS QUESTION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM

OUTCOMES ASSESSED:

- ✓ **E1 Appreciates the creativity, power and usefulness of mathematics to solve a broad range of problems**
- ✓ **E3 Uses the relationship between algebraic and geometric representations of complex numbers and of conic sections.**
- ✓ **E9 Communicates abstract ideas and relationships using appropriate notation and logical argument.**

	MARKS
QUESTION 1 Complex Numbers	**/15**
QUESTION 2 Complex Numbers	**/17**
	/32

This assessment task constitutes 10% of the Higher School Certificate Course Assessment.

Total marks - 32
Attempt questions 1 – 2

Answer each question on a separate writing booklet.

Question 1 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Given $z_1 = 3 - 2i$, $z_2 = 1 + 3i$. Find the value of

(i) $z_1 + z_2$ **1**

(ii) z_1^2 **1**

(iii) $\frac{z_1}{z_2}$. Leave your answer in Cartesian form. **2**

(b) $z = 1 - i$.

(i) Express z in modulus-argument form. **2**

(ii) Hence, determine the smallest value of n for which z^n is a purely imaginary number, where n is a positive integer. **2**

(c) Find the square root of $-4 + 3i$ **3**

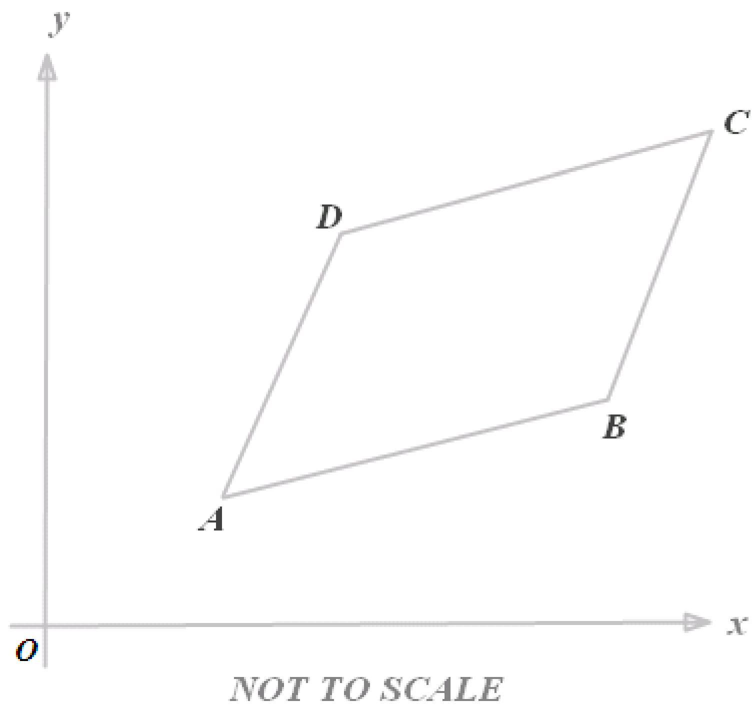
(d) Prove that for all complex numbers z and w, **4**

$$|z+w|^2 + |z-w|^2 = 2\left(|z|^2 + |w|^2\right).$$

Give a geometrical interpretation of this equation in the complex plane.
(*Use a diagram to aid your explanation*)

Question 2 (17 marks) Use a SEPARATE writing booklet **Marks**

(a) A and B are points in the Argand diagram representing the numbers $z_1 = 3 + i,\ z_2 = 3 + 2\sqrt{3} + 3i$ respectively.



(i) Find the complex number that represents the vector $\overrightarrow{AB}$. **1**

(ii) $ABCD$ is a parallelogram where $AB = 2AD$ and $\angle DAB = \dfrac{\pi}{6}$. **3**

Find the complex number that represents D.

Question 2 continues.....

Question 2 Cotinued..... **Marks**

(b) $Arg\left(z-2\right) = 2Arg\left(z\right)$

(i) Show vectors representing z, $z-2$ in an Argand diagram. **1**

(ii) If the point P represents z, O is the origin and A has coordinates (2, 0) **1**

in this Argand diagram, what is the nature of ΔOAP for non-real z?

(iii) Deduce that if z is non-real, then P lies on a circle and state its centre and radius. **2**

On a new diagram, sketch the locus in the Argand diagram of a point representing z satisfying $\text{Arg}\left(z-2\right) = 2\text{Arg}\left(z\right)$,
for both real and non-real z.

(c) (i) Use De Moivre's theorem to show that $\left(\cot\theta+i\right)^n+\left(\cot\theta-i\right)^n=\dfrac{2\cos n\theta}{\sin^n\theta}$. **1**

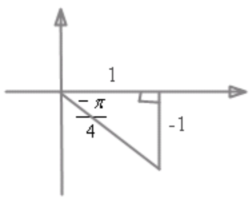
(ii) Show that the equation $\left(x+i\right)^5+\left(x-i\right)^5=0$ has the roots 0, $\pm cot\dfrac{\pi}{10}$, $\pm cot\dfrac{3\pi}{10}$. **3**

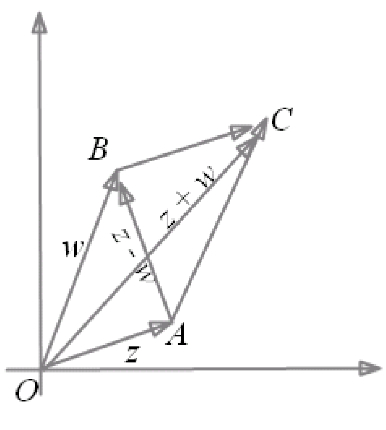
(iii) Hence, show that the equation $x^4-10x^2+5=0$ has roots $\pm cot\dfrac{\pi}{10}$, $\pm cot\dfrac{3\pi}{10}$. **3**

(iv) Hence show that $cot\dfrac{\pi}{10}=\sqrt{5+2\sqrt{5}}$. **2**

End of paper

Marking Scheme : Task 1 2011

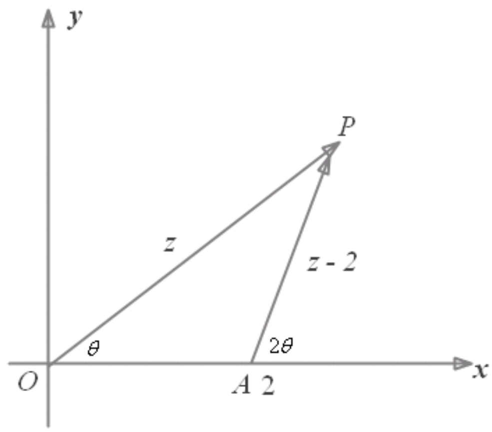
(a)(i)	$z_1 = 3 - 2i,\ \ z_2 = 1 + 3i.$ $z_1 + z_2 = 3 - 2i + 1 + 3i = 4 + i$	1 mark	*Done very well*
(ii)	$z_1^{\ 2} = (3-2i)^2$ $= 9 - 12i - 4$ $= 5 - 12i.$	1 mark for correct solution	*Done very well*
(iii)	$\dfrac{z_1}{\overline{z_2}} = \dfrac{3-2i}{1-3i}$ $= \dfrac{3-2i}{1-3i} \times \dfrac{1+3i}{1+3i}$ $= \dfrac{3+9i-2i+6}{1+9}$ $= \dfrac{9+7i}{10} = \dfrac{9}{10} + \dfrac{7i}{10}$	1 mark for $\overline{z_2}$ 1 mark for correct answer.	*Some students did not notice the bar on z_2.*
b)	$z = 1 - i.$ $= \sqrt{2}\ cis\ \dfrac{-\pi}{4}$  $z^n = \sqrt{2}^{\,n}\ cis\ \dfrac{-n\pi}{4}$ If z^n is purely imaginary number, then $Re\left(z^n\right) = 0$ $\text{Cos}\ \dfrac{-n\pi}{4} = 0$ ie. $\text{Cos}\ \dfrac{n\pi}{4} = 0$ $\therefore\ \dfrac{n\pi}{4} = \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{5\pi}{2}, \ldots\ldots$ $\therefore\ n = 2, 6, 10, \ldots..$ The smallest possible value of $n = 2$. Or $\arg\left(z^n\right) = \dfrac{k\pi}{2}$ where $k = 0, \pm 1, \pm 2 \ldots$ $\dfrac{-n\pi}{4} = \dfrac{k\pi}{2}$ $n = -2k$ $k = -1, \quad n = 2$	1 mark for modulus 1 mark for argument 1 mark for setting $\text{Cos}\ \dfrac{-n\pi}{4} = 0$ 1 mark for the correct answer	*The general solution of this question was incorrect some students. It was an obvious answer and therefore some students resorted to fudging the solution. All working must be shown.* *Substituting n = 1, 2, 3... is not a good strategy.*

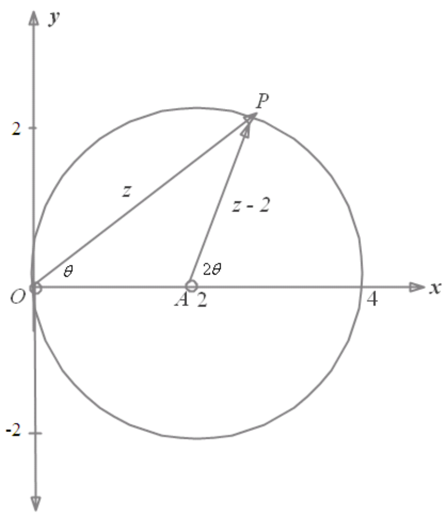
c)	Let $\sqrt{-4+3i} = x+iy$ $\therefore x^2-y^2=-4$ (1) $2xy=3 \Rightarrow xy=\frac{3}{2}$ (2) From (1), $x^4-x^2y^2=-4x^2$ Ie. $x^4-\frac{9}{4}=-4x^2$ (using (2)) $4x^4+16x^2-9=0$ $(2x^2+9)(2x^2-1)=0$ $\therefore x^2=-\frac{9}{2}$ (no solution) $x^2=\frac{1}{2}$ $\therefore x=\pm\frac{1}{\sqrt{2}}$ $\therefore y=\pm\frac{3}{\sqrt{2}}$ $\sqrt{-4+3i}=\pm\frac{1}{\sqrt{2}}(1+3i)$	1 mark for equations (1) and (2). 1 mark for the correct value of x. 1 mark	*Many students made the algebra of this too difficult due to silly errors and consequently got the incorrect answer*
d)	$\lvert z+w\rvert^2+\lvert z-w\rvert^2$ $=(z+w)\times(\bar{z}+\bar{w})+(z-w)\times(\bar{z}-\bar{w})$ $= z\bar{z}+z\bar{w}+w\bar{z}+w\bar{w}+$ $z\bar{z}-z\bar{w}-w\bar{z}+w\bar{w}$ $=2z\bar{z}+2w\bar{w}$ $=2(\lvert z\rvert^2)+2(\lvert w\rvert^2)$ $=2(\lvert z\rvert^2+\lvert w\rvert^2)$.  As given in the diagram, $z+w$ and $z-w$ represent the two diagonals of the parallelogram and z and w are the lengths of the two sides. $OA=BC=z$ and $OB=AC=w$. $\lvert z+w\rvert^2+\lvert z-w\rvert^2=2(\lvert z\rvert^2+\lvert w\rvert^2)$ means $OC^2+AB^2=OA^2+AC^2+OB^2+BC^2$	1 mark 1 mark 1 mark for the diagram and explaining $2(\lvert z\rvert^2+\lvert w\rvert^2)$ as the sum of squares of the sides.	Few students tried the algebraic method which is slightly more difficult than the method shown here. Diagram was shown by most students, but some had the arrows missing. Direction is essential when showing vectors. *Students must make the connections $OA=BC=z$ and $OB=AC=w$ before writing the final conclusion. Very poorly done.*

	$\therefore$ The relationship means that in a parallelogram, the sum squares of the two diagonals is equal to the sum of squares of its sides.	1 mark for the geometrical interpretation.	

Question 2

a) (i)	$z_1 = 3 + i,\quad z_2 = 3 + 2\sqrt{3} + 3i$ $\overrightarrow{AB} = z_2 - z_1$ $= 3 + 2\sqrt{3} + 3i - 3 + i$ $= 2\sqrt{3} + 2i$	1 mark	*Very poorly done by many students. When doing this question one must understand the transformation that is happening: Scale*
(ii)	$\overrightarrow{AB} = 2\sqrt{3} + 2i$ $\overrightarrow{AD} = \frac{1}{2}\overrightarrow{AB} \times cis\left(\frac{\pi}{6}\right)$ $= \frac{1}{2}(2\sqrt{3} + 2i) \times \left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$ $= \frac{1}{2}\left(\sqrt{3} + i\right)^2 = \frac{1}{2}\left(3 + 2\sqrt{3}i - 1\right)$ $= 1 + \sqrt{3}\,i$ $\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA}$ $\therefore \overrightarrow{OD} = \overrightarrow{AD} + \overrightarrow{OA}$ $= 1 + \sqrt{3}\,i + 3 + i$ $= 4 + \left(\sqrt{3} + 1\right)i$ $\therefore$ *D* represents the complex number $4 + \left(\sqrt{3} + 1\right)i$	1 mark for $\overrightarrow{AD} = \frac{1}{2}\overrightarrow{AB} \times cis\left(\frac{\pi}{6}\right)$ 1 mark 1 mark	*down $\overrightarrow{AB}$ to ½ the length and then rotate through $\left(\frac{\pi}{6}\right)$ in the anticlockwise direction (means multiply by $cis\left(\frac{\pi}{6}\right)$ which $= \left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$.* *Some students misunderstood "the complex number representing D" as $\overrightarrow{AD}$ rather than $\overrightarrow{OD}$.*

b)		1 mark for the diagram Vectors OP and AP represents z and $z - 2$ respectively. *(direction of vectors must be shown)*	z means from O to P and $z - 2$ means A to P. *Both vectors are joined to* ***P***. *Both are joined to P, they are not different points as some students had drawn. Directions were missing in many papers.*

ii)	$\angle AOP = \angle APO$. (exterior angle equals sum of the opp. Interior angles.) $\therefore \quad OA = AP$. $\therefore\ \Delta OAP$ is isosceles.  $\lvert OA\rvert = \lvert AP\rvert$ $2 = \lvert z - 2\rvert$ which is a circle with centre (2, 0) and radius 2. or Since $\angle PAx = 2\angle POx$, P must be on the circumference of the circle. (Angle subtended by arc at the centre equals double the angle at the circumference.). $OA = 2$. $\therefore$ the locus of P is a circle with centre (2, 0)radius 2.	1 mark 1 mark for the diagram. The points O and A must be excluded. 1 mark for the description with reasoning.	Very poorly done. Many students failed to use the information $\arg(z) = 2\arg(z - 2)$. In general it was noted that some students did not take sufficient time to understand the question and proceeded as $\arg(z) - \arg(z - 2) = \theta$.

c) (i)	$(\cot\theta+i)^n+(\cot\theta-i)^n$ $\left(\frac{\cos\theta}{\sin\theta}+i\right)^n+\left(\frac{\cos\theta}{\sin\theta}-i\right)^n$ $=\frac{(\cos\theta+i\sin\theta)^n+(\cos\theta-i\sin\theta)^n}{\sin^n\theta}$ $=\frac{(\cos n\theta+i\sin n\theta)+(\cos n\theta-i\sin n\theta)}{\sin^n\theta}$ $=\frac{2\cos n\theta}{\sin^n\theta}$	1 mark Applies DeMoivre theorem to simplify the expression	Well done in general. You must change the expression to cisθ form before applying DeMoivre theorem.
c) (ii)	$(x+i)^5+(x-i)^5$ Let $z=\cot\theta$ $(\cot\theta+i)^5+(\cot\theta-i)^5=\frac{2\cos 5\theta}{\sin^5\theta}=0$ ** Cos $5\theta=0$ $5\theta=2n\pi\pm\frac{\pi}{2}=(4n\pm1)\frac{\pi}{2}$, $\theta=(4n\pm1)\frac{\pi}{10}$, $n=0,\pm1,\pm2$ ** $n=0,\ \theta=\pm\frac{\pi}{10}$ $z=\cot\left(\pm\frac{\pi}{10}\right)=\pm\cot\left(\frac{\pi}{10}\right)$ $n=1,\ \theta=\frac{5\pi}{10}$ $z=\cot\left(\frac{5\pi}{10}\right)=0$ $\theta=\frac{3\pi}{10}$ $z=\cot\left(\frac{3\pi}{10}\right)$ $n=-1,\ \theta=\frac{-3\pi}{10}$ $z=-\cot\left(\frac{3\pi}{10}\right)$ $\therefore$ The solutions are $0,\ \pm\cot\left(\frac{\pi}{10}\right),\pm\cot\left(\frac{3\pi}{10}\right)$	1 mark 1 mark for the general solution. 1 mark for the correct solutions	Correctly identified the substitution and applied to the result from (i). General solution was again a problem and students were fudging answers. In <u>show that</u> question, you must show <u>all</u> working, even the ones you can do mentally

(ii)	$(x+i)^5+(x-i)^5$ $= x^5+5x^4i-10x^3-10x^2i+5x+i+$ $x^5-5x^4i-10x^3+10x^2i+5x-i$ $= 2\left(x^5-10x^3+5x\right)$ $= 2x\left(x^4-10x^2+5\right)$ ** $\therefore$ Solutions of $(x+i)^5+(x-i)^5=0$ ate the solutions of $2x\left(x^4-10x^2+5\right)=0$. Using (ii), the solutions of this polynomial are $0,\ \pm\cot\left(\frac{\pi}{10}\right), \pm\cot\left(\frac{3\pi}{10}\right)$. $2x\left(x^4-10x^2+5\right)=0$. $\therefore\ x=0$ and $x^4-10x^2+5=0$ has Solutions $\pm\cot\left(\frac{\pi}{10}\right), \pm\cot\left(\frac{3\pi}{10}\right)$.	1 mark for the correct expansion and simplifies the expression to get $2x\left(x^4-10x^2+5\right)$ 1 mark for $x=0$ 1 mark for solutions of $x^4-10x^2+5=0$	Most students correctly did the expansion.
(iv)	$x^4-10x^2+5=0$ $x^2=\frac{10\pm\sqrt{100-20}}{2}$ $=\frac{10\pm4\sqrt{5}}{2}$ $=5\pm2\sqrt{5}$ $\therefore\ x=\pm\sqrt{5\pm2\sqrt{5}}$ $\frac{\pi}{10}$ is in the first quadrant and $\therefore\cot\frac{\pi}{10}$ is positive. $\therefore\cot\frac{\pi}{10}=\sqrt{5+2\sqrt{5}}$	1 mark for the solutions of x 1 mark. Reasoning required.	This part was done well. Students drew the graph of y = cotθ to figure out the appropriate value of $\cot\frac{\pi}{10}$. Well done.