KILLARA HIGH SCHOOL

YEAR 12 – 2010

HSC ASSESSMENT 3

MATHEMATICS EXTENSION 2

Time allowed: 70 minutes

Total marks: 40

- **NOT ALL** questions are of equal value.
- Start each question on a new page.
- Show all necessary working.

Question 1 (20 marks) **Use a separate writing booklet**

a) Evaluate $\displaystyle\int_0^1 \frac{e^x}{(1+e^x)^2}$ 2

b) By completing the square find $\displaystyle\int \frac{1}{\sqrt{2x-x^2}}\,dx$ 2

c) Find $\displaystyle\int \frac{8}{(x+2)(x^2+4)}\,dx$ 5

d) Find $\displaystyle\int \frac{x}{\sqrt{1-x}}\,dx$ 3

e) Use substitution $t = \tan\frac{x}{2}$ to evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{2+\cos x}$ 4

f) If $\displaystyle I_n = \int_1^e x(\ln x)^n\,dx$, $n = 0, 1, 2, 3\ldots.$

i. show that $\displaystyle I_n = \frac{e^2}{2} - \frac{n}{2} I_{n-2}$, $n = 1, 2, 3\ldots$ 2

ii. Evaluate $\displaystyle\int_1^e x(\ln x)^3\,dx$ 3

Question 2 (20 marks) **Use a separate writing booklet**

a) For the ellipse $\frac{x^2}{16}+\frac{y^2}{25}=1$

i. Find the eccentricity. **1**

ii. Find the coordinates of the foci S and S′ **1**

iii. Find the equations of the directricies. **1**

iv. Sketch the curve $\frac{x^2}{16}+\frac{y^2}{25}=1$ **2**

b) $A(a,0)$ *and* $A'(-a,0)$ are the vertices of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,\ a>0$
$P(a\sec\theta, b\tan\theta)$ is any point on the hyperbola, $P\neq A$ *or* A'.

The tangent at P meets the tangent at A, A' *at* Q, R respectively.

i. Prove that the equation of the tangent at P is $\frac{\sec\theta}{a}x-\frac{\tan\theta}{b}y=1$. **3**

ii. Find the coordinates of Q *and* R. **2**

iii. Prove that the circle with QR as a diameter passes through the two foci of the hyperbola. **5**

c) The point $P\left(cp,\frac{c}{p}\right)$ and $Q\left(cq,\frac{c}{q}\right)$ lie on the rectangular hyperbola $xy=c^2$. The chord PQ subtends a right angle at another point $R\left(cr,\frac{c}{r}\right)$ on the hyperbola. Show that:

i) $pq=-\frac{1}{r^2}$ **2**

ii) the normal at R is parallel to PQ. **3**

Extension 2 - Mathematics _ T3 - 2010

Q4

(a) $\displaystyle\int_0^1 \frac{e^x}{(1+e^x)^2}\,dx = \left[-\frac{1}{1+e^x}\right]_0^1$ ✓

$= -\dfrac{1}{1+e} + \dfrac{1}{2}$

$= \dfrac{e-1}{2(1+e)}$ ✓

(b) $\displaystyle\int \frac{1}{\sqrt{2x-x^2}}\,dx = \int \frac{1}{\sqrt{1-(x-1)^2}}\,dx$ ✓

$= \sin^{-1}(x-1) + C$ ✓

(c) $\dfrac{a}{x+2} + \dfrac{bx+c}{x^2+4} = \dfrac{8}{(x+2)(x^2+4)} \Rightarrow a(x^2+4)+(bx+c)(x+2) = 8$

- let $x=-2 \Rightarrow 8a = 8$

$a = 1$

- coef. of constant terms : $4a+2c = 8$

$2c = 4 \Rightarrow c = 2$ ✓

- coef of x^2 terms : $a+b = 0 \Rightarrow b = -1$

$\therefore \displaystyle\int \frac{8}{(x+2)(x^2+4)}\,dx = \int \frac{1}{x+2} + \frac{2}{x^2+4} - \frac{x}{x^2+4}\,dx$ ✓

$= \ln(x+2) +$ ✓ $\tan^{-1}\dfrac{x}{2}$ ✓ $- \dfrac{1}{2}\ln(x^2+4) + C$ ✓

(d) let $u = \sqrt{1-x} \Rightarrow du = \dfrac{-dx}{2\sqrt{1-x}}$ ✓

and $x = 1-u^2$

$\therefore \displaystyle\int \frac{x}{\sqrt{1-x}}\,dx = -2\int 1-u^2\,du$

$= -2\left(u - \frac{1}{3}u^3\right)$ ✓

$= \dfrac{2}{3}(1-x)^{3/2} - 2\sqrt{1-x} + C$ ✓

(e) let $t = \tan \frac{x}{2} \Rightarrow dt = \frac{1}{2}\sec^2 \frac{x}{2}\, dx$

$= \frac{1}{2}(1+t^2)\, dx \Rightarrow dx = \frac{2dt}{1+t^2}$ ✓

when $x = 0$, $t = 0$

$x = \pi/2$, $t = \tan \pi/4 = 1$

$$\therefore \int_0^{\pi/2} \frac{dx}{2+\cos x} = \int_0^1 \frac{1}{2+\frac{1-t^2}{1+t^2}} \cdot \frac{2dt}{1+t^2}$$

$$= 2\int_0^1 \frac{1}{2+2t^2+1-t^2}\, dt \checkmark$$

$$= 2\int_0^1 \frac{1}{t^2+3}\, dt$$

$$= \frac{2}{\sqrt{3}}\left[\tan^{-1}\frac{t}{\sqrt{3}}\right]_0^1 \checkmark$$

$$= \frac{2}{\sqrt{3}} \cdot \frac{\pi}{6} = \frac{\pi\sqrt{3}}{9} \checkmark$$

(f) $I_n = \int_1^e x(\ln x)^n\, dx$, $n = 0, 1, 2 \ldots$

$$= \left[\tfrac{1}{2}x^2(\ln x)^n\right]_1^e - \frac{1}{2}\int x^2 \cdot n \cdot \frac{1}{x}(\ln x)^{n-1}\, dx \checkmark$$

$$= \frac{e^2}{2} - \frac{n}{2}\int_1^e x(\ln x)^{n-1}\, dx \checkmark$$

$$\therefore I_n = \frac{e^2}{2} - \frac{n}{2} I_{n-1} \qquad n = 1, 2, 3 \ldots$$

(ii) $I_3 = \int_1^e x(\ln x)^3$

$$= \frac{e^2}{2} - \frac{3}{2}\left(\frac{e^2}{2} - 1\cdot\left(\frac{e^2}{2} - \frac{1}{2}I_0\right)\right) \checkmark$$

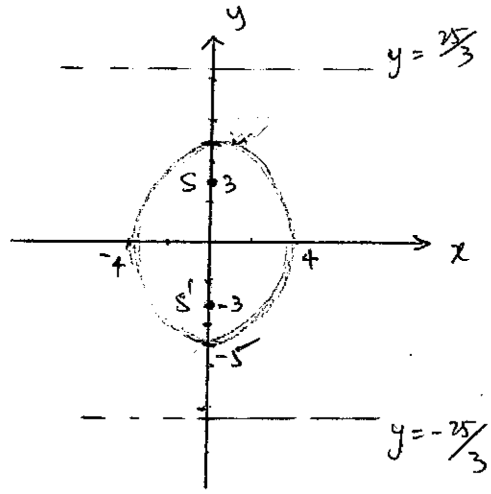
where $I_0 = \int_1^e x\, dx = \left[\frac{x^2}{2}\right]_1^e = \frac{e^2}{2} - \frac{1}{2}$ ✓

$$\therefore I_3 = \frac{e^2}{2} - \frac{3e^2}{4} + \frac{3e^2}{4} - \frac{3}{4}\cdot\frac{e^2}{2} + \frac{3}{8}$$

$$= \frac{e^2}{8} + \frac{3}{8} = \frac{1}{8}(e^2+3) \checkmark$$

Q2

(a) $\frac{x^2}{16} + \frac{y^2}{25} = 1$



(i) $16 = 25(1 - e^2)$

$e^2 = 1 - \frac{16}{25}$

$= \frac{9}{25}$

$e = \frac{3}{5}$ ✓

ii) $S\left(0, \pm 5 \times \frac{3}{5}\right) \Rightarrow S(0, \pm 3)$ ✓

iii) $y = \pm \frac{b}{e} \Rightarrow y = \pm 5 \div \frac{3}{5} \Rightarrow y = \pm \frac{25}{3}$ ✓

(b)

gradient of the tangent at P

$\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{b^2x}{a^2y}$

at $\begin{cases} x = a\sec\theta \\ y = b\tan\theta \end{cases}$

$\therefore \; m = \frac{b\sec\theta}{a\tan\theta}$ ✓

equation of the tangent at P

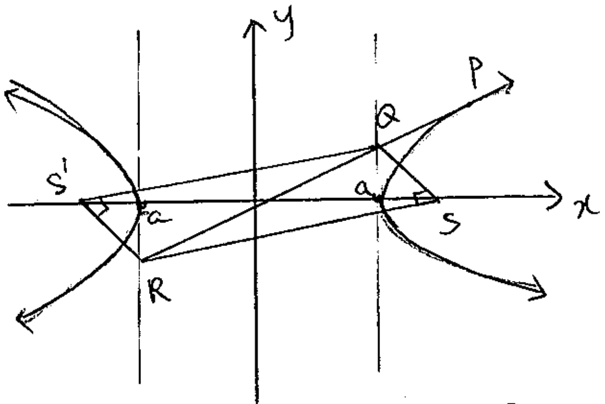
$y - b\tan\theta = \frac{b\sec\theta}{a\tan\theta}(x - a\sec\theta)$ ✓

$\frac{y}{b}\tan\theta - \frac{\cancel{b}\tan^2\theta}{\cancel{b}} = \frac{x}{a}\sec\theta - \frac{\cancel{a}\sec^2\theta}{\cancel{a}}$

$\Rightarrow \frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = \tan^2\theta - \sec^2\theta$

$\frac{\sec\theta}{a}x - \frac{\tan\theta}{b}y = 1$ ✓

(ii)



from (i) equation of the tangent at P is $\frac{\sec\theta}{a}x - \frac{\tan\theta}{b}y = 1$ ①

Sub. $x=a$ into ① $\Rightarrow$ $\frac{\cancel{a}}{\cancel{a}}\sec\theta - \frac{y}{b}\tan\theta = 1$

$\Rightarrow y = (\sec\theta - 1)\frac{b}{\tan\theta}$

$\therefore$ Coordinates of Q is $\left(a, \frac{b}{\tan\theta}(\sec\theta - 1)\right)$ ✓

Sub. $x=-a$ into ① $\Rightarrow$ $-\sec\theta - \frac{y}{b}\tan\theta = 1$

$\Rightarrow y = -\frac{b}{\tan\theta}(\sec\theta + 1)$

$\therefore$ Coordinates of R is $\left(-a, \frac{-b}{\tan\theta}(\sec\theta + 1)\right)$ ✓

Now $m_{QS} = \frac{b}{\tan\theta}(\sec\theta - 1) \div (a - ae)$

and $m_{RS} = \frac{\cancel{-}b}{\tan\theta}(\sec\theta + 1) \div \cancel{-}(a + ea)$

$m_{QS} \cdot m_{RS} = \frac{b^2}{\cancel{\tan^2\theta}}(\cancel{\sec^2\theta - 1}) \div (a^2 - a^2e^2)$ ✓

$= b^2 \div -a^2(e^2 - 1)$

$= b^2 \div -b^2$

$= -1$ ✓

$\therefore \angle QRS = 90^\circ$ ($QS \perp RS$).

similarly

$$m_{RS'} = \frac{-b}{\tan\theta}(\sec\theta + 1) \div (-a + ae)$$

$$m_{QS'} = \frac{b}{\tan\theta}(\sec\theta - 1) \div (a + ae)$$

$$m_{RS'} \cdot m_{QS'} = \frac{b^2}{\tan^2\theta}(\sec\theta - 1) \div (a^2 - a^2e^2)$$

$$= b^2 \div \left[-a^2(e^2-1)\right]$$

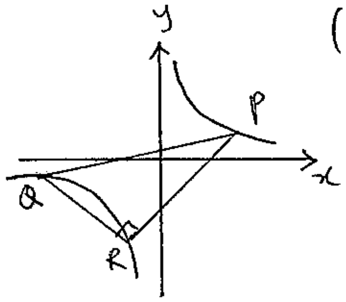
$$= -1$$

$\therefore RS' \perp QS'$.

both $\angle QS'R$ and $\angle QSR$ are angles in a semi circle with diameter QR

$\therefore S, S'$ lie on the circle with diameter QR.

(c)



(i) $m_{PR} = \dfrac{\frac{c}{p} - \frac{c}{R}}{cp - cR} = \dfrac{\cancel{c}(\cancel{R-P})}{\cancel{c}pR(\cancel{P-R})} = -\dfrac{1}{pr}$

similarly

$$m_{QR} = -\frac{1}{qr}$$

Since $QR \perp PR$, $m_{QR} \times m_{PR} = -1$

$$\Rightarrow -\frac{1}{pr} \cdot -\frac{1}{qr} = -1 \quad \Rightarrow \frac{1}{pqr^2} = -1$$

$$\therefore pq = -\frac{1}{r^2}$$

(ii) Now $m_{PQ} = -\dfrac{1}{pq}$. sub . $pq = \dfrac{-1}{r^2}$

$$\Rightarrow m_{PQ} = -\frac{1}{-\frac{1}{r^2}} = r^2$$

Gradient of the tangent at P

$$x\frac{dy}{dx} + y = 0 \Rightarrow \frac{dy}{dx} = -\frac{y}{x}$$

at $x = cr$, $y = \dfrac{c}{r}$ $\Rightarrow \dfrac{dy}{dx} = -\dfrac{c}{r} \div cr$

$$= -\frac{1}{r^2}$$

$\therefore$ gradient of the normal at R is r^2

$$m_{PQ} = m_R = r^2$$

$\therefore$ PQ is // the tangent at R.