Killara HIGH SCHOOL

STUDENT NUMBER

MATHEMATICS
Extension 2

ASSESSMENT TASK NO: 1
Date: 2nd December 2011

TIME ALLOWED: *Reading time: 5 minutes*
Working time: 60 minutes

INSTRUCTIONS: Start each question in a new booklet
Marks may not be given for untidy and poorly arranged work.
All relevant working should be shown.

THIS QUESTION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM

OUTCOMES ASSESSED:

E1	Appreciates the creativity, power and usefulness of mathematics to solve a broad range of problems.
E3	Uses the relationship between algebraic and geometric representations of complex numbers and of conic sections.
E9	Communicates abstract ideas and relationships using appropriate notation and logical argument.

		MARKS
Section 1	Multiple Choice, Show answers in your working booklet.	/4
Section 2	Complex Numbers, four questions	/39
		/43

SECTION 1 Multiple Choice

Question 1

The real root of $z^3 - 4z^2 + 14z - 20 = 0$ is 2. The other roots are:

a) 2 + 3i, 2 – 3i b) -1 + 3i, -1 – 3i

c) 1 + 3i, 1 – 3i d) -2 + 3i, -2 – 3i

Question 2

The real numbers ‘a’ and ‘b’ that satisfy the equation

a(2 – i) + b(1 + 4i) = 4 – 11i are

a) a = 2, b = -3 b) a = 3, b = -2

c) a = 2, b = -2 d) a = 3, b = -3

Question 3

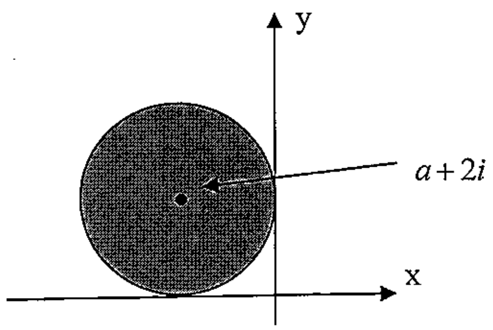
If $z = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ then a possible value for z in modulus-argument form is

a) $(\cos\frac{\pi}{15} + i\sin\frac{\pi}{15})^{-10}$ b) $\left[\cos(-\frac{\pi}{15}) - i\sin(-\frac{\pi}{15})\right]^{-10}$

c) $(\cos\frac{\pi}{15} - i\sin\frac{\pi}{15})^{-10}$ d) $(\cos\frac{\pi}{15} - i\sin\frac{\pi}{15})^{10}$

Question 4

Consider the Argand diagram below.



Which inequality could define the shaded region?

a) $|z-(a+2i)| \geq 1$ b) $|z-(a+2i)| \leq 1$

c) $|z+a-2i| \leq 1$ d) $|z+a-2i| \geq 1$

SECTION 2

Marks

Question 1

a) If $z=2-5i$, find in the form of $a+ib$: 4

i. $\bar{z}$ ii. $z+\bar{z}$ iii. $z\bar{z}$ iv. z^{-1}

b) i. If $z=-1+i$, then express z in modulus-argument form. 2

ii. Determine the smallest value of n for which z^n is a real number, where n is a positive integer. 3

c) Show that $(1+i)^{2n}+(1-i)^{2n}=2^{n+1}$ where $\frac{n}{2}$ is even. 3

Question 2

a) Let $z=-1+i\sqrt{3}$. Find the square root of z. **Do not** use De Moivre's thorem. 3

b) Solve $z^2+4(1+i\sqrt{3})z-4(1-i\sqrt{3})=0$ 2

c) Find the five fifth roots of unity in modulus-argument form. 3

Marks

Question 3

a) Draw neat labelled sketches on separate Argand diagrams to indicate the locus of the complex number z.

i. $|z-2-3i|<1$ and $1<\mathrm{Re}(z)\leq 2$. 3

ii. $|z+3-i|=|z-4|$ 2

b) Describe the locus given by $2|z|=z+\bar{z}+4$ 2

c) If $z=x+iy$ then find the locus of $\dfrac{z-1}{z-2i}$ where this ratio is purely imaginary. 2

Question 4

a) Show that $\cos 4\theta$ as a polynomial in terms of $\cos\theta$ is given by $8\cos^4\theta-8\cos^2\theta+1$. 3

b) Explain why $x=\cos\dfrac{(2k-1)\pi}{8}$ where k = 1, 2, 3 and 4 are the roots of $8x^4-8x^2+1=0$. 3

c) Consider $\arg(z-1)-\arg(z+2)=\dfrac{\pi}{2}$

i. Illustrate the locus of z on an Argand diagram. 1

ii. Find the locus of z in cartesian form. 3

THE END

Start here for Question No.

Section 2.

Question 1.

a) $Z = 2-5i$

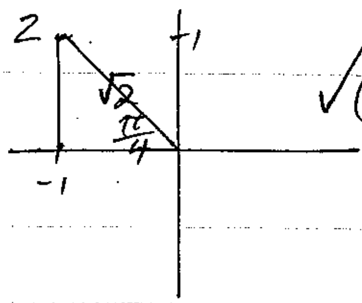
i $\bar{Z} = 2+5i$

ii $Z+\bar{Z} = 4$

iii $Z\bar{Z} = 4+25$
$= 29$

iv $Z^{-1} = \frac{2}{29} + \frac{5}{29}i$

b) i $Z = -1+i$
$= \sqrt{2}\ cis\ \frac{3\pi}{4}$ ✓



$\sqrt{}$ (or $\sqrt{2}(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$

ii $Z = -1+i$
$= \sqrt{2}\ cis\ \frac{3\pi}{4}$
$\therefore Z^n = (\sqrt{2}\ cis\ \frac{3\pi}{4})^n$
$= (\sqrt{2})^n\ cis\ \frac{3n\pi}{4}$ ✓

if Z^n is real then
$Im\ Z^n = 0$

ie $\sin\frac{3n\pi}{4} = 0$ ✓

for $\frac{3n\pi}{4} = \pi k$
$n = \frac{4}{3}k$ where $k = 1, 2, 3, \ldots$

for n to be a positive integer, $k = 3$
ie $n = 4$ is the smallest ✓
positive integer

e) $1+i = \sqrt{2}\,\text{cis}\frac{\pi}{4}$

$1-i = \sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)$

$(1+i)^{2n} = 2^n\,\text{cis}\left(\frac{2n\pi}{4}\right)$

$(1-i)^{2n} = 2^n\,\text{cis}\left(\frac{-2n\pi}{4}\right)$

$\therefore (1+i)^{2n}+(1-i)^{2n} = 2^n\,\text{cis}\left(\frac{n\pi}{2}\right)+2^n\,\text{cis}\left(-\frac{n\pi}{2}\right)$ ✓

$= 2^{n+1}\left[\text{cis}\left(\frac{n\pi}{2}\right)+\text{cis}\left(-\frac{n\pi}{2}\right)\right]$

$= 2^n\left[2\times\cos\frac{n\pi}{2}\right]$ ✓

$= 2^{n+1}\times\cos\frac{n\pi}{2}$

$= 2^{n+1}$ if $\frac{n}{2}$ is even ($\frac{n}{2}$ even $\Rightarrow \cos\frac{n\pi}{2}=1$) ✓

Question 2.

a) let $\omega^2 = -1+i\sqrt{3}$ where $\omega = a+ib$

$\therefore (a+ib)^2 = -1+i\sqrt{3}$

$a^2-b^2+i2ab = -1+i\sqrt{3}$

Equating real and Imaginary parts

$a^2-b^2=-1$ and $2ab=\sqrt{3}$ ✓

ie $4a^4-4a^2b^2=-4a^2$ — ① and $4a^2b^2=3$ — ②

Adding ① to ② gives

$4a^4 = -4a^2+3$

$4A^2+4A-3=0$ ✓ where $A=a^2$

$(2A-1)(2A+3)=0$

$A=\frac{1}{2}$ or $A=-\frac{3}{2}$

$A\neq-\frac{3}{2}$

$\therefore a^2=\frac{1}{2}\Rightarrow a=\pm\frac{1}{\sqrt{2}}$

sub into ② gives

$b^2=\frac{3}{2}\Rightarrow b=\pm\sqrt{\frac{3}{2}}$

$\therefore \sqrt{-1+i\sqrt{3}} = \frac{1}{\sqrt{2}}+i\frac{\sqrt{3}}{\sqrt{2}},\ -\frac{1}{\sqrt{2}}-i\frac{\sqrt{3}}{\sqrt{2}}$ ✓

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Question 2 b) $z^2 + 4(1+i\sqrt{3})z - 4(1-i\sqrt{3}) = 0$

$\triangle = b^2 - 4ac = 16(1-3+i2\sqrt{3}) + 16(1-i\sqrt{3})$

$= -32 + i32\sqrt{3} + 16 - i16\sqrt{3}$

$= -16 + i16\sqrt{3}$

$= 16(-1+i\sqrt{3})$ ✓

$\therefore z = \dfrac{-4(1+i\sqrt{3}) \pm \sqrt{\triangle}}{2}$

$= \left[-4 - i4\sqrt{3} \pm 4\left(\frac{1}{\sqrt{2}} + i\frac{\sqrt{3}}{\sqrt{2}}\right)\right] \div 2$

$= -2 - i2\sqrt{3} \pm 2\left(\frac{1}{\sqrt{2}} + i\frac{\sqrt{3}}{\sqrt{2}}\right)$

✓ $= -2 - i2\sqrt{3} \pm (\sqrt{2} + i\sqrt{6})$

$= -2+\sqrt{2} - i2\sqrt{3} + i\sqrt{6}, \ -2 - i2\sqrt{3} - \sqrt{2} - i\sqrt{6}$

$= (-2+\sqrt{2}) + i(\sqrt{6} - 2\sqrt{3})$ or $-(2+\sqrt{2}) - i(\sqrt{6} + 2\sqrt{3})$

c) Let $z^5 = 1$ where $z = r\,\text{cis}\,\theta$.

$\therefore (r\,\text{cis}\,\theta)^5 = 1$

$r^5\,\text{cis}\,5\theta = 1$ $\qquad |r^5\,\text{cis}\,5\theta| = |r^5| = 1$

✓ $\therefore \text{cis}\,5\theta = 1 + i0$ $\qquad \therefore r = 1.$

$\therefore \cos 5\theta = 1$ and $\sin 5\theta = 0$

✓ $\therefore 5\theta = 2k\pi$ where $k = 0, \pm 2, \pm 3, \ldots$

$\theta = \frac{2k\pi}{5}$

$\therefore \theta = 0, \frac{2\pi}{5}, -\frac{2\pi}{5}, \frac{3\pi}{5}, -\frac{3\pi}{5}$

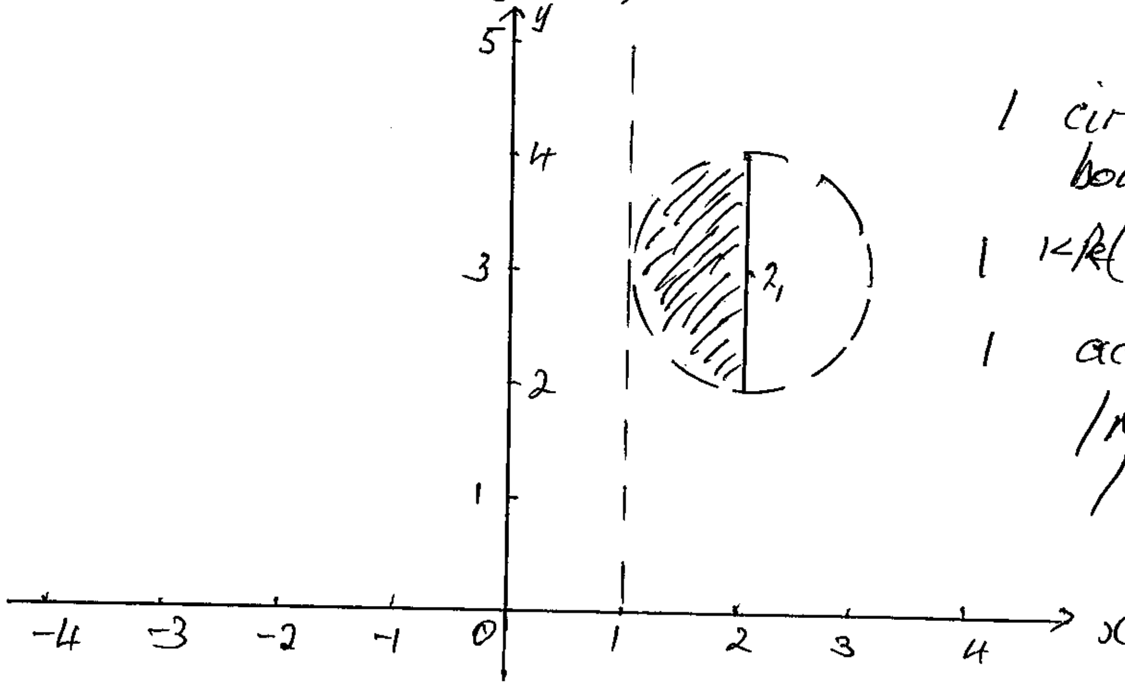
✓ $\therefore z = 1, \ \text{cis}\left(\pm\frac{2\pi}{5}\right), \ \text{cis}\left(\pm\frac{3\pi}{5}\right)$

Question 3

a) i $|z-2-3i|<1$

$|z-(2+3i)|<1$ and $1<Re(z)\le 2$

a circle centre $(2,3)$, radius 1.



1 circle and boundary.

1 $1<Re(z)\le 2$

1 accuracy / neatness / size.

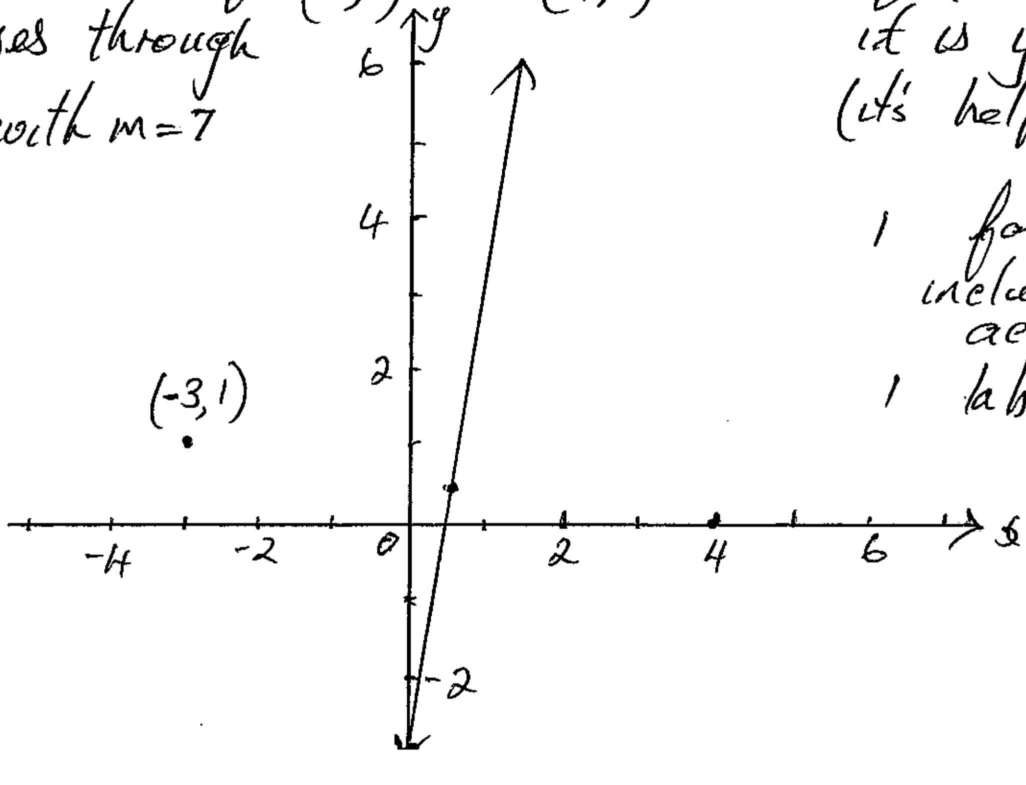
ii $|z+3-i|=|z-4|$

$|z-(-3+i)|=|z-4|$

z is the perpendicular bisector of $(-3,1)$ and $(4,0)$

$\therefore$ passes through $\left(\frac{1}{2},\frac{1}{2}\right)$ with $m=7$

Not asked for the cartesian equation however it is $y=7x-3$ (it's helpful).



1 for line include accuracy.

1 labelled points

Question 3

b) let $z = x + iy \quad \therefore \bar{z} = x - iy$

$$\therefore 2|z| = z + \bar{z} + 4$$

becomes $2\sqrt{x^2+y^2} = x + iy + x - iy + 4$

$$= 2x + 4$$

$$4(x^2+y^2) = 2^2(x+2)^2$$

$$4x^2 + 4y^2 = 4(x^2 + 4x + 4)$$

$$= 4x^2 + 16y + 16$$

$$4y^2 = 16\overset{x}{\cancel{y}} + 16$$

$$y^2 = 4(\overset{x}{\cancel{y}} + 1) \quad ✓$$

This is a parabola with vertex $(-1, 0)$ ✓ focal length 1, focus $(0,0)$, directrix $\overset{x}{\cancel{y}} = -2$

c) $z = x + iy \quad \therefore \dfrac{z-1}{z-2i} = \dfrac{x+iy-1}{x+iy-2i} = \dfrac{(x-1)+iy}{x+i(y-2)} \times \dfrac{x-i(y-2)}{x-i(y-2)}$

$$= \frac{x^2 - x + y^2 - 2y + i[xy - (x-1)(y-2)]}{x^2 + (y-2)^2} \quad ✓$$

because the ratio is purely imaginary then

$$Re\left(\frac{z-1}{z-2i}\right) = 0$$

ie $\quad x^2 - x + y^2 - 2y = 0 \quad ✓$

$$x^2 - x + \left(\tfrac{1}{2}\right)^2 + y^2 - 2y + 1 = 1\tfrac{1}{4} = \tfrac{5}{4} = \left(\tfrac{\sqrt{5}}{2}\right)^2$$

$$\left(x - \tfrac{1}{2}\right)^2 + (y-1)^2 = \left(\tfrac{\sqrt{5}}{2}\right)^2$$

Question 4

a) Consider $(\cos\theta + i\sin\theta)^4$

By De Moivre's Thm $(\cos\theta + i\sin\theta)^4 = \cos 4\theta + i\sin 4\theta$ ①

Now, $(\cos\theta + i\sin\theta)^4$

$$= \cos^4\theta + i4\cos^3\theta\sin\theta + i^2 6\cos^2\theta\sin^2\theta + i^3 4\cos\theta\sin^3\theta + i^4\sin^4\theta$$

✓ $$= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta + i(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta)$$ ②

$\therefore$ ① = ②

Equating real coefficients of ① and ② gives

✓ $$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$$
$$= \cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2$$
✓ $$= \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta$$
$$= 8\cos^4\theta - 8\cos^2\theta + 1$$

b) * $\cos 4\theta = 0$ when $4\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \ldots$ ✓
$$\theta = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \ldots$$

* let $x = \cos\theta$, sub into $8\cos^4\theta - 8\cos^2\theta + 1$

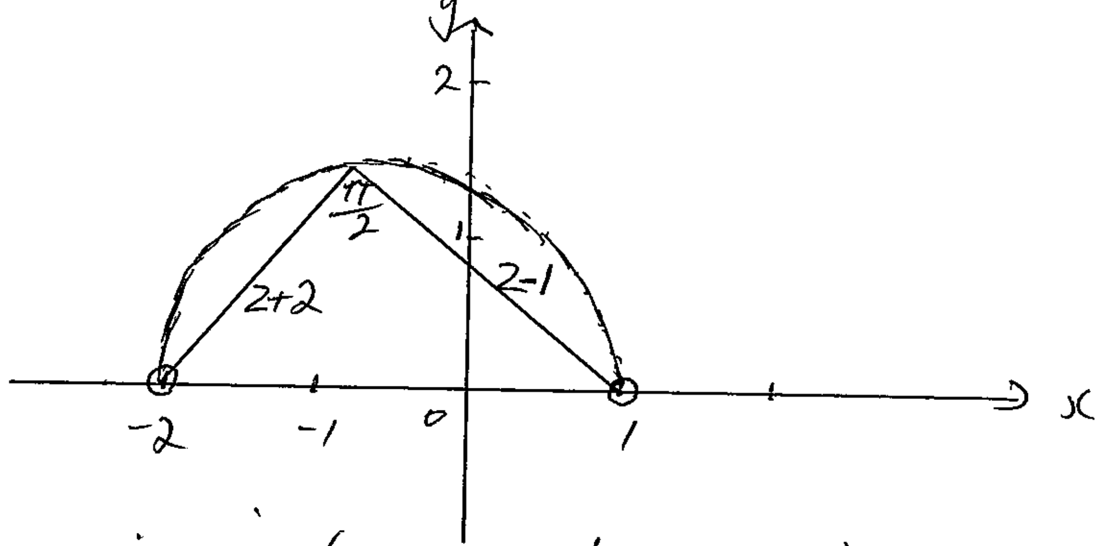
✓ $\therefore$ $8\cos^4\theta - 8\cos^2\theta + 1 = \cos 4\theta = 0$ gives
$8x^4 - 8x^2 + 1 = 0$ has roots
$x = \cos\frac{\pi}{8}, \cos\frac{3\pi}{8}, \cos\frac{5\pi}{8}, \cos\frac{7\pi}{8}$

✓ $\therefore$ the roots are $x = \cos\frac{(2k-1)\pi}{8}$
where $k = 1, 2, 3, 4$ (eqnⁿ is of degree 4)

Question 4

c) i $\arg(z-1) - \arg(z+2) = \frac{\pi}{2}$.

This is a semi circle $\text{Im } z \geq 0$
passing through $(-2, 0)$ and $(1, 0)$



iii semi circle centre $(-\frac{1}{2}, 0)$ of radius $\frac{3}{2}$
as angle in semi circle is $\frac{\pi}{2}$ ✓

Equ^n is $\left(x+\frac{1}{2}\right)^2 + (y-0)^2 = \left(\frac{3}{2}\right)^2$ ✓

$\left(x+\frac{1}{2}\right)^2 + y^2 = \frac{9}{4}$ where $y > 0$ ✓