Killara High School
Extension 2 Mathematics

Year 12
JUNE 2011

TOTAL TIME: 60 minutes
TOTAL MARKS: 40

Instructions

1. Show all necessary working.
2. Marks may be deducted for a lack of working or poor setting out of solutions.
3. Recommended time per question: question 1, 34 minutes and question 2, 26 minutes.

NAME ______________________________

Question 1 (23 marks) **Marks**

a) Find $\int \frac{dx}{\sqrt{3-2x-x^2}}$ 2

b) Find $\int \frac{dx}{(x-2)(x-3)}$ 2

c) Find $\int 2x\ln(x^2+1)dx$ 2

d) Evaluate $\int_0^{\frac{1}{\sqrt{2}}} \frac{x^2 dx}{\sqrt{1-x^2}}$ 3

e) i. Prove that if $I_n = \int \tan^n\theta \; d\theta$, where n is a non-negative number, then $I_{n+2} = \frac{\tan^{n+1}\theta}{n+1} - I_n$ 3

ii. Hence find I_4 1

f) Evaluate $\int_1^2 \frac{25dx}{(x+2)(2x-1)^2}$ 4

g) Evaluate $\int_0^{\frac{\pi}{4}} x\sin 2x dx$ 2

h) Use the 't' method to evaluate $\int_0^{\frac{\pi}{3}} \frac{\tan x}{1+\cos x}dx$. 4

End of Question 1

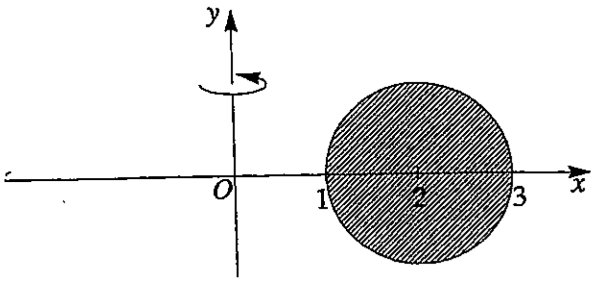
Question 2 (17 marks) **Marks**

a) The base of a certain solid is an equilateral triangle of side length S, with one vertex at the origin and an altitude along the x-axis of length p units.
Each plane cross-section perpendicular to the x-axis is a square of side length $2y$ units, one side of which lies in the base of the solid.

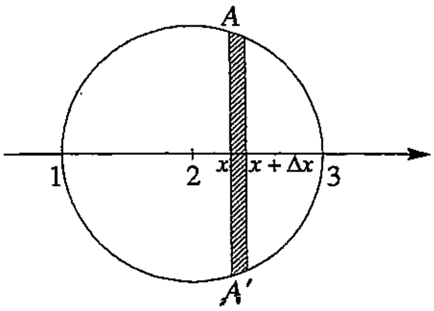
i. Show that the volume V of the solid is given by
$V = 4\int_0^p y^2 \, dx$ 3

ii. Calculate the volume of the solid. 2

b)



The circular region $(x-2)^2 + y^2 \le 1$ is rotated about the y-axis.



i. Show that $AA' = 2\sqrt{1-(x-2)^2}$. 2

ii. Show that the volume ΔV obtained when a typical strip of height AA' and thickness Δx is rotated about the y-axis is given by
$\Delta V = 4\pi x\sqrt{1-(x-2)^2}\,\Delta x$. 1

iii. Find the total volume of the solid generated. 3

Question 2 continues over the page

Question 2 continued

c) The region bounded by the curve $y = x^3$, the line $y = 1$ and the y-axis is rotated about the line $y = -1$.

i. Show that if slices are taken perpendicular to the x-axis then the volume of a typical slice is given by:

$$\Delta V = \pi(3 - 2x^3 - x^6)dx$$ 3

ii. Calculate the volume of the solid generated. 3

The End

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

Mx2 2011 Ex 2 Solutions

a $\int \frac{dx}{\sqrt{3-2x-x^2}}$

$= \int \frac{dx}{\sqrt{4-(x+1)^2}}$ ✓

$= \sin^{-1}\left(\frac{x+1}{2}\right) + c$ ✓

b $\int \frac{dx}{(x-2)(x-3)}$

$= \int \frac{-1}{x-2}\,dx + \int \frac{1}{x-3}\,dx$ ✓

$= -\ln(x-2) + \ln(x-3) + c$

$= \ln\left(\frac{x-3}{x-2}\right) + c$ ✓

d $\int_0^{\frac{1}{\sqrt{2}}} \frac{x^2}{\sqrt{1-x^2}}\,dx$

$\boxed{\begin{array}{ll} \text{let } x = \sin\theta & \therefore dx = \cos\theta\, d\theta \\ \text{when } x = \frac{1}{\sqrt{2}}, & \theta = \frac{\pi}{4} \\ \text{when } x = 0 & \theta = 0 \end{array}}$

$= \int_0^{\frac{\pi}{4}} \frac{\sin^2\theta \cos\theta\, d\theta}{\sqrt{1-\sin^2\theta}}$ ✓

$= \int_0^{\pi/4} \frac{\sin^2\theta \cos\theta\, d\theta}{\cos\theta}$

$= \int_0^{\frac{\pi}{4}} \sin^2\theta\, d\theta$

$= \frac{1}{2}\int_0^{\pi/4} (1-\cos 2\theta)\, d\theta$ ✓

$= \frac{1}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi/4}$

$= \frac{1}{2}\left[\left(\frac{\pi}{4} - \frac{1}{2}\right) - (0 - 0)\right]$

$= \frac{\pi - 2}{8}$ ✓

c $\int 2x \ln(x^2+1)\, dx$

$\int \tan^{-1} x\, dx$ see over

$= \int 1 \cdot \tan^{-1} x\, dx$ let $v' = 1$, $v = x$

$= x\tan^{-1}x - \int x \cdot \frac{1}{1+x^2}\, dx$ let $u = \tan^{-1}x$, $u' = \frac{1}{1+x^2}$

$= x\tan^{-1}x - \frac{1}{2}\int \frac{2x}{1+x^2}\, d\theta$

$= x\tan^{-1}x - \frac{1}{2}\ln(1+x^2) + c$

e i $I_n = \int \tan^n\theta\, d\theta$

$= \int \tan^{n+2}\theta \cdot \cot^2\theta\, d\theta$

$= \int \tan^{n+2}\theta(\operatorname{cosec}^2\theta - 1)\, d\theta$

$= \int \tan^{n+2}\theta \operatorname{cosec}^2\theta\, d\theta - \int \tan^{n+2}\theta\, d\theta$ ✓

$= \int \tan^{n+2}\theta \operatorname{cosec}^2\theta\, d\theta - I_{n+2}$

$= \int \tan^n\theta \sec^2\theta\, d\theta - I_{n+2}$ ✓

$I_n = \frac{\tan^{n+1}\theta}{n+1} - I_{n+2}$

$I_{n+2} = \frac{\tan^{n+1}\theta}{n+1} - I_n$ ✓

$\int 2x \ln(x^2+1)\, dx$

let $u = x^2 + 1$
$\therefore du = 2x\, dx$.

$= \int \ln u\, du$

$= \int 1 \cdot \ln u\, du$

$= \int \frac{d(u)}{du} \ln u\, du$

$= u\ln u - \int u \cdot \frac{1}{u}\, du$ ✓

$= u\ln u - \int 1\, du$

$= u\ln u - u + c$

$= (x^2+1)\ln(x^2+1) - (x^2+1) + c$ ✓

$(x^2+1)\ln(x^2+1) - x^2 + c$.

If $I_{n+2} = \frac{\tan^{n+1}\theta}{n+1} - I_n$

then $I_4 = \frac{\tan^3\theta}{3} - I_2$

$= \frac{\tan^3\theta}{3} - \int \tan^2\theta \, d\theta$

$= \frac{\tan^3\theta}{3} - \int (\sec^2\theta - 1) \, d\theta$ ✓

$= \frac{\tan^3\theta}{3} - \tan\theta + \theta + c$

f

let $\frac{25}{(x+2)(2x-1)^2} = \frac{a}{x+2} + \frac{b}{2x-1} + \frac{c}{(2x-1)^2}$

$\therefore 25 = a(2x-1)^2 + b(x+2)(2x-1) + c(x+2)$ ✓

$x = \frac{1}{2}$, $25 = \frac{5}{2}c$, $c = 10$

$x = -2$, $25 = 25a$, $a = 1$

$x = 0$, $25 = 1 - 2b + 20$, $b = -2$

$\therefore \frac{25}{(x+2)(2x-1)^2} = \frac{1}{x+2} + \frac{-2}{2x-1} + \frac{10}{(2x-1)^2}$ ✓

$\therefore \int_1^2 \frac{25\,dx}{(x+2)(2x-1)^2} = \int \frac{1}{x+2} - \frac{2}{2x-1} + \frac{10}{(2x-1)^2} \, dx$ ✓

$= \left[\ln(x+2) - \ln(2x-1) - 5(2x-1)^{-1}\right]_1^2$

$\left[\ln\left|\frac{x+2}{2x-1}\right| - \frac{5}{2x-1}\right]_1^2$

$\left(\ln\left|\frac{4}{3}\right| - \frac{5}{3}\right) - \left(\ln\frac{3}{1} - 5\right)$

$\ln\left|\frac{4}{9}\right| + \frac{10}{3}$

$= \left(\ln 4 - \ln 3 - \frac{5}{3}\right) - \left(\ln 3 - \ln 1 - 5\right)$

$= \frac{10}{3} + 2\ln\left(\frac{2}{3}\right)$ ~~or $\frac{10}{3}$ +~~ ✓

g. $\int x \sin 2x \, dx$

let $u = x$, $v' = \sin 2x$

$u' = 1$, $v = -\frac{1}{2}\cos 2x$

$= \left[(x)(-\frac{1}{2}\cos 2x)\right]_0^{\pi/4} - \int_0^{\pi/4} (-\frac{1}{2}\cos 2x) \, dx$ ✓

$= \left(-\frac{\pi}{8}\cos\frac{\pi}{2}\right) - (0) + \frac{1}{2}\int_0^{\pi/4} \cos 2x \, dx$

$= 0 + \frac{1}{2} \times \frac{1}{2}\left[\sin 2x\right]_0^{\pi/4}$

$= \frac{1}{4}\left[\sin\frac{\pi}{2} - 0\right]$ ✓

$= \frac{1}{4}$

h $\int_0^{\pi/3} \frac{\tan x}{1+\cos x} \, dx$

let $t = \tan\frac{x}{2}$

$\therefore \frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$

$dx = \frac{2\,dt}{\sec^2\frac{x}{2}}$

$dx = \frac{2\,dt}{1+t^2}$ ✓

$\int_0^{\pi/3} \frac{\tan x}{1+\cos x} \, dx$

$= \int_0^{\sqrt{3}} \frac{\frac{2t}{1-t^2}}{1 + \frac{1-t^2}{1+t^2}} \times \frac{2\,dt}{1+t^2}$ ✓

$= \int_0^{\sqrt{3}} \left(\frac{4t}{1-t^2}\right) \div \left(\frac{2}{1+t^2}\right)\left(\frac{1+t^2}{1}\right) dt$

$= \int_0^{\sqrt{3}} \frac{2t}{1-t^2} \, dt$ ✓

$= \left[-\ln(1-t^2)\right]_0^{1/\sqrt{3}}$

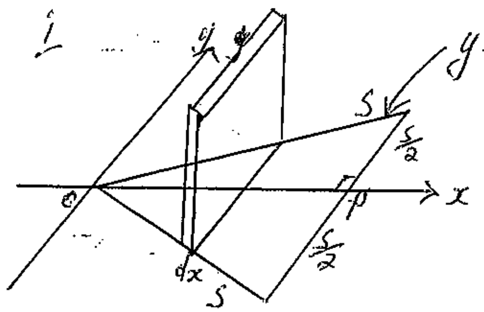
$= -\ln\frac{2}{3} + \ln 0$

$= -\ln\frac{2}{3}$

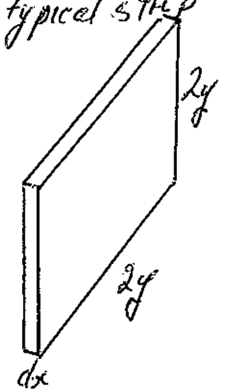
$= \ln\frac{3}{2}$ ✓

Question 2

(a) i



Consider a typical strip



$dV \doteqdot 4y^2\,dx$

Total Volume $\doteqdot \sum_{dx=0}^{P} 4y^2\,dx$ ✓

$= \lim_{dx \to 0} \sum_{x=0}^{P} 4y^2\,dx$ ✓

$= 4\int_0^P y^2\,dx.$ ✓

ii

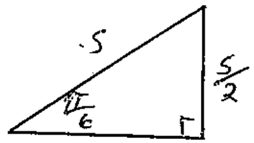
Total Volume $= 4\int_0^{\frac{s\sqrt{3}}{2}} \frac{1}{3}x^2\,dx$ ✓

Must be just in terms of S or P

$= \frac{4}{3}\left[\frac{x^3}{3}\right]_0^{\frac{s\sqrt{3}}{2}}$

$= \frac{4}{3}\left[\frac{1}{3}\cdot s^3 \cdot \frac{3\sqrt{3}}{8} - 0\right]$

$= \frac{s^3\sqrt{3}}{6}$ ✓



$\cos\frac{\pi}{6} = \frac{P}{S}$

$P = \frac{S\sqrt{3}}{2}$

Question 2

b i $(x-2)^2 + y^2 = 1$

$y^2 = 1-(x-2)^2$

$y = \pm\sqrt{1-(x-2)^2}$

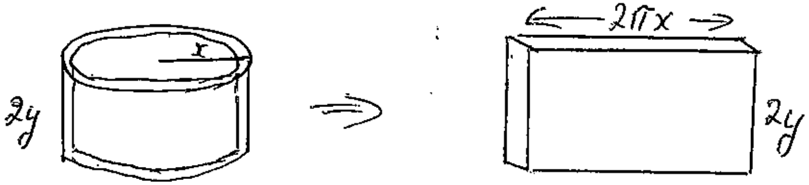
∴ 2 points A and A' on the circle are

$A\left(x, \sqrt{1-(x-2)^2}\right)$ and $A'\left(x, -\sqrt{1-(x-2)^2}\right)$ ✓

∴ $AA' = \sqrt{1-(x-2)^2} - \left(-\sqrt{1-(x-2)^2}\right)$

$= 2\sqrt{1-(x-2)^2}$ ✓

ii



where $AA' = 2y$

$\Delta V \doteqdot (2\pi x)(2y)\,dx$

$= 2\pi x\left(2\sqrt{1-(x-2)^2}\right)dx$

$= 4\pi x\sqrt{1-(x-2)^2}\,dx$ ✓

iii Volume $= \lim_{dx\to 0}\sum_{x=1}^{3}\Delta V$ ✓

$= \int_1^3 4\pi x\sqrt{1-(x-2)^2}\,dx$

Let $u = x-2$, $\therefore du = dx$, $x = u+2$, $x=1$ $u=-1$, $x=3$ $u=1$

$= 4\pi\int_{-1}^{1}(u+2)\sqrt{1-u^2}\,du$ ✓

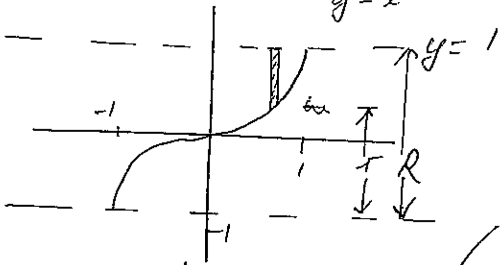
$= 4\pi\left[\int_{-1}^{1}\left(u\sqrt{1-u^2}\right)du + 2\int_{-1}^{1}\sqrt{1-u^2}\,du\right]$

odd function   semicircle

$= 4\pi[0+\pi] = 4\pi^2$

Question 2

c i



$R = 2$

$r = 1 + y$.

$\Delta V = \pi (R^2 - r^2) dx$

$= \pi [4 - (1+y)^2] dx$

$= \pi [4 - (1 + 2y + y^2)] dx$.

$= \pi [3 - 2y - y^2] dx$

$= \pi [3 - 2x^3 - x^6] dx$.

ii Volume $= \lim_{dx \to 0} \sum_{x=0}^{1} \pi (3 - 2x^3 - x^6) dx$

$= \pi \int_0^1 (3 - 2x^3 - x^6)\, dx$.

$= \pi \left[3x - \frac{2x^4}{4} - \frac{x^7}{7} \right]_0^1$

$= \pi \left[3 - \frac{1}{2} - \frac{1}{7} - 0 \right]$

$= \frac{33\pi}{14}$ units^3