Killara HIGH SCHOOL

STUDENT NUMBER

MATHEMATICS
Extension 2

ASSESSMENT TASK NO: 2
Date: 11/3/2011

TIME ALLOWED: *Reading time:* 5 *minutes*
Working time: 3 *hours*

INSTRUCTIONS: This paper consists of 8 questions
ANSWER all 8 questions
Start each question in a new booklet
Marks may not be given for untidy and poorly arranged work.
All relevant working should be shown.

THIS QUESTION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM

OUTCOMES ASSESSED:

E1	appreciates of the scope, usefulness, beauty and elegance of mathematics to solve a broad range of problems
E3	uses the relationship between algebraic and geometric representations of complex numbers
E4	uses efficient techniques for the algebraic manipulation required in dealing with questions such as those involving conic sections and polynomials
E6	combines the ideas of algebra and calculus to determine the important features of the graphs of a wide variety of functions
E9	communicates abstract ideas and relationships using appropriate notation and logical argument

This assessment task constitutes 25% of the Higher School Certificate Course Assessment.

Total Mark − 120
Attempt Questions 1 – 8
All questions are of equal value

Answer each question on a SEPARATE writing booklet. Extra writing booklets are available

Question 1 (15 marks) Use a SEPARATE writing booklet **Marks**

a) $z = 1 + 2i$. Find in the form $a + ib$

(i) $z + \bar{z}$ **1**

(ii) $\dfrac{\bar{z}}{(1+z)^2}$ **1**

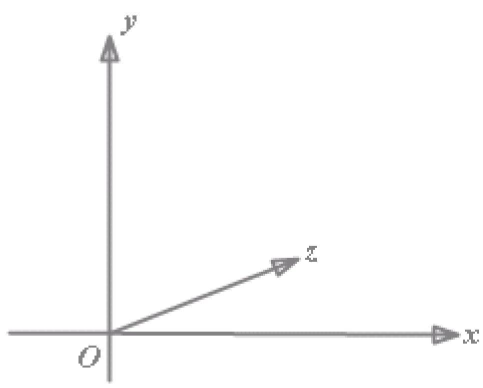
b) $z_1 = 1 - i$ and $z_2 = -\sqrt{3} + i$

(i) Write both z_1 and z_2 in modulus/ argument form. **2**

(ii) Find $z_1 z_2$ in modulus/ argument form. **2**

(iii) Hence express $\cos \dfrac{7\pi}{12}$ as a surd. **1**

c)



Copy this Argand diagram and also show vectors representing iz and $iz - z$. **2**

Question 1 continues on next page

d) α, β are complex numbers such that $\dfrac{\alpha+\beta}{\alpha-\beta}$ is an imaginary number.

(i) Show the vectors for α, β, $\alpha+\beta$ and $\alpha-\beta$ on an Argand diagram which illustrate this condition. **2**

(ii) Deduce that $|\alpha| = |\beta|$. **1**

e) z is a complex number such that $|z-i| \leqslant 1$ and $0 \leqslant \arg z \leqslant \dfrac{\pi}{4}$.

(i) Sketch the locus of z on an Argand diagram. **2**

(ii) Find the minimum value of $|z-i|$. **1**

End of question 1

Question 2 (15 marks) Use a SEPARATE writing booklet **Marks**

a) When the polynomial $P(x)$ is divided by $(x-2)$ the remainder is 5, While division of $P(x)$ by $(x+2)$ leaves a remainder of -1. Find the remainder when $P(x)$ is divided by (x^2-4). **2**

b) $P(x) = 2x^3 - 7x^2 + ax + b$, where a and b are real, and $2 - i$ is a zero of $P(x)$.

(i) Find the remaining two zeros of $P(x)$. **2**

(ii) Find the values of a and b. **2**

(iii) Factor $P(x)$ into irreducible factors with real coefficients. **1**

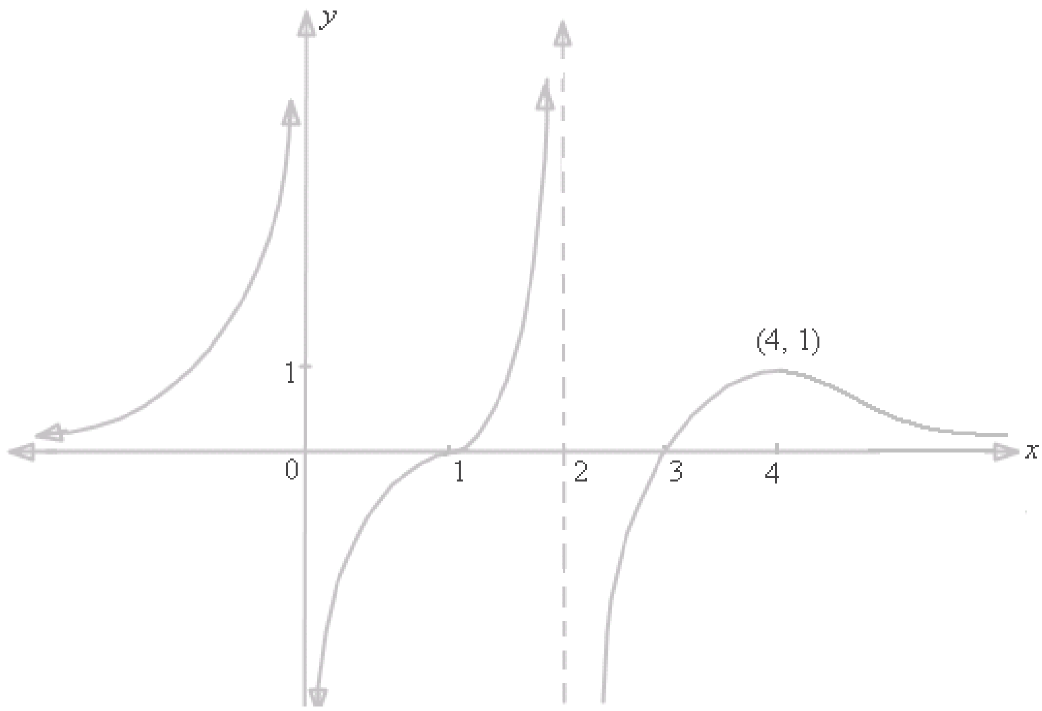
c) (i) Explain why $\cos 3\theta = \text{Re}\{(\cos\theta + i\sin\theta)^3\}$ and hence show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. **2**

(ii) Hence, solve $8x^3 - 6x - 1 = 0$. **3**

d) Show that $1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots + \dfrac{x^n}{n!}$ has no multiple zero for any integer $n > 1$. **3**

Question 3 (15 marks) Use a SEPARATE writing booklet **Marks**

a) The graph of $y = f(x)$ is shown below.



Draw separate sketches for the following:

(i) $y = \dfrac{1}{f(x)}$ **3**

(ii) $y = |f(x)|$ **2**

(iii) $y = f(|x|)$ **2**

(iv) $y = (f(x))^2$ **2**

(v) $y = \sqrt{f(x)}$ **2**

b) For the polynomial equation $x^3 + 4x^2 + 2x - 3 = 0$ with roots α, β and γ. Find:

(i) The equation whose roots are $\alpha^2,\ \beta^2,\ \gamma^2$. **2**

(ii) The equation whose roots are $\dfrac{1}{\alpha\beta}, \dfrac{1}{\beta\gamma}, \dfrac{1}{\alpha\gamma}$. **2**

Question 4 (15 marks) Use a SEPARATE writing booklet **Marks**

a) The angles of elevation of the top of a tower P measured from three points A, B, C are α, β, γ respectively.

A, B, C are in a straight line such that $AB = BC = a$, but the line AC does not pass through S, the base of the tower.

(i) If $\angle ABS = \theta$, show that

$$(CS)^2 = a^2 + h^2 \cot^2 \beta + 2ah \cot \beta \cos \theta. \qquad \textbf{2}$$

(ii) Prove that the height of the tower is

$$\frac{a\sqrt{2}}{\left\{\cot^2 \alpha + \cot^2 \gamma - 2\cot^2 \beta\right\}^{\frac{1}{2}}}. \qquad \textbf{3}$$

b) Consider the curve $y = \cos\left(\sqrt{x}\right),\ 0 \leqslant x \leqslant 4\pi^2$

(i) Find $\frac{dy}{dx}$ and the gradient of the limiting tangent as $x \to 0$. **2**

(ii) Find any stationary points and determine their nature. **3**

(iii) Sketch $y = \cos\left(\sqrt{x}\right),\ 0 \leqslant x \leqslant 4\pi^2$, then complete the sketch to show the graphs of $y = \cos\left(\sqrt{|x|}\right),\ -4\pi^2 \leqslant x \leqslant 4\pi^2$, showing any intercepts on the coordinate axes, the limiting tangents (including their equations) at any critical points, and the coordinates of any stationary points. **5**

Question 5 (15 marks) Use a SEPARATE writing booklet **Marks**

Find:

(i) $\int \tan^2 x \, dx$ **2**

(ii) Using the substitution $x = \dfrac{1}{u}$, find $\displaystyle\int \frac{dx}{\left(x^2+1\right)^{\frac{3}{2}}}$. **3**

(iii) Use the substitution $x = 10\sqrt{2}\sin\theta$ to show that $\displaystyle\int_{-10}^{10} \sqrt{200 - x^2}\, dx = 100 + 50\pi$, **4**

then use a geometrical argument to verify this result.

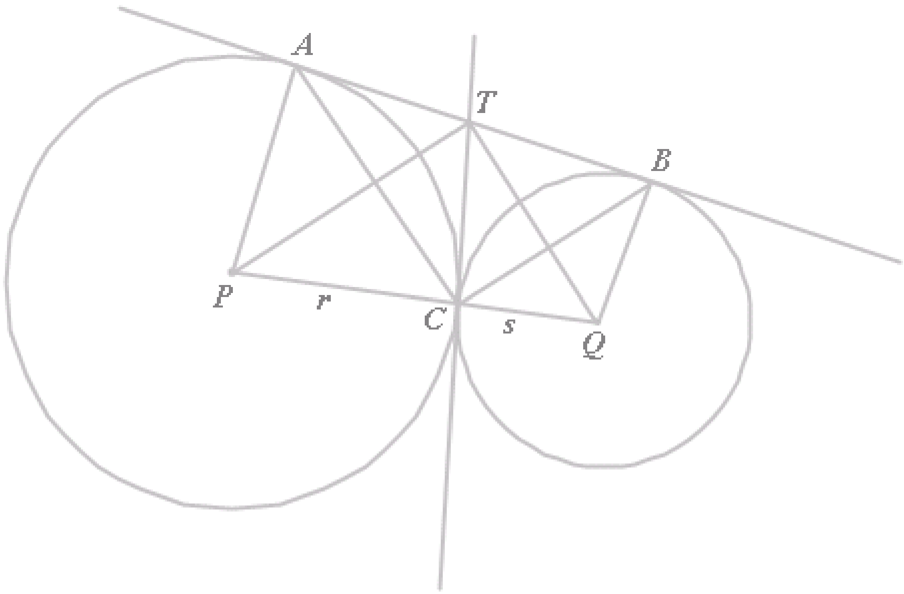
(iv) Use the substitution $u^6 = x$ to find $\displaystyle\int \frac{1}{x^{\frac{1}{2}} - x^{\frac{1}{3}}}\, dx$. **3**

(You may use the table of standard integrals)

(v) Write $\dfrac{x^3+1}{x\left(x^2+1\right)}$ in the form $a + \dfrac{b}{x} + \dfrac{cx+d}{x^2+1}$. **3**

Question 6 (15 marks) Use a SEPARATE writing booklet **Marks**

a)



Two circles touch at C. One circle has centre P and radius r, the other has centre Q and radius s. AB is a common tangent touching the first and second circles at A and B respectively. T is the intersection of the common tangent at C with tangent AB.

(i) Explain why $TA = TC = TB$. Deduce that $\angle ACB = 90^\circ$. **2**

(ii) Explain why $\angle TAC = \angle TPC$ and $\angle TBC = \angle TQC$. **2**

(iii) Hence, show that $\Delta TPC \parallel\!| \Delta TQC$. **1**

(iv) Deduce that $TC^2 = rs$. **1**

Question 6 continued **Marks**

b) (i) Use mathematical induction to prove that **4**

$$\sum_{r=0}^{n} \frac{1}{(r+1)(r+2)(r+3)} = \frac{(n+1)(n+4)}{4(n+2)(n+3)}.$$

(ii) Hence, find $\displaystyle \underset{n \to \infty}{Lim} \sum_{r=0}^{n} \frac{1}{(r+1)(r+2)(r+3)}$. **2**

c) A, B and C are the angles of a triangle. **3**
Show that $(CosA + i\sin A)(CosB + i\sin B) + (CosC - i\sin C) = 0$.

End of question 6

Question 7 (15 marks) Use a SEPARATE writing booklet **Marks**

a) For the hyperbola $\dfrac{x^2}{9} - \dfrac{y^2}{16} = 1$, find:

(i) The eccentricity. **1**

(ii) The coordinates of the foci **1**

(iii) The equations of the directrices **1**

(iv) Equations of the asymptotes **1**

(v) Hence, sketch the hyperbola, showing all the major features. **2**

b) Two possible forms of a cubic polynomial function $y = f(x)$ are sketched below, where a and b are the x- coordinates of the turning points.

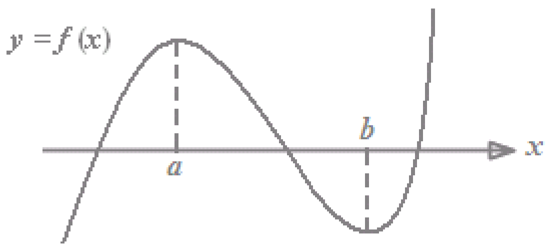


Fig. 1

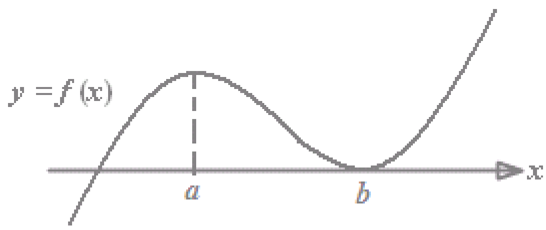


Fig. 2

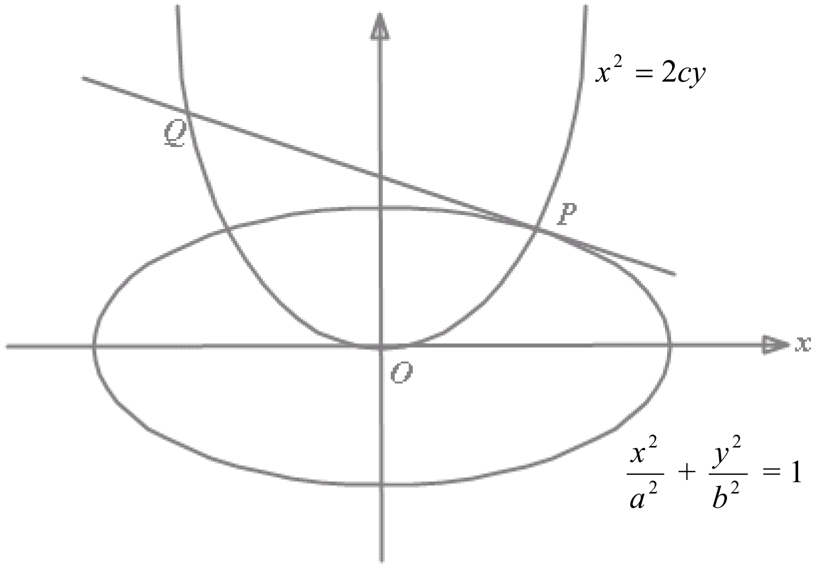
(i) Comment on the possible values of $f(a) \times f(b)$ in the figures 1 and 2. **1**

(ii) For what range of values of k will the equation $2x^3 - 3x^2 - 36x + 3k = 0$ have three real roots, not all necessarily distinct. (Hint: Find the stationary points for this function.) **3**

(iii) For $P(x) = x^3 - 3\text{m}^2 x + n$ where $m, n > 0$ show that the roots are real and distict if $n < 2m^3$. **3**

(iv) If the roots of $P(x)$ in part (iii) are in the ratio 2 : −3 : 5. Show that $90\sqrt{3}\,m^3 = 11\sqrt{11}\,n$. **2**

Question 8 (15 marks) Use a SEPARATE writing booklet **Marks**



The parabola $x^2 = 2cy$ intersects the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point $P(2cp, cp^2)$ in the first quadrant, where $a > b > 0$ and $c > 0$.

The tangent to the ellipse at P meets the parabola again at $Q(2cq, cq^2)$.

(i) Show that the tangent to the ellipse at P has equation $\frac{2cpx}{a^2} + \frac{2cp^2y}{b^2} = 1$. **3**

(ii) If this tangent meets the parabola at $(2ct, ct^2)$ show that **2**

$\frac{p^2t^2}{b^2} + \frac{4pt}{a^2} - \frac{1}{c^2} = 0$ and deduce that, considered as an equation in t, this equation has roots p and q.

(iii) If PQ subtends a right angle at the origin, show that $pq = -4$ and deduce that $\frac{1}{b^2} = \frac{1}{a^2} + \frac{1}{(4c)^2}$. **3**

(iv) Hence, show that if PQ subtends a right angle at the origin, then $p = 2e$ where e is the eccentricity of the ellipse. **4**

(v) Find a set of positive rational numbers a, b, c with $a > b$ so that PQ subtends a a right angle at the origin, and sketch the corresponding parabola and ellipse showing equations of the curves, the equation of the tangent at P, and the coordinates of P and Q. (A simple sketch would suffice.) **3**

End of paper

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

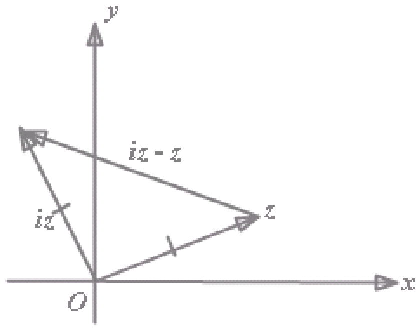
$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

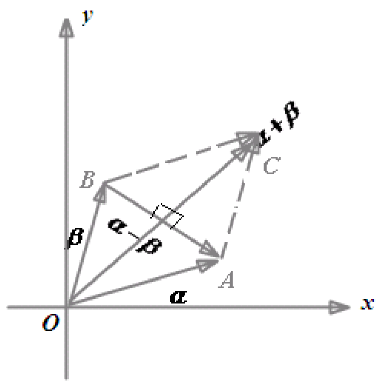
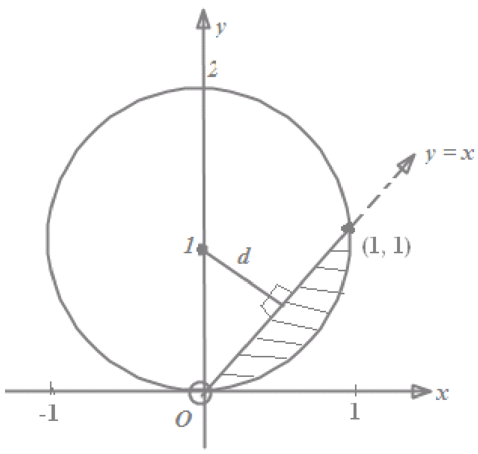
$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

2011 Ext 2 marking Scheme

Question 1

a) (i) $z = 1 + 2i$. $z + \bar{z} = 1 + 2i + 1 - 2i$ $= 2$ (ii) $(1+z)^2 = 4(1+i)^2 = 8i$ $\dfrac{\bar{z}}{(1+z)^2} = \dfrac{1-2i}{8i} = \dfrac{i+2}{-8}$ $= \dfrac{-1}{4} - \dfrac{1}{8}i$	1 mark for correct answer 1 mark for correct answer	
b) $z_1 = 1 - i$ and $z_2 = -\sqrt{3} + i$ (i) $z_1 = \sqrt{2}\{\cos\left(-\dfrac{\pi}{4}\right) + i\sin\left(-\dfrac{\pi}{4}\right)\}$ and $z_2 = -\sqrt{3} + i$ $= 2\{\cos\left(\dfrac{5\pi}{6}\right) + i\sin\left(\dfrac{5\pi}{6}\right)\}$ (ii) $\lvert z_1 z_2 \rvert = 2\sqrt{2}$ Arg $(z_1 z_2) = -\dfrac{\pi}{4} + \dfrac{5\pi}{6} = \dfrac{7\pi}{12}$ $z_1 z_2 = 2\sqrt{2}\ \{\cos\left(\dfrac{7\pi}{12}\right) + i\sin\left(\dfrac{7\pi}{12}\right)\}$ (iii) $z_1 z_2 = (1 - i) \times (-\sqrt{3} + i)$ $= (-\sqrt{3} + 1) + i(\sqrt{3} + 1)$ Comparing real parts, $2\sqrt{2}\ \cos\dfrac{7\pi}{12} = \text{Re}(z_1 z_2) = -\sqrt{3} + 1$ $\therefore \text{Cos}\ \dfrac{7\pi}{12} = \dfrac{1-\sqrt{3}}{2\sqrt{2}}$	1 mark for each mod-arg form *(2 marks)* Modulus 1 mark Argument 1 mark Expands $z_1 z_2$ and compares real parts 1 mark	
c) Diagram  	• iz with $\dfrac{\pi}{2}$, equal lengths 1 mark • $iz - z$ 1 mark	

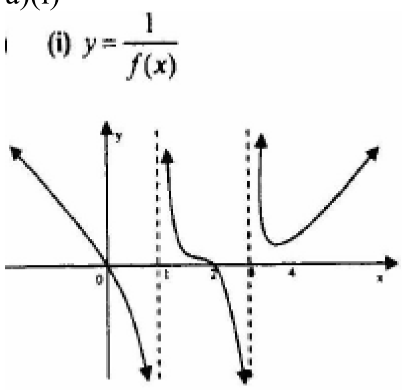
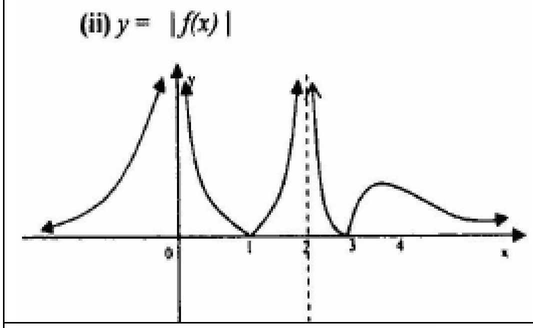
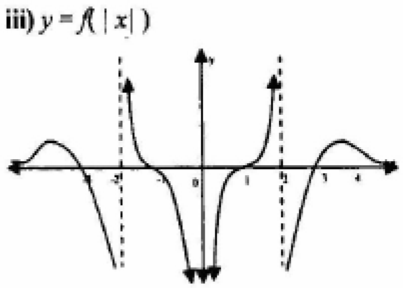
d)(i) Diagram  (ii) $\dfrac{\alpha+\beta}{\alpha-\beta}$ = ki α + β = ki (α − β) ∴ arg (α + β) = $\dfrac{\pi}{2}$ + arg(α − β) ∴ angle between diagonals = $\dfrac{\pi}{2}$ ∴ parallelogram OACB is a rhombus. ∴ OA = OB ∴ |α| = |β|	• shows vectors forming parallelogram with diagonals α + β and α − β ***1 mark*** • Shows $\dfrac{\pi}{2}$ between diagonals **1 mark** (ii)Deduces ||logram is rhombus when |α| = |β| **1 mark**	
e) (i) Diagram  (ii) $d =$ min. $\lvert z-i \rvert$ $= \sin\dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}}$ minimum value of $\lvert z-i \rvert = \dfrac{1}{\sqrt{2}}$	e)(i) • Inside circle **1 mark** • Correct segment **1 mark** (ii) minimum value **1 mark**	

Question 2

a) $P(x) \equiv (x^2-4)Q(x) \;+\; ax+b$ $P(2)=5 \Rightarrow 5 = 0+2a+b$ (1) $P(-2)=-1 \Rightarrow 0-2a+b$ (2) (1) + (2) gives $2b=4$ $\therefore b=2$ $a=\frac{3}{2}$ remainder is $\frac{3}{2}x+2$ b) $P(x)=2x^3-7x^2+ax+b$ a and b are real (i) $2-i$ is a zero $\Rightarrow 2+i$ is also a zero. Let α be the third zero. $\alpha+(2+i)+(2-i) = \frac{7}{2} \Rightarrow \alpha = -\frac{1}{2}$ (ii) $\frac{a}{2} = (2+i)(2-i) - \frac{1}{2}(2+i) - \frac{1}{2}(2+i)$ $\Rightarrow \frac{a}{2}=3 \quad \therefore a=6$ $\frac{-b}{2} = (2+i)(2-i)(-\frac{1}{2}) \Rightarrow \frac{-b}{2} = \frac{-5}{2}$ $\therefore b=5$ (iii) $P(x)=(2x+1)(x^2-4x+5)$	• Writes division transformation and obtains simul. equations **1 mark** • Solves to write the remainder **1 mark** (b)(i) Writes complex conjugates as root **1 mark** (ii) Value of a **1 mark** Value of b **1 mark** (iii) Factorisation **1mark**	
c)(i) by De Moivre's theorem, $(\cos\theta+i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$ Equating real parts, $\text{Cos } 3\theta = \text{Re}\{(\cos\theta+i\sin\theta)^3\}$ $= \cos^3\theta + 3\cos\theta\,(i\sin\theta)^2$ $= \cos^3\theta - 3\cos\theta\,(1-\cos^2\theta)$ $= 4\cos^3\theta - 3\cos\theta$ (ii) $8x^3-6x-1=0.$ $4x^3-3x = \frac{1}{2}$ Let $x=\cos\theta$ $\therefore$ Using (i), $\cos 3\theta = \frac{1}{2}$ $\therefore 3\theta = \pm\frac{\pi}{3}, \pm\frac{5\pi}{3}, \pm\frac{7\pi}{3}, \ldots$ $\therefore \theta = \pm\frac{\pi}{9}, \pm\frac{5\pi}{9}, \pm\frac{7\pi}{9}, \ldots$ $\therefore x = \cos\frac{\pi}{9}, \cos\frac{5\pi}{9}, \cos\frac{7\pi}{9}$.	• Quotes De Moivre theorem **1mark** • Finds real part after expansion and simplifies **1 mark** (ii) • Lets $x=\cos\theta$ and obtains $\cos 3\theta = \frac{1}{2}$ **1 mark** • Writes solutions for θ **1 mark** • hence writes the solutions of x. **1 mark**	

c) Let $P(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!}$ Then, $P'(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^{n-1}}{(n-1)!}$ $\therefore\ P(x) - P'(x) = \frac{x^n}{n!}$ (1) Suppose α is a multiple zero of $P(x)$. Then $P(\alpha) = P'(\alpha) = 0$ and $P(\alpha) - P'(\alpha) = \frac{\alpha^n}{n!} = 0 \Rightarrow \alpha = 0$, (using (1). But $P(0) \neq 0$ Hence P(x) has no multiple zero.	(c) * States that if α is a multiple zero of $P(x)$. then $P(\alpha) = P'(\alpha) = 0$ **1 mark** • Finds expression for $P(\alpha) - P'(\alpha)$ and deduces that $\alpha = 0$ **1 mark** • Explains $P(0) \neq 0$ and hence no multiple zeros **1 mark**	

Question 3

a)(i) 	(a)(i) • Correct graph (award **3 marks)** • Graph drawn in correct regions and asymptotes correct **(2 marks)** • Correct asymptotes **(1 mark)**	
(ii) 	(ii) • Correct graph **(2 marks)** • Correct graph for $x > 2$ or $x < 2$ **(1 mark)**	
	(iii) • Correct graph **2 marks** • Correct graph for $x > 0$ or $x < 0$ **(1 mark)**	

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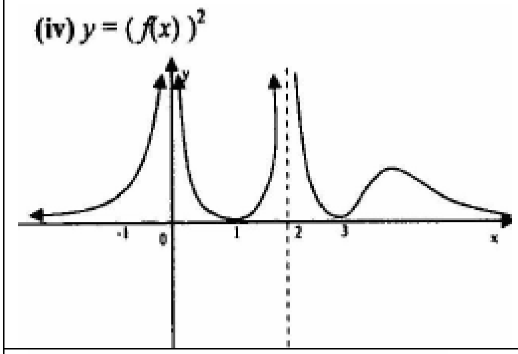
<td></td>
<td>(iv)

• Correct graph (2 marks)

• Correct stationary points (1 mark)</td>
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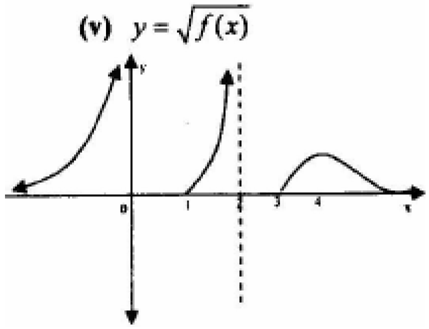
<td></td>
<td>(v)

• Correct shape for at least 2 branches AND correct domain
2 marks
• Correct shape for at least 2 branches or correct domain
1 mark</td>
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<td>b) $x^3 + 4x^2 + 2x - 3 = 0$ with roots α, β and γ
(i) equation with roots α^2, β^2, γ^2.

Let $y = x^2$
$\therefore\ x = \sqrt{y}$
$\therefore\ \sqrt{y}$ satisfies the equation
$x^3 + 4x^2 + 2x - 3 = 0$

ie. $(\sqrt{y}\,)^3 + 4(\sqrt{y}\,)^2 + 2(\sqrt{y}\,) - 3 = 0$

$(\sqrt{y}\,)^3 + 2(\sqrt{y}\,) = 3 - 4y$
Squaring both sides,

$y^3 + 4y^2 + 4y = 9 - 24y + 16y^2$
$y^3 - 12y^2 + 28y - 9 = 0$
$\therefore$ Required equation is
$x^3 - 12x^2 + 28x - 9 = 0$</td>
<td>b(i)

• Substitutes $x = \sqrt{y}$ into the equation
1 mark

• Manipulates the equation and gives the correct answer in terms of x.

1 mark</td>
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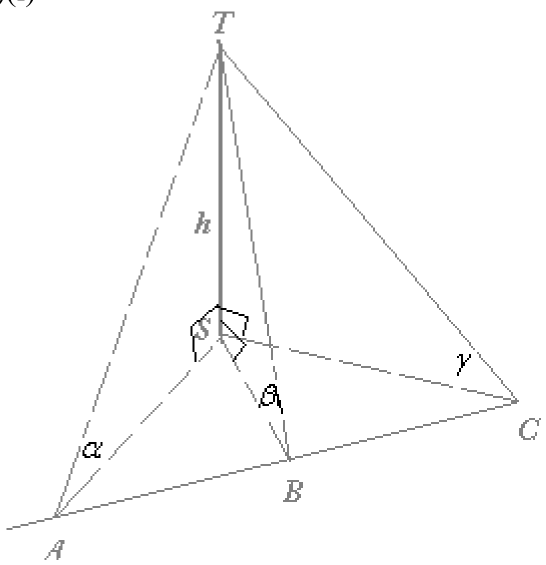
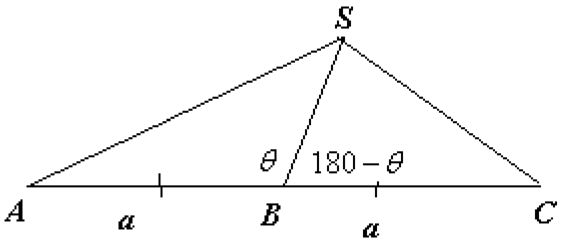
ii) $x^3 + 4x^2 + 2x - 3 = 0$ with roots α, β and γ $\alpha\beta\gamma = 3$ $\frac{1}{\alpha\beta} = \frac{\gamma}{\alpha\beta\gamma} = \frac{\gamma}{3}$ $\therefore \frac{1}{\alpha\beta}, \frac{1}{\beta\gamma}, \frac{1}{\alpha\gamma} = \frac{\gamma}{3}, \frac{\alpha}{3}, \frac{\beta}{3}$ Let $y = \frac{x}{3}$ $\quad \therefore x = 3y$ $\therefore (3y)^3 + 4(3y)^2 + 2(3y) - 3 = 0$ $\therefore 27y^3 + 36y^2 + 6y - 3 = 0$ Required equation is $27x^3 + 36x^2 + 6x - 3 = 0$	b)(ii) • Evaluates $\alpha\beta\gamma = 3$ and expresses each of the roots as $\frac{\gamma}{3}, \frac{\alpha}{3}, \frac{\beta}{3}$. **1 mark** • Substitutes $x = 3y$ and gives the correct equation **1 mark.**	

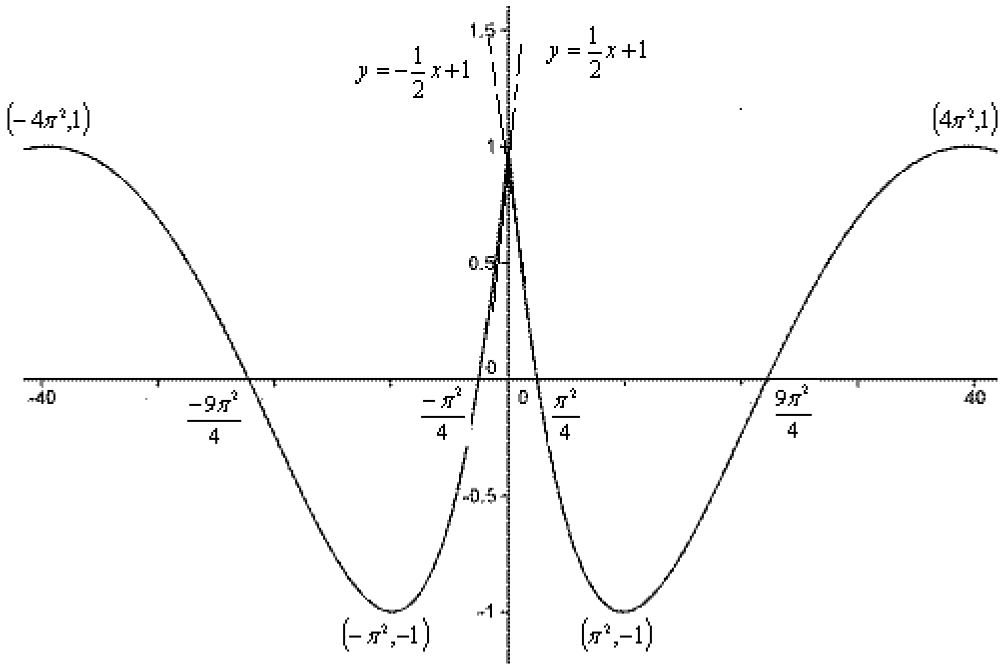
Question 4

a)(i)   In ΔBST, $\tan\beta = \frac{h}{BS}$ $BS = h\cot\beta$ $(CS)^2 = a^2 + h^2\cot^2\beta - 2ah\cot\beta\times\cos(180 - \theta)$ $= a^2 + h^2\cot^2\beta + 2ah\cot\beta\cos\theta$	b)(ii) **1 mark** for the correct diagram with all the information shown on it. • Applies cosine rule and obtains the correct expression for (CS). **1 mark.**	

a)(ii) In ΔCST, CS = $h \cot \gamma$ Using (i), $(CS)^2 =$ $a^2 + h^2 \cot^2 \beta + 2ah \cot\beta \cos \theta = h^2 \cot^2 \gamma$ (1) In ΔABS, $\text{Cos } \theta = \dfrac{a^2 + h^2 \cot^2 \beta - h^2 \cot^2 \alpha}{2ah \cot \beta}$ Substitute cosθ in (1), $a^2 + h^2 \cot^2 \beta +$ $2ah \cot\beta \times \dfrac{a^2 + h^2 \cot^2 \beta - h^2 \cot^2 \alpha}{2ah \cot \beta} = h^2 \cot^2 \gamma$ $a^2 + h^2 \cot^2 \beta + a^2 + h^2 \cot^2 \beta - h^2 \cot^2 \alpha =$ $h^2 \cot^2 \gamma$ $2a^2 + 2h^2 \cot^2 \beta - h^2 \cot^2 \alpha = h^2 \cot^2 \gamma$ $2a^2 = h^2 \cot^2 \gamma + h^2 \cot^2 \alpha - 2h^2 \cot^2 \beta$ $= h^2 (\cot^2 \gamma + \cot^2 \alpha - 2 \cot^2 \beta)$ $\therefore a^2 = \dfrac{2a^2}{\{\cot^2 \alpha + \cot^2 \gamma - 2\cot^2 \beta\}}$ $\therefore a = \dfrac{a\sqrt{2}}{\{\cot^2 \alpha + \cot^2 \gamma - 2\cot^2 \beta\}^{\frac{1}{2}}}$	b)(ii) **1 mark** for the correct expression for cosθ **1 mark** for equation (1) and substitute cosθ in it. **1 mark** for the correct solution of the resultant equation	
b)(i) $y = \cos\left(\sqrt{x}\right)$ $\dfrac{dy}{dx} = -\dfrac{1}{2}\dfrac{\sin\sqrt{x}}{\sqrt{x}} \to -\dfrac{1}{2}$ as $x \to 0$. Hence, the limiting tangent as $x \to 0$ has gradient $-\dfrac{1}{2}$. (ii) $\dfrac{dy}{dx} = 0$ as $\sin\sqrt{x} = 0$ ie. $x = \pi(^2,\ 4\pi^2$ sign of $\dfrac{dy}{dx} = -\dfrac{1}{2}\dfrac{\sin\sqrt{x}}{\sqrt{x}}$	**1 mark** for $\dfrac{dy}{dx}$ **1 mark** for limiting gradient. (ii) **1 mark** for Minimum turning point **1 mark** for maximum turning point	

x	π	π^2	$2\pi^2$	$4\pi^2$
$\frac{dy}{dx}$	-0.28	0	0.11	0

Minimum turning point at $(\pi^2, -1)$ Maximum turning point at $(4\pi^2, 1)$	(*must use the gradient test or the second derivative to determine the turning points*	
(iii) y- intercept (0, 1) x-intercept: let y = 0. $\cos\left(\sqrt{x}\right) = 0 \quad \therefore \sqrt{x} = \frac{\pi}{2} \quad \therefore x = \frac{\pi^2}{4}$ $\sqrt{x} = \frac{3\pi}{2} \quad \therefore x = \frac{9\pi^2}{4}$	**1 mark** for y-intercept	
x-intercepts are $\left(\frac{\pi^2}{4}, 0\right)$ and $\left(\frac{9\pi^2}{4}, 0\right)$	**1 mark** for x-intercepts.	
Critical point $x = 0$. $m = -\frac{1}{2} \quad (0, 1)$ Equation of the tangent $y = -\frac{1}{2}x + 1$	**1 mark** for equations of tangents.	



1 mark for shape
1 mark for the details.

Question 5

(i) $\int \tan^2 x\, dx$ $\int (\sec^2 x - 1)dx$ $= \tan x - x + C$	Expresses $\tan^2 x$ as $\text{Sec}^2 x - 1$ **1 mark** Integrates correctly **1 mark**	
(ii) $\int \frac{dx}{(x^2+1)^{\frac{3}{2}}}$ Let $x = \frac{1}{u}$ $dx = \frac{-1}{u^2}du$ $\int \frac{dx}{(x^2+1)^{\frac{3}{2}}} = \int -\frac{1}{u^2}.\frac{du}{\left(\frac{1}{u^2}+1\right)^{\frac{3}{2}}}$ $= \int -\frac{1}{u^2}.\frac{u^3\, du}{(1+u^2)^{\frac{3}{2}}}$ **1 mark** $= -\int \frac{u\, du}{(1+u^2)^{\frac{3}{2}}}$ $= -\frac{1}{2}\int \frac{2u\, du}{(1+u^2)^{\frac{3}{2}}}$ $= -\frac{1}{2}\frac{(1+u^2)^{-\frac{3}{2}+1}}{-\frac{3}{2}+1} + C$ **1 mark** $= -\frac{1}{2}\frac{(1+u^2)^{-\frac{1}{2}}}{-\frac{1}{2}} = (1+u^2)^{-\frac{1}{2}} + C$ $= \frac{1}{\sqrt{1+u^2}} + C$ $= \frac{1}{\sqrt{1+\frac{1}{x^2}}} + C = \frac{x}{\sqrt{1+x^2}} + C$	Makes the transformation to u **1 mark** Correctly integrates **1 mark** Gives the answer in terms of x **1 mark**	

(iii) $\int_{-10}^{10} \sqrt{200 - x^2}$

Let $x = 10\sqrt{2}\sin\theta$

$dx = 10\sqrt{2}\cos\theta \, d\theta$

$x = 10, \ \theta = \dfrac{\pi}{4}$

$\int_{-10}^{10} \sqrt{200 - x^2} \ = \ 2 \times \int_{0}^{10} \sqrt{200 - x^2}$ being is an even function.

$$\int_{0}^{10} \sqrt{200 - x^2} = \int_{0}^{\frac{\pi}{4}} \sqrt{200(1 - \sin^2\theta)}10\sqrt{2}\cos\theta \, d\theta$$

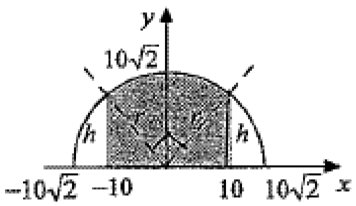
$$= \int_{0}^{\frac{\pi}{4}} 200\cos^2\theta \, d\theta \quad \textbf{1 mark}$$

$$= \int_{0}^{\frac{\pi}{4}} 200\frac{1+\cos 2\theta}{2} \, d\theta \quad \textbf{1 mark}$$

$$= \ 100\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{4}}$$

$$= \ 100\left[\tfrac{\pi}{4} + \tfrac{1}{2}\right] \ = \ 25\pi + 50$$

$$2 \times \int_{0}^{10} \sqrt{200 - x^2} \ = \ 50\pi + 100 \quad \textbf{1 mark}$$



Geometrically, integral gives the shaded area comprising ¼ circle (r = $10\sqrt{2}$) and two right triangles (h = 10).

Shaded area is ¼ π ($10\sqrt{2}$)^2 + 2 × ½ × 10 × 10

= 50π + 100 **1 mark**

- Converting dx to dθ and obtaining integrand for dθ in terms of $\cos^2\theta$
- Finding the primitive function
- Substituting appropriate limits to evaluate the integral
- Verifying the result geometrically

1 mark for each point.

(iv) $\int \frac{1}{x^{\frac{1}{2}} - x^{\frac{1}{3}}}\,dx$ $u^6 = x \qquad dx = 6u^5\,\mathrm{du}$ $x^{\frac{1}{2}} = u^3$ and $x^{\frac{1}{3}} = \mathrm{u}^2$ **1 mark** $\int \frac{1}{x^{\frac{1}{2}} - x^{\frac{1}{3}}}\,dx = \int \frac{6u^5\,du}{u^3 - u^2} = \int \frac{6u^3\,du}{u-1}$ $= 6\int u^2 + u + 1 + \frac{1}{u-1}\,du$ **1 mark** $= 6\left[\frac{u^3}{3} + \frac{u^2}{2} + u + \ln(u-1)\right] + C$ $= 6\left[\frac{\sqrt{x}}{3} + \frac{\sqrt[3]{x}}{2} + \sqrt[6]{x} + \ln(\sqrt[6]{x} - 1)\right] + C$ **1 mark**	• Expressing the integral in terms of u. • Performing the polynomial division and expressing the integral as addition of simpler terms • Performs the integration correctly.	

(v) $\frac{x^3+1}{x(x^2+1)} = a + \frac{b}{x} + \frac{cx+d}{x^2+1}$. $x^3 + 1 \equiv ax(x^2+1) + b(x^2+1) + x\,(cx+d)$ **1 mark** Comparing constants, $b = 1$ Comparing coefficients of x^2, $0 = b + c$ $0 = 1 + c \Rightarrow c = -1$ Comparing coefficients of x^3, $a = 1$. Comparing coefficients of x, $0 = a + d$. $0 = 1 + d \Rightarrow d = -1$ $\therefore \frac{x^3+1}{x(x^2+1)} = 1 + \frac{1}{x} - \frac{x+1}{x^2+1}$. `	• Writes the equivalent equation • **2 marks** if all the coefficients are correct and rewrites the partial fractions using the correct coefficients • **Award one mark** if atleast two of the coefficients are correct and rewrites the partial fractions using the correct coefficients	

Question 6

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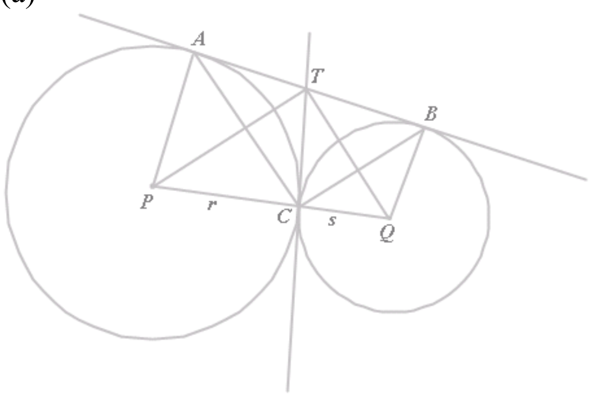
<td>(a)


(i) $TA = TC$
$TC = TB$. (tangents from an external point T are equal)
$TA = TC = TB$. Means T is the centre of a circle through A, B and C.

$\therefore$ AB is the diameter.
$\therefore$ $\angle ACB = 90^\circ$ ($\angle$ in a semicircle is 90°)</td>
<td>(i)
• Identifies equal tangents 1 mark

• Identifies $\angle$ in a semicircle)</td>
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<td>(ii) $\angle$ PAT = 90º
$\angle$ PCT = 90º (tangent is $\perp$ to the line from the centre drawn to the point of contact.
$\therefore$ ATCP is a cyclic quadrilateral (opposite angles are supplementary)

$\therefore$ $\angle$ TAC = $\angle$ TPC ($\angle$ s subtended at the circumference in the same segment by chord TC are equal.)

Similarly, $\angle$ TBC = $\angle$ TQC.</td>
<td>(ii)
• Identifies cyclic quadrilateral 1 mark

• Uses angles on the same chord subtended at the circumference 1mark</td>
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<td>(iii) In ΔTPC and ΔQTC,
$\angle$ TPC = $\angle$ TAC = 90º − $\angle$ TBC (since $\angle$ ACB = 90º, and $\angle$ sum of ΔABC = 180º)

= 90º − $\angle$ TQC
= $\angle$ QTC.
Also, $\angle$ TCP = $\angle$ TCQ (both 90º, and tangent $\perp$ to Radius)
$\Delta TPC \parallel\!| \Delta TQC$ (equiangular)</td>
<td>(iii)
Finds equal angles
1mark</td>
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(iv) $\Delta TPC \parallel\!\mid \Delta TQC$ $\therefore \dfrac{TC}{QC} = \dfrac{PC}{TC}$ (corresponding sides in similar Δs are proportional) $\therefore TC^2 = PC \times QC$ $TC^2 = rs$	(iv) Uses sides in proportion	
(b) Let S_n be the statement $\sum_{r=0}^{n} \dfrac{1}{(r+1)(r+2)(r+3)} = \dfrac{(n+1)(n+4)}{4(n+2)(n+3)}$. Step 1: let n = 0 LHS = $\dfrac{1}{(0+1)(0+2)(0+3)} = \dfrac{1}{6}$ RHS = $\dfrac{(0+1)(0+4)}{4(0+2)(0+3)} = \dfrac{1}{6}$ LHS = RHS and $\therefore S_0$ is true. Step 2: Assume S_k is true. $\sum_{r=0}^{k} \dfrac{1}{(r+1)(r+2)(r+3)} = \dfrac{(k+1)(k+4)}{4(k+2)(k+3)}$.	• Proves the statement is true for n = 0 **1mark** • Writes the S_k statement correctly.	

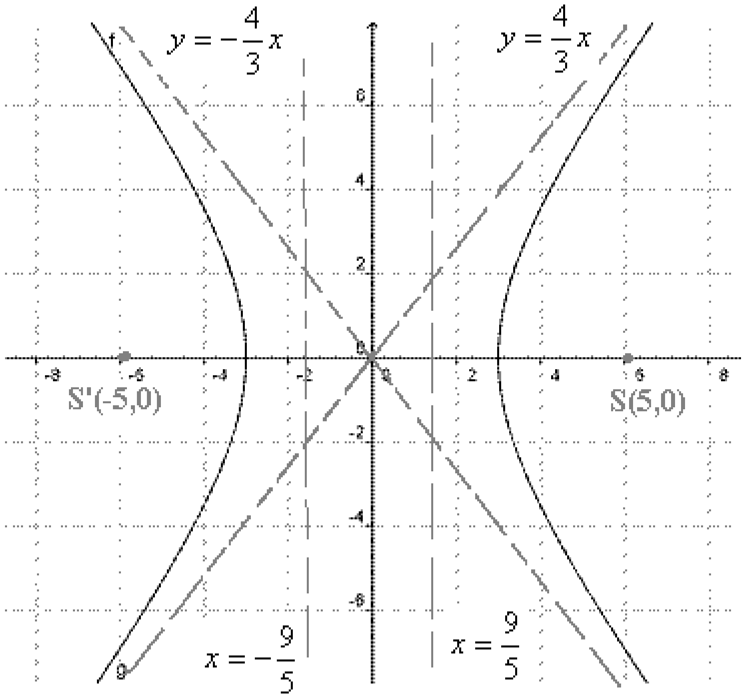
Step 3: We need to prove S_{k+1} is true. $\therefore \sum_{r=0}^{k} \dfrac{1}{(r+1)(r+2)(r+3)} + \dfrac{1}{(k+2)(k+3)(k+4)}$ $= \dfrac{(k+1)(k+4)}{4(k+2)(k+3)} + \dfrac{1}{(k+2)(k+3)(k+4)}$ $= \dfrac{1}{4(k+2)(k+3)(k+4)}\left[(k+1)(k+4)^2+4\right]$ Consider $\left[(k+1)(k+4)^2+4\right]$ $= \left[(k+1)(k^2+8k+16)+4\right]$ $= k^3 + 9k^2 + 24k + 20$ $= (k+2)(k^2 + 7k + 10)$ $= (k+2)^2(k+5)$ $\therefore \dfrac{1}{4(k+2)(k+3)(k+4)}\left[(k+1)(k+4)^2+4\right]$ $= \dfrac{1}{4(k+2)(k+3)(k+4)} \times (k+2)^2(k+5)$	• Adds $\dfrac{1}{(k+2)(k+3)(k+4)}$ to the S_k expression. **1mark** • Correct algebraic simplification to arrive at the expression $S_k = \dfrac{(k+2)(k+5)}{4(k+3)(k+4)}$ **1 mark** *Note: A mark must be deducted in the absence of a reasonable concluding statement in step 5.*	

$= \dfrac{(k+2)(k+5)}{4(k+3)(k+4)}$ $\therefore$ If S_k is true, then S_{k+1} is true. Step 4: We have proved S_0 is true, and then using the result in step 4, S_1 , S_2 ... S_n is true for all integral values of n.		
b)(ii) $\lim_{n\to\infty}\sum_{r=0}^{n}\dfrac{1}{(r+1)(r+2)(r+3)} = \lim_{n\to\infty}\dfrac{(n+1)(n+4)}{4(n+2)(n+3)}$ $= \lim_{n\to\infty}\dfrac{n^2+5n+4}{4(n^2+5n+6)}$ $= \lim_{n\to\infty}\dfrac{n^2+5n+4}{4(n^2+5n+6)}$ $= \dfrac{1}{4}\lim_{n\to\infty}\dfrac{1+\frac{5}{n}+\frac{4}{n^2}}{\left(1+\frac{5}{n}+\frac{6}{n^2}\right)}$ $= \dfrac{1}{4}\times\dfrac{1+0+0}{(1+0+0)} = \dfrac{1}{4}$	(b)(ii) • Uses the result from (i) and substitutes for $\dfrac{1}{(r+1)(r+2)(r+3)}$ **1 mark** • Divides by the highest power of n both numerator and denominator and demonstrates the understanding $\lim_{n\to\infty}\dfrac{1}{n}=0$.**1mark**	

c) A + B = π – C Cos (A + B) = − Cos C Sin (A + B) = sin C $(CosA + i\sin A)(CosB + i\sin B) =$ Cos A cos B + i cos A sin B + i sinA cos B – sin A sin B $= (CosAcoB - \sin A\sin B) + i(\sin ACosB + Cosa\sin B)$ = cos(A + B) + i Sin (A + B) = − (cos C – i sin C) $\therefore\ (CosA + i\sin A)(CosB + i\sin B)(CosC - i\sin C)$ = − (cos C – i sin C) + $(CosC - i\sin C)$ = 0	c) • Correct expansion of $(CosA + i\sin A)\times(CosB + i\sin B)$ **1 mark** • Recognising the expansions of sine and cosine of an angle sum **1 mark** • Expressing trig. Ratios of A+B in terms of trig. Ratios of C. **1 mark**	

Question 7

a) $\dfrac{x^2}{9} - \dfrac{y^2}{16} = 1$ a = 3, b = 4 (i) $b^2 = a^2(e^2 - 1)$ $16 = 9(e^2 - 1)$ $e^2 - 1 = \dfrac{16}{9} \quad \therefore e^2 = \dfrac{25}{9} \quad \therefore e = \dfrac{5}{3}$	**1 mark**	
(ii) foci $(\pm ae, 0) = \left(\pm 3 \times \dfrac{5}{3}, 0\right)$ $= (\pm 5, 0)$	**1 mark**	
(iii) equations of the directrices is $x = \pm\dfrac{a}{e}$ $= \pm\dfrac{3}{5/3} = \pm\dfrac{9}{5}$	**1 mark**	
(iv) Equations of the asymptotes: $y = \pm\dfrac{b}{a}x$ $y = \pm\dfrac{4}{3}x$	**1 mark**	

(v)  	(v) Shape and directrices **1mark** Asymptotes, foci and vertices **1mark**	
(b)(i) $f(a) \times f(b) < 0$ in fig. 1 $f(a) \times f(b) = 0$ in fig. 2 (ii) $f(x) = 2x^3 - 3x^2 - 36x + 3k$ $f'(x) = 6x^2 - 6x - 36$ $f'(x) = 0$ for stationary points. $6x^2 - 6x - 36 = 0$ $6(x^2 - x - 6) = 0$ $(x - 3)(x + 2) = 0$ $x = 3,\ x = -2$ $f(3) = 2\times(3)^3 - 3(3)^2 - 36\times(3) + 3k$ $-81 + 3k$ $f(-2) = 2\times(-2)^3 - 3(-2)^2 - 36\times(-2) + 3k$ $44 + 3k$ From (i) $\Rightarrow$ 3 real roots if $f(a)\times f(b) \le 0$ **1 mark** So, $f3)\times f(-2) \le 0$ $(3k - 81)(3k + 44) \le 0$ $-\frac{44}{3} \le k \le 27$ **1 mark** (iii) $P(x) = x^3 - 3m^2x + n$ $P'(x) = 3x^2 - 3m^2$ $3(x + m)(x - m) = 0$ for stationary points. $X = \pm m$ **1 mark** $P(m) = m^3 - 3m^3 + n = -2m^3 + n.$ $P(-m) = -m^3 + 3m^3 + n = 2m^3 + n.$		

For 3 real and distinct roots $P(m)\times P(-m) < 0$ $-4m^6 + n^2 < 0$ $n^2 < 4m^6$ $n < 2m^3$ as required. (iv) Let the roots be 2α, -3α and 5α. Sum of roots two at a time $= -6\alpha^2 - 15\alpha^2 + 10\alpha^2$ $\therefore \frac{c}{a} = -3m^2 = -11\alpha^2$ $\alpha^2 = \frac{3m^2}{11}$ (1) **1 mark** Product of roots $-\frac{d}{a} = -n = 2\alpha \times -3\alpha \times 5\alpha$ $-n = -30\alpha^3$ $\alpha^3 = \frac{n}{30}$ (2) From (1) $\alpha^3 = \frac{3\sqrt{3}m^3}{11\sqrt{11}}$ ….(3) From (2) and (3), $\frac{n}{30} = \frac{3\sqrt{3}m^3}{11\sqrt{11}}$ $90\sqrt{3}m^3 = 11\sqrt{11}\,n$ **1 mark**		

Question 8

(i) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$; $\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$ $\frac{dy}{dx} = -\frac{x}{a^2} \Big/ \frac{y}{b^2}$ Gradient at (x_1, y_1) is $-\frac{x_1}{a^2} \Big/ \frac{y_1}{b^2}$ Equation of the tangent at (x_1, y_1) is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = k$ where k is a constant. (x_1, y_1) lies on the tangent $\Rightarrow k = \frac{x_1^{\ 2}}{a^2} + \frac{y_1^{\ 2}}{b^2} = 1$ Hence, tangent at $(2cp, cp^2)$ has equation $\frac{2cpx}{a^2} + \frac{cp^2y}{b^2} = 1$	8 (i) • Finding the gradient of the tangent • Finding the equation of the tangent • Substitution of the coordinates of P.	

(ii) $(2ct, ct^2)$ lie on tangent at $P \Rightarrow$ $\frac{2cp(2ct)}{a^2} + \frac{cp^2(ct^2)}{b^2} = 1$ $\therefore \frac{4c^2pt}{a^2} + \frac{p^2c^2t^2}{b^2} = 1$ $\frac{4pt}{a^2} + \frac{p^2t^2}{b^2} = \frac{1}{c^2}$ $\frac{p^2t^2}{b^2} + \frac{4pt}{a^2} - \frac{1}{c^2} = 0$ (1) **1 mark** Since $(2ct, ct^2)$ lies on the tangent for both $t = p$ (at P) and $t =$ q (at Q), equation (1) considered as a quadratic in t has roots p and q **1 mark**	(ii) • Obtaining requird deduction • Explaining required deduction	
(iii) Gradient of $OP = \frac{cp^2}{2cp} = \frac{p}{2}$ $P(2cp, cp^2)$ and $Q(2cq, cq^2) \Rightarrow$ gradient of $OP \times$ gradient of OQ $= \frac{p}{2} \times \frac{q}{2} = -1$ $OP \perp OQ \quad \therefore pq = -4$ **1 mark** Since, q is a root of (1), substituting $t = q$ gives $\frac{(pq)^2}{b^2} + \frac{4pq}{a^2} - \frac{1}{c^2} = 0$ Since $pq = -4$, $\frac{(-4)^2}{b^2} + \frac{4(-4)}{a^2} - \frac{1}{c^2} = 0$ **1 mark** Hence, $\angle POQ = 90° \Rightarrow \frac{1}{b^2} = \frac{1}{a^2} + \frac{1}{(4c)^2}$ **1 mark**	(iii) • Showing $pq = -4$ • Substitution $t = q$ and $pq = -4$ in expression in (ii) • Rearrangement to obtain the required result.	
(iv) The quadratic in t, $\frac{p^2t^2}{b^2} + \frac{4pt}{a^2} - \frac{1}{c^2} = 0$ (1) has roots p and q. (from (ii)) $\therefore$ product of roots $= pq = \dfrac{\frac{1}{c^2}}{\frac{p^2}{b^2}} = \frac{-b^2}{c^2p^2}$ **1 mark** $\therefore pq = -4 \Rightarrow p^2 = \frac{b^2}{4c^2}$ **1 mark**	(iv) • Obtaining an expression for pq as the product of roots of the quadratic in (ii) • Expressing p in terms of b and c.	

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Since, p is in the first quadrant, $p > 0$.
Hence, $\angle POQ = 90° \Rightarrow p = \dfrac{b}{2c}$
The eccentricity e of the ellipse is given
by $e^2 = 1 - \dfrac{b^2}{a^2}$.
From (iii), $\angle POQ = 90° \Rightarrow$
$\dfrac{1}{b^2} = \dfrac{1}{a^2} + \dfrac{1}{(4c)^2} \Rightarrow 1 = \dfrac{b^2}{a^2} + \dfrac{b^2}{(4c)^2}$
$\therefore 1 - \dfrac{b^2}{a^2} = \dfrac{b^2}{(4c)^2} \quad \therefore e^2 = \left(\dfrac{b}{4c}\right)^2$
$\therefore e = \dfrac{b}{4c}$ 1 mark
Now, $p = \dfrac{b}{2c}$ and $e = \dfrac{b}{4c} \Rightarrow p = 2e$. 1 mark
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• Expressing e in terms of b and c.
• Dedudting $p = 2e$.
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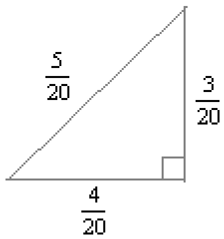
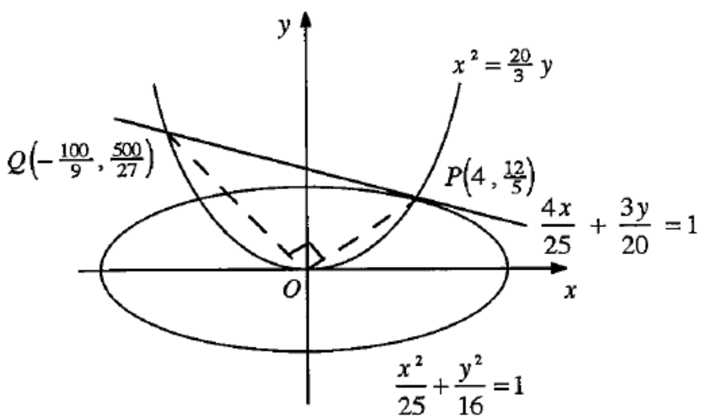
(v) A suitable set of a, b, c can be formed from any Pythagorean triad using
$\dfrac{1}{b^2} = \dfrac{1}{a^2} + \dfrac{1}{(4c)^2}$
Using 3, 4, 5 and dividing by the product of 4 and 5,
$\dfrac{1}{b} = \dfrac{5}{20} = \dfrac{1}{4}; \quad \dfrac{1}{a} = \dfrac{4}{20} = \dfrac{1}{5}; \quad \dfrac{1}{4c} = \dfrac{3}{20} = \dfrac{1}{4}$ satisfy
$\dfrac{1}{b^2} = \dfrac{1}{a^2} + \dfrac{1}{(4c)^2}$
Now, $a = 5$, $b = 4$ and $c = \dfrac{5}{3}$
1 mark


and $\therefore p = \dfrac{b}{2c} = \dfrac{6}{5}$.
Also, $pq = -4 \Rightarrow q = \dfrac{-10}{3}$. 1 mark


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(v)
• Finding an appropriate set of values for a, b, c.
• Finding the corresponding values of p and q.
• Sketch the required information.
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