Killara HIGH SCHOOL

CONSERVA PROGREDERE

STUDENT NUMBER

MATHEMATICS

EXTENSION 2

ASSESSMENT TASK NO:1

Date: 5th December 2012

TIME ALLOWED: *Reading time: 5 minutes*

Working time: 60 minutes

INSTRUCTIONS: **SECTION I:** 4 Multiple Choice questions: Circle your correct choice

SECTION II: Answer all 4 questions

Start each question in a new booklet

Marks may not be given for untidy and poorly arranged work.

All relevant working should be shown.

THIS QUESTION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM

OUTCOMES ASSESSED:

E1: Appreciates the creativity, power and usefulness of mathematics to solve a broad range of problems
E2: Chooses appropriate strategies to construct arguments and proofs in both concrete and abstract settings
E3: Uses the relationship between algebraic and geometric representations of complex numbers

	MARKS
SECTION I Multiple Choice questions	/4
SECTION II:	
QUESTION 1 Complex Numbers	/11
QUESTION 2 Complex Numbers	/13
QUESTION 3 Complex Numbers	/14
QUESTION 4 Complex Numbers	/ 8
	/50

This assessment task constitutes 10% of the Higher School Certificate Course Assessment.

Section I

4 marks
Attempt Questions 1 – 4

Use the Multiple Choice answer sheet for Questions 1 – 4.

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1. Simplify $\frac{1}{i^9}$

   (A) i
   (B) $-i$
   (C) 1
   (D) -1

2. If $z = r(\cos\theta + i\sin\theta)$, which of the following is z^{-1}?

   (A) $r(\cos\theta - i\sin\theta)$
   (B) $-r(\cos\theta - i\sin\theta)$
   (C) $\frac{1}{r}(\cos\theta - i\sin\theta)$
   (D) $\frac{1}{r}(\cos(-\theta) - i\sin(-\theta))$

3. $z^2 + z^{-2}$ is equivalent to

   (A) $2\cos\theta$
   (B) $2\cos 2\theta$
   (C) $2i\sin\theta$
   (D) $4i\sin\theta\cos\theta$

4. If $z = a + ib$, where $a \neq 0$ and $b \neq 0$,

   Which of the following is false?

   (A) $z - \bar{z} = 2bi$
   (B) $|z|^2 = |z||\bar{z}|$
   (C) $|z| + |\bar{z}| = |z + \bar{z}|$
   (D) $\arg(z) + \arg(\bar{z}) = 0$

Section II

46 Marks
Attempt Questions 5 – 8

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 5 − 8, your responses should include relevant mathematical reasoning and/or calculations.

Question 5 (11 marks) Use a SEPARATE writing booklet. **Marks**

a) If $z = 3 - 4i$ and $\omega = 2 + i$, find in the form $x + iy$:

(i) $z + \omega$ **1**

(ii) $z\overline{\omega}$ **1**

b) Consider the complex numbers $z_1 = \sqrt{2}\left(1 + i\sqrt{3}\right)$ and $z_2 = \sqrt{6}(1 + i)$

(i) Express $\dfrac{z_1}{z_2}$ exactly in the form $x + iy$ where x and y are real numbers. **2**

(ii) Write z_1, z_2 and $\dfrac{z_1}{z_2}$ in the modulus/argument form. **2**

(iii) Hence, find the exact value of $\cos\dfrac{\pi}{12}$. **2**

c) If $z = x + iy$ (x and y real) and $z^2 = -5 - 12i$, find z. **3**

End of Question 5

Question 6 (13 marks) Use a SEPARATE writing booklet. **Marks**

a) On separate diagrams, draw a neat sketch of the locus specified by:

(i) $\arg\left(z-\left(1+\sqrt{3}i\right)\right) = \dfrac{\pi}{3}$ **2**

(ii) $z^2 - \bar{z}^2 \leq 16i$ **2**

b) (i) Sketch the locus of $\arg\left(\dfrac{z+1}{z-1}\right) = \dfrac{\pi}{4}$ **2**

(ii) Find the equation of this locus. **2**

(iii) Find the maximum value of $|z|$. **1**

c) The complex number $z = x + iy$, where x and y are real, satisfies the parametric equation $z = 1+2i+t(3-4i)$ where t is a real parameter.

(i) Show that the Cartesian equation of the locus of P which represents in an Argand diagram is given by $4x+3y = 10$. **2**

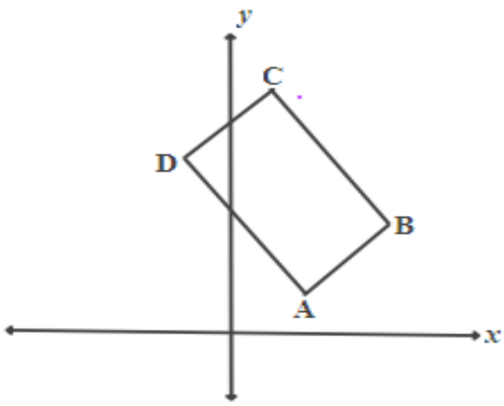
(ii) Hence, find the minimum value of $|z|$. **2**

End of Question 6

Question 7 (14 marks) Use a SEPARATE writing booklet. **Marks**

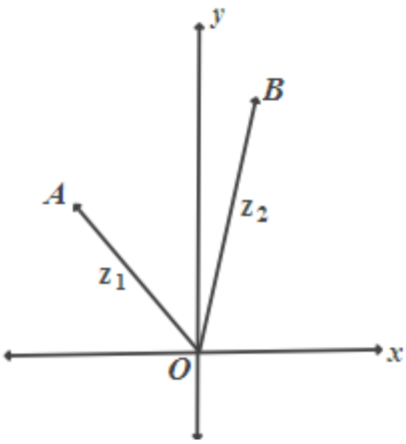
a) In the Argand diagram, ABCD is a rectangle and AD = 2 AB. **3**

The vertices A and B correspond to the complex numbers u and v.



Write the complex number that is represented by the point C.

b)



In an Argand diagram, vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ represent the complex numbers

$z_1 = 2\left(\cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}\right)$ and $z_2 = 2\left(\cos\frac{7\pi}{15} + i\sin\frac{7\pi}{15}\right)$ respectively.

(i) Show that ΔOAB is equilateral. **2**

(ii) Express $z_2 - z_1$ in modulus/argument form. **2**

Question 7 continues on next page

c) (i) By solving the equation $z^3 = 1$ find the three cube roots of 1 in modulus/argument form. **2**

(ii) If ω is a complex root of unity, prove that ω^2 also is a root. **1**

(iii) Hence, verify that $\omega^3 = 1$ and $1 + \omega + \omega^2 = 0$. **1**

(iv) Find the quadratic equation with integer coefficients that has roots $(4 + \omega)$ and $(4 + \omega^2)$ **3**

End of Question 7

Question 8 (8 marks) Use a SEPARATE writing booklet. **Marks**

a) Find the locus of β, if $\beta = \dfrac{z-2+i}{z+2-i}$ given $|z| = 1$ **3**

b) The complex numbers that correspond to the points A, B, C and D are z_1, z_2, z_3 and z_4.

(i) Interpret geometrically the following results:

(α) $z_1 - z_2 + z_3 - z_4 = 0$ **2**

(β) $z_1 + 2iz_2 - z_3 - 2iz_4 = 0$ **2**

(ii) What is the nature of the quadrilateral $ABCD$ if *both* relations are satisfied? **1**

End of paper

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Multiple Choice Answer Sheet

Completely fill the response oval representing the most correct answer.

1. A ⬭ B ⬭ C ⬭ D ⬭

2. A ⬭ B ⬭ C ⬭ D ⬭

3. A ⬭ B ⬭ C ⬭ D ⬭

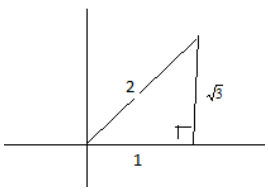
4. A ⬭ B ⬭ C ⬭ D ⬭

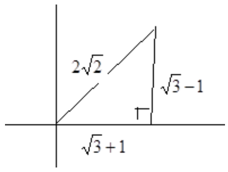
Marking scheme

Multiple Choice: **1.** B **2.** C **3.** B **4.** C

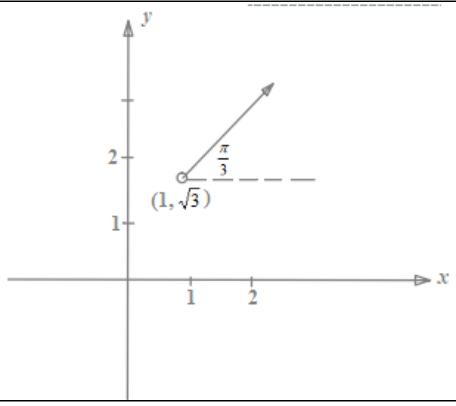
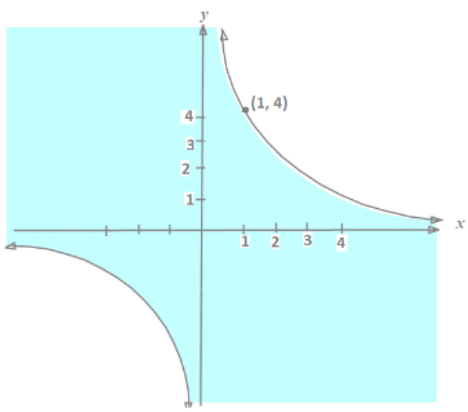
Question 5

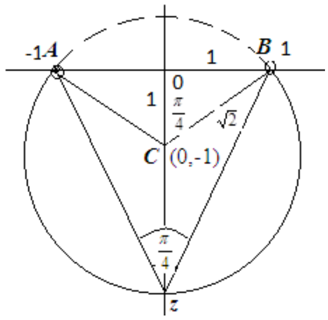
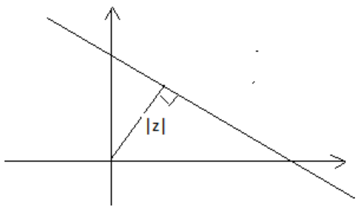
a) (i)

a)(i)	$z = 3 - 4i$ and $\omega = 2 + i$, $z + \omega = 3 + 2 + I(-4 + 1)$ $= 5 - 2i$	1 mark (correct answer	
(ii)	$z + \overline{\omega} = (3 - 4i)(2 - i)$ $= 6 - 4 - 3i - 8i$ $= 2 - 11i$	1 mark (correct answer)	
b)(i)	$z_1 = \sqrt{2}\left(1 + i\sqrt{3}\right)$ and $z_2 = \sqrt{6}(1 + i)$ $\frac{z_1}{z_2} = \frac{\sqrt{2}\left(1+i\sqrt{3}\right)}{\sqrt{6}(1+i)}$ $= \frac{\sqrt{2}\left(1+i\sqrt{3}\right)(1-i)}{\sqrt{6}(1+i)(1-i)}$ $= \frac{\sqrt{2}\left(1+i\sqrt{3}-i+\sqrt{3}\right)}{\sqrt{6}\times 2}$ $= \frac{1\left(1+\sqrt{3}+i\left(\sqrt{3}-1\right)\right)}{\sqrt{3}\times 2}$ $z = \frac{1+\sqrt{3}}{2\sqrt{3}} + i\frac{\sqrt{3}-1}{2\sqrt{3}}$	1 mark: Multiplies by the conjugate $(1-i)$ both the numerator and denominator. 1 mark: correctly simplifies and expresses the answer in the x + iy form.	
b)(ii)	$z_1 = \sqrt{2}\left(1 + i\sqrt{3}\right)$  $= \sqrt{2} \times 2cis\frac{\pi}{3}$ $z_2 = \sqrt{6}(1+i)$ $= \sqrt{6} \times \sqrt{2}cis\frac{\pi}{4}$ $= 2\sqrt{3}cis\frac{\pi}{4}$	1 mark : correct representation of z_1, z_2 in the modulus/argument form	

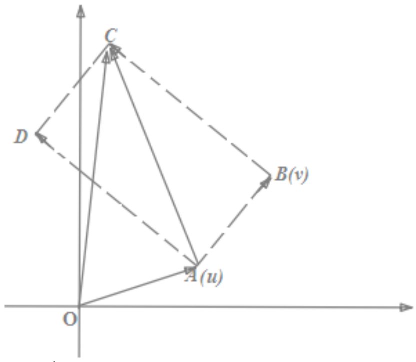
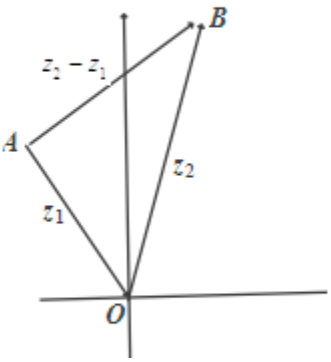
	$z = \dfrac{1+\sqrt{3}}{2\sqrt{3}} + i\dfrac{\sqrt{3}-1}{2\sqrt{3}}$ $= \dfrac{1}{2\sqrt{3}}\left(1+\sqrt{3}+i(\sqrt{3}-1)\right)$  $2\sqrt{2}$ cis θ, where θ = $\tan^{-1}\dfrac{\sqrt{3}-1}{\sqrt{3}+1}$ OR Using De Moivre's theorem, $\dfrac{z_1}{z_2} = \dfrac{2\sqrt{2}}{2\sqrt{3}} cis\left(\dfrac{\pi}{3}-\dfrac{\pi}{4}\right)$ $= \dfrac{\sqrt{2}}{\sqrt{3}} cis\left(\dfrac{\pi}{12}\right)$ $= \dfrac{\sqrt{6}}{3} cis\left(\dfrac{\pi}{12}\right)$	1 mark: gives $\dfrac{z_1}{z_2}$ in the modulus/argument form	
b(iii)	From(i) and (ii), $\dfrac{\sqrt{2}}{\sqrt{3}} cis\left(\dfrac{\pi}{12}\right)$ $= \dfrac{1+\sqrt{3}}{2\sqrt{3}} + i\dfrac{\sqrt{3}-1}{2\sqrt{3}}$ Comparing real parts, $\dfrac{\sqrt{2}}{\sqrt{3}}\cos\dfrac{\pi}{12} = \dfrac{1+\sqrt{3}}{2\sqrt{3}}$ $\therefore \cos\dfrac{\pi}{12} = \dfrac{\sqrt{3}}{\sqrt{2}} \times \dfrac{1+\sqrt{3}}{2\sqrt{3}}$ $= \dfrac{1+\sqrt{3}}{2\sqrt{2}}$	1 mark: Equates the two forms of $\dfrac{z_1}{z_2}$ 1 mark: Compares the real parts to give the answer.	

Question 6

a)	$z^2 = -5 - 12i$ $z = \sqrt{2^2 + (3i)^2 - 2\times 2\times 3i}$ $= \sqrt{(2-3i)^2}$ $= \pm(2 - 3i)$ Method 2 Let $-5 - 12i = (x - iy)^2$ $= x^2 - y^2 - 2xy\,i$ $X^2 - y^2 = -5$ **1 mark** $xy = 6$ $(x^2 + y^2)^2 = (x^2 - y^2)^2 + 4x^2y^2$ $= 25 + 4 \times 36 = 169$ $X^2 + y^2 = \pm 13$ $X^2 + y^2 \neq -13$ being sum of perfect squares of real numbers. $X^2 - y^2 = -5$ $X^2 + y^2 = 13$ $2x^2 = 8 \quad x^2 = 4$ $X = 2, \quad y = -3$ $X = -2, \quad y = 3$ $Z = \pm(2 - 3i)$	1 mark: forms the two simultaneous equations correctly. 1 mark: correctly solves for x 1 mark: using the x value, correctly solves for x and expresses the z.	
b)(i)		1 mark: correct head point and correct direction. 1 mark: the point $(1, \sqrt{3})$ is excluded and correct argument labelled.	
b)(ii)	$z^2 - \bar{z}^2 \leq 16i$ $(z + \bar{z})(z - \bar{z}) \leq 16i$ $2x.\,2iy \leq 16i$ $xy \leq 4$ **1 mark** 	1 mark: for correct locus equation 1 mark: Correct graph	

c)(i)	$\arg\left(\dfrac{z+1}{z-1}\right) = \dfrac{\pi}{4}$ 	1 mark: the major arc is in the correct quadrants 1 mark: the angle between $z - 1$ and $z + 1$ is labelled as $\dfrac{\pi}{4}$ and the points (-1, 0) and (1, 0) excluded	
(ii)	The angle subtended by chord at the circumference is $\dfrac{\pi}{4}$. Then the angle at the centre is $\dfrac{\pi}{2}$. $\therefore \angle OCB = \dfrac{\pi}{4}$ The locus is a major circle with centre (0,1) and radius $\sqrt{2}$. **1 mark** The locus is $x^2 + (y + 1)^2 = 2$.	1 mark: Calculates correct centre and radius of the circle 1 mark: Gives the correct locus	
(iii)	Maximum value of $\lvert z\rvert = 1 + \sqrt{2}$.	1 mark: correct answer	
b)	$Z = x + iy = 1+2i+t(3-4i)$ $x = 1 + 3t$ $\therefore t = \dfrac{x-1}{3}$ $y = 2 - 4t$ $= 2-4\left(\dfrac{x-1}{3}\right)$ $3y = 6 - 4x + 4$ $4x + 3y - 10 = 0$	1 mark: compares the real and imaginary parts. Gives $x = 1 + 3t$ and $y = 2 - 4t$ 1 mark: eliminates t and gives the correct locus.	
(ii)	 Min. value of $\lvert z\rvert$ is the perpendicular distance from (0,0) to the line $4x + 3y - 10 = 0$. Perpendicular distance = $\dfrac{\lvert 0+0-10\rvert}{\sqrt{4^2+3^2}} = 2$ Min. value of $\lvert z\rvert = 2$.	2 marks: correct answer 1 mark only: attempts to find the perpendicular distance from (0,0) to the line.	

Question 7

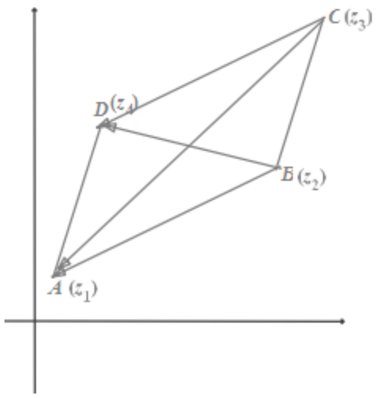
a)	 $\overrightarrow{AB} = v - u$ $\overrightarrow{AD} = 2i\,\overrightarrow{AB}$ $= 2i(v-u)$ **1 mark** $\overrightarrow{BC} = \overrightarrow{AD}$ $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ $\therefore\ \overrightarrow{AC} = v-u+2i(v-u)$ $\overrightarrow{OA} + \overrightarrow{AC} = \overrightarrow{OC}$ $\therefore \overrightarrow{OC} = u+v-u + 2i(v-u)$ C represents the complex number $v + 2i(v-u)$	3 mark: correct solution with all steps of working shown 2 mark: All correct steps to evaluate $\overrightarrow{AC}$ 1 mark: correctly writes $\overrightarrow{AB}$ and hence $\overrightarrow{AD}$ in terms of *v* and *u*.	*Poorly done. Many of the students realised* $\overrightarrow{AD} = 2i(v-u)$. *But after that adding the appropriate vectors to get* $\overrightarrow{AC}$ *and* $\overrightarrow{OC}$ *the caused a lot of problems.*
b)(i)	 $\|z_1\| = \|z_2\| = 2$ $\arg\left(\frac{z_1}{z_2}\right) = \frac{4\pi}{5} - \frac{7\pi}{15} = \frac{\pi}{3}$ $\therefore \angle OBA = \angle OAB = \left(\pi - \frac{\pi}{3}\right) \div 2 = \frac{\pi}{3}$ (angle sum of a triangle is 180^0) $\therefore$ OA = OB = AB = 2 sides opposite equal angles of a triangle are equal.	1 mark: evaluates $\arg\left(\frac{z_1}{z_2}\right) = \frac{\pi}{3}$ 1 mark: proves that all the sides of the triangle are equal giving reasons.	*Equilateral means all the* <u>*three*</u> *sides are equal.* $\therefore$ *Saying two sides are equal and one angle is* $\frac{\pi}{3}$ *would not be sufficient to prove* $\therefore \Delta OAB$ *is equilateral.* *Remember: the marker does not figure out the implication of your statements for you!*

	$\therefore \Delta OAB$ is equilateral.		
(ii)	$\vec{OA} = \vec{OB}\ cis\dfrac{\pi}{3}$ $\vec{AB} = z_2 - z_1$ Arg($z_2 - z_1$) = $\dfrac{\pi}{3} - (\pi - \arg z_1)$ **1 mark** $= \dfrac{\pi}{3} - (\pi - \dfrac{4\pi}{5}) = \dfrac{\pi}{3} - \dfrac{\pi}{5}$ $= \dfrac{2\pi}{15}$ $\lvert z_2 - z_1 \rvert = 2$ $z_2 - z_1 = 2cis\left(\dfrac{2\pi}{15}\right)$	1 mark: correctly evaluates arg($z_2 - z_1$) **1 mark** 1 mark: Correctly gives $\lvert z_2 - z_1 \rvert$ and expresses $z_2 - z_1$ in mod/arg form	*Many students had* $z_2 - z_1 =$ $2cis\dfrac{7\pi}{15} - 2cis\dfrac{4\pi}{5}$ $= 2cis\dfrac{-\pi}{15}$ *This is not a mathematically correct statement.* $\arg\left(\dfrac{z_1}{z_2}\right) = cis(\theta_1 - \theta_2)$ *and* $\left\lvert\dfrac{z_1}{z_2}\right\rvert = \dfrac{\lvert z_1\rvert}{\lvert z_2\rvert}$
c)(i)	$z^3 = 1$ Let z = r (cos θ + i sin θ) $\therefore$ $z^3 = r^3$ (cos 3θ + i sin 3θ) = 1 + 0i (using De Moivre's theorem) $\therefore$ r = 1 Comparing real and imaginary parts, Cos 3θ = 1 and sin 3θ = 0 **1 mark** $\therefore$ 3θ = 2kπ $\therefore$ θ = $\dfrac{2k\pi}{3}$ where k = 0, −1, 1 k = 0, z_0 = cos 0 + I sin 0 = 1 k = −1, z_1 = $\cos\dfrac{-2\pi}{3} + i\sin\dfrac{-2\pi}{3}$ k = 1, z_2 = $\cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}$ **1 mark**	1 mark: uses De Moivre's theorem to expand z^3 and equates the real and imaginary parts. 1 mark: solves cos 3θ = 1 and gives the three correct solutions in mod/arg form.	*This being a familiar question, many students skipped many steps. You must show all lines of working.* *In HSC you have little control on the marking scheme and you do not want to lose marks on questions that you know.*
(ii)	$Z^3 = 1$ The three roots are 1, $\cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}$ $\cos\dfrac{-2\pi}{3} + i\sin\dfrac{-2\pi}{3} = \cos\dfrac{4\pi}{3} + i\sin\dfrac{4\pi}{3}$. ω is a complex root and hence ≠ 1 Let ω = $\cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}$ *Then* $\cos\dfrac{4\pi}{3} + i\sin\dfrac{4\pi}{3} = \omega^2$	1 mark: Shows that the three roots are 1, ω and ω^2, stating $\omega \neq 1$.	*Well done.* *You should acknowledge that* ω ≠ 1*as it is a complex root.*

(iii)	$z^3 - 1 = 0$ Hence product of roots = c/a = 1 $\therefore 1\times\omega\times\omega^2 = 1$ $\omega^3 = 1$ Sum of roots = −b/a = 0 $\therefore 1+\omega+\omega^2 = 0$	1 mark: proves sum of roots = 0 and product of roots 1	*Some students assumed that* $\omega^3 = 1$. *You needed to prove it.*
(iii)	$(4+\omega)$ and $(4+\omega^2)$ sum of roots = $4+\omega+4+\omega^2$ = 8 + $\omega++\omega^2$ = 8 + −1 = 7 1 + $\omega+\omega^2$ = 0 from (ii) Product roots = $(4+\omega)(4+\omega^2)$ = 16 + 4ω^2 + 4ω + ω^3 = 16 + 4($\omega+\omega^2$) + 1 = 16 − 4 + 1 = 13 Then the quadratic is $z^2 - z(7) + 13 = 0$ $z^2 - 7z + 13 = 0$	1 mark: evaluates the sum of root = 7 1 mark: evaluates the product of roots = 13 1 mark: writes the quadratic	*Students who realised this was a bout sum and product of roots did very well. However, a few students failed to understand the the basic form of a quadratic equation.* $z^2 - (Sum) + (\Pr od) = 0$ *not +*

Question 8

a)	Find the locus of β if $\beta = \dfrac{z-2+i}{z+2-i}$ given $\lvert z\rvert$ = 1 We solve the equation for z and use $\lvert z\rvert = 1$ $\therefore \beta z + 2\beta - i\beta = z - 2 + i$ $\therefore z(\beta-1) = \beta(i-2) + (i-2)$ $ie\ \ z(\beta-1) = (i-2)(\beta+1)$ **1 mark** Now $ie\ \ \lvert z(\beta-1)\rvert = \lvert(i-2)(\beta+1)\rvert$ $ie\ \ \lvert z\rvert\lvert(\beta-1)\rvert = \lvert(i-2)\rvert\lvert(\beta+1)\rvert$ $ie\ \ 1\times\lvert(\beta-1)\rvert = \sqrt{(-2)^2+1^2}\,\lvert(\beta+1)\rvert$ As $\lvert z\rvert$ = 1 and $\lvert(i-2)\rvert = \sqrt{5}$ ie. $\lvert(\beta-1)\rvert = \sqrt{5}\lvert(\beta+1)\rvert$ **1 mark** We need to find the locus of β. Let $\beta = x + iy$ The distance of β from (1,0) = $\sqrt{5}$ times the distance from (−1, 0) I $\sqrt{(x-1)^2+y^2} = \sqrt{5}\left(\sqrt{(x+1)^2+y^2}\right)$ $(x-1)^2 + y^2 = 5\left((x+1)^2+y^2\right)$ ie $x^2+y^2+3x+1=0$	1 mark: makes z subject of the formula 1 mark: uses $\lvert z\rvert$ = 1 and writs the parametric equation of β. 1 mark: finds the locus equation of β.	*Very poorly done. The algebraic method is too difficult for this question. You may want to practise some locus questions both algebraically and geometrically to figure out the efficient method.*

	$x^2 + +3x + \frac{9}{4} + y^2 = -1 + \frac{9}{4}$ $\left(x + \frac{3}{2}\right)^2 + y^2 = \frac{5}{4}$ Which is a circle with centre ($\left(-\frac{3}{2}, 0\right)$ and radius $\frac{\sqrt{5}}{2}$		
b)(i) (α)	 $z_1 - z_2 + z_3 - z_4 = 0$ ie $z_1 - z_2 = z_4 - z_3$ $\overrightarrow{BA} = \overrightarrow{CD}$ $\lvert z_1 - z_2 \rvert = \lvert z_4 - z_3 \rvert$ Arg($z_1 - z_2$) = arg($z_4 - z_3$) ie. Opposite sides of the quadrilateral are equal and parallel. ABCD is a parallelogram.	1 mark: states that $z_1 - z_2 = z_4 - z_3$ 1 mark: demonstrates that opposite sides of ABCD are equal and parallel, hence a parallelogram.	*Those of you who got* $z_1 - z_2 = z_4 - z_3$ *did OK; While if you took* $z_1 + z_3 = z_2 + z_4$ *there is nowhere to go.* *Students needed to interpret the results in terms of the quadrilateral to get the full marks.*
(β)	$z_1 + 2iz_2 - z_3 - 2iz_4 = 0$ $z_1 - z_3 = 2iz_4 - 2iz_2$ $z_1 - z_3 = 2i(z_4 - z_2)$ $\overrightarrow{CA} = 2i\left(\overrightarrow{BD}\right)$ Ie. CA is double the length of BD and perpendicular to each other. Ie. One diagonal is double the length of the other and perpendicular to each other.	1 mark: proves that $\overrightarrow{CA} = 2i\left(\overrightarrow{BD}\right)$ 1 mark: makes the correct geometric interpretation.	*"interpret geometrically" was the question. You must explain your result.*
(ii)	If both conditions are satisfied it is a parallelogram with diagonals perpendicular to each other. Hence it is a rhombus with diagonals in the ratio 1:2.	1 mark	*Reasoning requires for this question.*