Killara HIGH SCHOOL

STUDENT NUMBER

MATHEMATICS
Extension 2

ASSESSMENT TASK NO:1
Date: 28th November 2013

TIME ALLOWED: *Reading time: 5 minutes*
Working time: 1 hour

INSTRUCTIONS: This paper consists of 3 questions
ANSWER all 3 questions
Start each question in a new booklet
Marks may not be given for untidy and poorly arranged work.
All relevant working should be shown.

THIS QUESTION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM

OUTCOMES ASSESSED:

E1	Appreciates the creativity, power and usefulness of mathematics to solve a broad range of problems
E3	Uses the relationship between algebraic and geometric representations of complex numbers
E4	uses efficient techniques for the algebraic manipulation required in dealing with questions such as those involving polynomials
E9	Communicates abstract ideas and relationships using appropriate notation and logical argument

	MARKS
QUESTION 1 Complex Number	/15
QUESTION 2 Complex Number	/12
QUESTION 3 Polynomials	/13
	/40

This assessment task constitutes 10% of the Higher School Certificate Course Assessment.

Answer each question on a SEPARATE writing booklet. Extra working booklets are available.

In Questions 1 – 3, your responses should include relevant mathematical reasoning and/or calculations.

Question 1 (15 marks) Use a SEPARATE writing booklet.

(a) Given that $A = 3 + 4i$ and $B = 2 - i$,

Express the following in the form $x + iy$

(i) $A\overline{B}$ **2**

(ii) $\dfrac{B}{A}$ **2**

(b) (i) If $z = x + iy$ such that $(x + iy)^2 = 7 + 24i$, find z. **2**

(ii) Hence, or otherwise solve the equation $z^2 + 3iz - (4 + 6i) = 0$. **2**

(c) Sketch the region on the Argand diagram where the inequalities **3**

$$|z - \overline{z}| < 2 \quad \text{and} \quad |z - 1| \geq 1$$

hold simultaneously.

(d) If $P = \sqrt{3} + i$

(i) Express P in the form $r(\cos\theta + i\sin\theta)$. **2**

(ii) Hence, or otherwise, find P^4 in the form $x + iy$. **2**

End of Question 1

Question 2 (12 marks) Use a SEPARATE writing booklet.

(a) Let $z = \cos\theta + i\sin\theta$ where $-\frac{\pi}{2} < \theta < 0$.

(i) Show that the imaginary part of $\frac{1}{z+i}$ is independent of θ. **2**

(ii) Given that $\operatorname{Im}\left(\frac{1}{z+i}\right) = -\frac{1}{2}$ and $\operatorname{Im}\left(\frac{1}{z+i}\right) = \frac{5}{6}\operatorname{Im} z$, **3**

express the complex number $\frac{1}{z+i}$ in the form $x+iy$.

(b) (i) If $\frac{x}{y} = \frac{a}{b}$, then prove that $\frac{x-a}{y-b} = \frac{a}{b}$ **1**

(ii) A, B and C are three points on an Argand diagram representing the complex **3**

numbers z_1, z_2, z_3 respectively.

Using the result from (i) or otherwise, prove that :

If $\frac{z_3 - z_2}{z_3 - z_1} = \frac{z_3 - z_1}{z_2 - z_1}$, then ΔABC is equilateral.

(c) Find the equation, in Cartesian form, of the locus of the point z if **3**

$$\operatorname{Re}\left(\frac{z-4}{z}\right) = 0$$

End of Question 2

Question 3 (13 marks) Use a SEPARATE writing booklet.

(a) When $P(x) = x^4 + ax^2 + bx + 2$ is divided by $x^2 + 1$, the remainder is $-x + 1$. **2**

Find the values of a and b.

(b) (i) Show that if α is a root of multiplicity r of $P(x) = 0$, $r > 1$, then α is a root of multiplicity $r - 1$ of $P'(x) = 0$. **2**

(ii) Show that $x = 3$ is a root of multiplicity two, of $x^4 - 5x^3 + 4x^2 + 3x + 9 = 0$. **1**

(iii) Hence, solve $x^4 - 5x^3 + 4x^2 + 3x + 9 = 0$ completely. **3**

(c) If $P(z) = z^8 - \frac{5}{2}z^4 + 1$. The complex number α is a root of $P(z) = 0$.

(i) Show that $i\alpha$ and $\frac{1}{\alpha}$ are also roots of $P(z) = 0$. **2**

(ii) Find one of the roots of $P(z) = 0$ in exact form. **1**

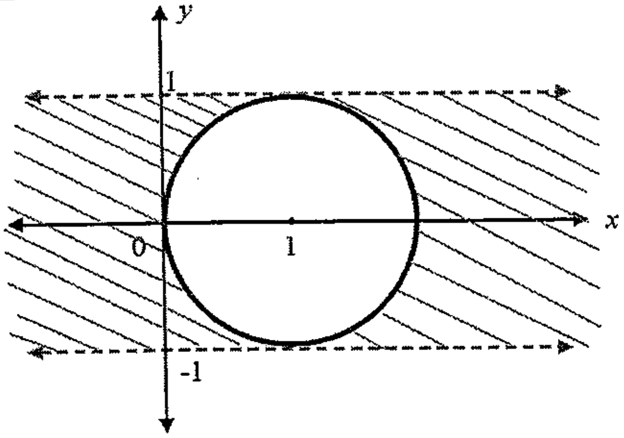
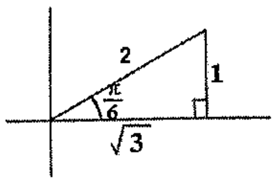
(iii) Hence, find all roots of $P(z) = 0$. **2**

End of paper

Marking scheme

Question 1

a)(i)	$A = 3+4i$ and $B = 2-i$, $A\overline{B} = (3+4i)(2+i)$ $= 6+3i+8i-4$ $= 2+11i$	1 mark: gives the correct conjugate $\overline{B}$ 1 mark: Correctly performs the multiplication.
(ii)	$\frac{B}{A} = \frac{2-i}{3+4i} = \frac{(2-i)(3-4i)}{(3+4i)(3-4i)}$ $= \frac{6-8i-3i-4}{9+\cancel{25}\ 16}$ $= \frac{2-11i}{\cancel{34}\ 25}$	1 mark: Multiplies the numerator and the denominator by the conjugate. 1mark: Simplifies correctly.
b)(i)	$(x+iy)^2 = 7+24i$ $x^2 - y^2 + 2xyi = 7+24i$ $x^2 - y^2 = 7$ $xy = 12$ **1 mark** $x = 4, \quad y = 3$ $7+24i = (4+3i)^2$ $\sqrt{7+24i} = \pm(4+3i)$	(i) 1 mark: correctly equates the real and imaginary parts to give the equations $x^2 - y^2 = 7 \quad and\ xy = 12$ 1 mark: gives the correct solutions for *x* and *y* and hence the correct solutions.
(ii)	$z^2 + 3iz - (4+6i) = 0$ $Z = \frac{-3i \pm \sqrt{(3i)^2 + 4(4+6i)}}{2}$ $= \frac{-3i \pm \sqrt{-9+16+24i}}{2}$ $= \frac{-3i \pm \sqrt{7+24i}}{2}$ $= \frac{-3i \pm (4+3i)}{2}$ $= 2, \quad -3i-2$	(ii) 1 mark: correctly applies the quadratic formula to solve the equation 1 mark: uses result from (i) $\sqrt{7+24i}$ to give the roots in simplest form.

c)	 $\|z-\bar{z}\| < 2$ $\|2iy\| < 2$ $\|y\| < 1$	1 mark: correct sketch of $\|z-\bar{z}\| < 2$. 1 mark: correct sketch of $\|z-1\| \geq 1$ 1 mark: correct combined region.
d) (i) (ii)	$P = \sqrt{3} + i$  $= 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$ $P^4 = 2^4\left(\cos\frac{4\pi}{6} + i\sin\frac{4\pi}{6}\right)$ $= 2^4\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$ **1 mark** $= 16\left(\frac{-1}{2} + i\frac{\sqrt{3}}{2}\right)$ **1 mark**	2 mark: correct answer Award mark: correct r or θ and expresses the answer in $rcis\theta$ form. (ii) 1 mark: applies the De Moivre's theorem to give P^4 in mod-argument form. 1 mark: Converts P^4 in $a+ib$ form.
Question 2		
a(i)	$z = \cos\theta + i\sin\theta$ $z + i = \cos\theta + i\sin\theta + i$ $= \cos\theta + i(\sin\theta + 1)$	1 mark: correctly expresses $\frac{1}{z+i}$ in simplified form with real denominators

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$$\frac{1}{z+i} = \frac{1}{\cos\theta + i(\sin\theta+1)}$$

$$= \frac{1}{\cos\theta + i(\sin\theta+1)} \times \frac{\cos\theta - i(\sin\theta+1)}{\cos\theta - i(\sin\theta+1)}$$

$$= \frac{\cos\theta - i(\sin\theta+1)}{\cos^2\theta + (\sin\theta+1)^2} \quad \textbf{1 mark}$$

$$\therefore im\left(\frac{1}{z+i}\right) = \frac{-(\sin\theta+1)}{\cos^2\theta + (\sin\theta+1)^2}$$

$$= \frac{-(\sin\theta+1)}{\cos^2\theta + \sin^2\theta + 2\sin\theta + 1}$$

$$= \frac{-(\sin\theta+1)}{1+2\sin\theta+1} \quad \text{as } |z| = 1$$

$$= \frac{-(\sin\theta+1)}{2(1+\sin\theta)}$$

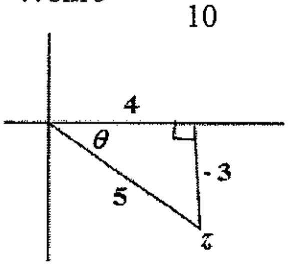
$$= \frac{-1}{2} \text{ which is independent of } \theta. \textbf{ 1 mark}$$

$$\text{Im}\left(\frac{1}{z+i}\right) = \frac{5}{6}\text{Im}\, z$$

$$\therefore \frac{-1}{2} = \frac{5}{6}\text{Im}\, z$$

$$\frac{-1}{2} = \frac{5\sin\theta}{6}$$

$$\therefore \sin\theta = \frac{-6}{10} = \frac{-3}{5} \quad \textbf{1 mark}$$



$$\text{Re}\left(\frac{1}{z+i}\right) = \frac{\cos\theta}{2(1+\sin\theta)} = \frac{4/5}{2(1+{-3}/5)} = 1$$

1 mark

$$\frac{1}{z+i} = 1 - \frac{1}{2}i \quad \textbf{1 mark}$$

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1 mark: proves that the imaginary part is independent of θ.

(ii)

1 mark: Uses $\text{Im}\left(\frac{1}{z+i}\right)$ from (i) to find $\sin\theta$

1 mark: Uses the $\sin\theta$ value to evaluate $\text{Re}\left(\frac{1}{z+i}\right)$

1 mark: writes $\text{Re}\left(\frac{1}{z+i}\right)$ in $a+ib$ form

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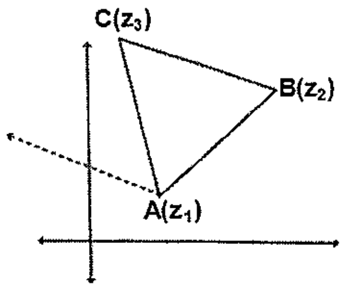
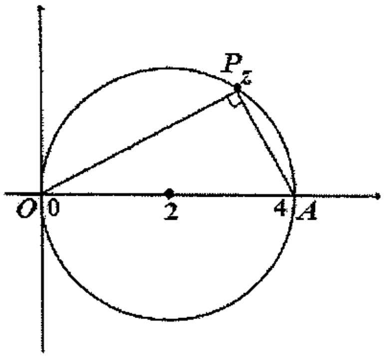
b(i)	$\dfrac{x}{y}=\dfrac{a}{b}$ $\therefore bx=ay$ $bx-ab=ay-ab$ $b(x-a)=a(y-b)$ $\dfrac{x-a}{y-b}=\dfrac{a}{b}$	1 mark: correct proof
(ii)	 $\dfrac{z_3-z_2}{z_3-z_1}=\dfrac{z_3-z_1}{z_2-z_1}$ Using the result from (i) $\dfrac{z_3-z_2-z_3+z_1}{z_3-z_1-z_2+z_1}=\dfrac{z_3-z_1}{z_2-z_1}$ $\dfrac{z_1-z_2}{z_3-z_2}=\dfrac{z_3-z_1}{z_2-z_1}$ Ie. $\dfrac{z_3-z_2}{z_3-z_1}=\dfrac{z_3-z_1}{z_2-z_1}=\dfrac{z_1-z_2}{z_3-z_2}$ $\therefore \arg\left(\dfrac{z_3-z_2}{z_3-z_1}\right)=\arg\left(\dfrac{z_3-z_1}{z_2-z_1}\right)=\arg\left(\dfrac{z_1-z_2}{z_3-z_2}\right)$ $\angle BCA = \angle CAB = \angle ABC$ ΔABC is equilateral.	1 mark: proves that $\dfrac{z_3-z_2}{z_3-z_1}=\dfrac{z_3-z_1}{z_2-z_1}=\dfrac{z_1-z_2}{z_3-z_2}$ 1 mark: states that $\arg\left(\dfrac{z_3-z_2}{z_3-z_1}\right)=\arg\left(\dfrac{z_3-z_1}{z_2-z_1}\right)=\arg\left(\dfrac{z_1-z_2}{z_3-z_2}\right)$ 1 mark: states each of the angles of ΔABC are equal. Gives correct proof.
c.	$\mathrm{Re}\left(\dfrac{z-4}{z}\right)=0$ $\dfrac{z-4}{z}$ is imaginary. $\dfrac{z-4}{z} = \pm i$	1 mark: states that $\dfrac{z-4}{z}$ is imaginary

Question 3

(a)	$P(x) = x^4 + ax^2 + bx + 2$ is divided by $x^2 + 1$ $\therefore P(i) = (i)^4 + a(i)^2 + b(i) + 2 = -i + 1$ $1 - a + bi + 2 = -i + 1$ $3 - a + bi = -i + 1$ Comparing real and imaginary parts, $3 - a = 1 \quad \therefore \ a = 2$ and $\quad b = -1$	2 marks: correct answer from correct working. Award 1 mark: if demonstrates $P(i) = -i + 1$ of $P(-i) = i + 1$ (demonstrates correct application of remainder theorem)
b(i)	If α is a root of multiplicity r of $P(x)$, then $P(x) = (x - \alpha)^r Q(x) = 0$ where $Q(\alpha) \neq 0$. Then $P'(x) = (x - \alpha)^r Q'(x) +$ $r(x - \alpha)^{r-1} Q(x)$ $= (x - \alpha)^{r-1} [(x - \alpha) Q'(x) + rQ(x)]$ $= (x - \alpha)^{r-1} Q_1(x) = 0$, where $Q_1(\alpha) \neq 0$ Hence, if α is a root of multiplicity r of $P(x) = 0$ then α is a root of multiplicity $r - 1$ of $P'(x) = 0$.	1 mark: correctly expresses $P(x)$ in the division transformation form with α having a multiplicity of r. 1 mark: Uses product rule to establish that $P'(x)$ has a root of multiplicity $r - 1$.
(ii)	Let $P(x) = x^4 - 5x^3 + 4x^2 + 3x + 9$ $P(x) = 3^4 - 5 \times 3^3 + 4 \times 3^2 + 3 \times 3 + 9$ $= 0$ $\therefore x = 3$ is a root of $P(x) = 0$ $P'(x) = 4x^3 - 15x^2 + 8x + 3$ $P'(3) = 4 \times 3^3 - 15 \times 3^2 + 8 \times 3 + 3$ $= 0$ $\therefore \ x = 3$ is a root of $P'(x) = 0$. $P''(x) = 12x^2 - 30x + 8$ $P''(3) = 12 \times 3^2 - 30 \times 3 + 8$	1 mark: Correctly proves $P(3) = 0$ and $P'(3) = 0$ 1 mark: Establishes that $P''(3) \neq 0$ and hence $x = 3$ is root of multiplicity 2.

	$z-4 = \pm iz$ $\therefore PA \perp OP$ Angle in a semi-circle = $90°$.  $\therefore$ locus of *P* is a circle with centre (2, 0) and radius 2 cm. Locus of *P* is $(x-2)^2 + y^2 = 4$ Or Z = $x+iy$ $\frac{z-4}{z} = 1-\frac{4}{z}$ $= 1-\frac{4}{x+iy}$ $= 1-\frac{4(x-iy)}{x^2+y^2}$ $\text{Re}\left(\frac{z-4}{z}\right) = \text{Re}\left(\frac{x^2+y^2-4x+4iy}{x^2+y^2}\right) = 0$ $= \frac{x^2+y^2-4x}{x^2+y^2} = 0$ $= x^2+y^2-4x = 0$ $= x^2-4x+4+y^2 = 0+4$ $(x-2)^2 + y^2 = 4$	1 mark: interprets that $PA \perp OP$ 1 mark: gives the correct locus of P giving reasons.

	Hence, the other roots are $\frac{1}{-\sqrt[4]{2}}, \frac{1}{\sqrt[4]{2}}, \frac{1}{-\sqrt[4]{2}i}, \frac{1}{\sqrt[4]{2}i}.$ $\therefore z = \sqrt[4]{2}i,\ \sqrt[4]{2}i,\ \sqrt[4]{2},\ -\sqrt[4]{2}$	

c) (i)	$P(z)=z^8-\frac{5}{2}z^4+1$ $P(\alpha)=\alpha^8-\frac{5}{2}\alpha^4+1 \quad = \quad 0$ $P(i\alpha)=(i\alpha)^8-\frac{5}{2}(i\alpha)^4+1$ $= \quad \alpha^8-\frac{5}{2}\alpha^4+1=0$ $\therefore i\alpha$ is a root of $P(z)=0$ **1 mark** $P\left(\frac{1}{\alpha}\right)= \left(\frac{1}{\alpha}\right)^8-\frac{5}{2}\left(\frac{1}{\alpha}\right)^4+1$ $= \left(\frac{1}{\alpha^8}\right)-\frac{5}{2}\left(\frac{1}{\alpha^4}\right)+1=$ $\frac{1}{\alpha^8}\left(1-\frac{5}{2}\alpha^4+\alpha^8\right)$ $\alpha\neq 0. \quad 1-\frac{5}{2}\alpha^4+\alpha^8=0$ $P\left(\frac{1}{\alpha}\right)=0$ $\frac{1}{\alpha}$ is a root. **1 mark**	1 mark: substitutes $i\alpha$ into the polynomial and gives the correct proof. 1 mark: substitutes $\frac{1}{\alpha}$ into the polynomial and gives the correct proof.
(ii)	$z^8-\frac{5}{2}z^4+1=0$ $2z^8-5z^4+2=0$ $(z^4-2)(2z^4-1)=0$ $\therefore z=\sqrt[4]{2}i$ is a root	1 mark: correctly factorises to give one root
(iii)	$P(z)$ is an even function. Hence, $\therefore z=-\sqrt[4]{2}i$ is also a root. Using (i) if α is a complex root, then $i\alpha$ is also a root. $\therefore z=i(\sqrt[4]{2}i) = -\sqrt[4]{2}$ is a root. Similarly, if $z=-\sqrt[4]{2}i$ is a root, then $z=i(-\sqrt[4]{2}i)$ $= \sqrt[4]{2}$ is a root. Using (i), if α, then $\frac{1}{\alpha}$ is also a root.	1 mark: Uses the even function property or the conjugate root property, since the coefficients of $P(z)=0$ are real. 1 mark: refers to result from (i) to give $i\alpha$ as a root 1 mark: refers to result from (i) to give $\frac{1}{\alpha}$ as a root

	$= 26 \neq 0$ Hence $x = 3$ is a root of multiplicity 2.	
(iii)	$x^4 - 5x^3 + 4x^2 + 3x + 9 = (x-3)^2 Q(x)$ $$\begin{array}{r l} & x^2 + x + 1 \\ x^2 - 6x + 9 \,\big) & x^4 - 5x^3 - 4x^2 + 3x + 9 \\ & \underline{x^4 - 6x^3 + 9x^2} \\ & x^3 - 5x^2 + 3x \\ & \underline{x^3 - 6x^2 + 9x} \\ & x^2 - 6x + 9 \\ & \underline{x^2 - 6x + 9} \\ & 0 \end{array}$$ $\therefore P(x) = (x-3)^2(x^2+x+1) = 0$ $X = 3,\ 3,$ $\dfrac{-1 \pm \sqrt{1-4}}{2} = \dfrac{-1 \pm \sqrt{3}i}{2}$	1 mark: Performs correct polynomial division 2 ~~1~~ mark: Correctly solves the quadratic to give the correct solutions of *x*