Killara
HIGH SCHOOL

STUDENT NUMBER

MATHEMATICS
EXTENSION 2

ASSESSMENT TASK NO: 2

Date: 13^{th} March 2013

TIME ALLOWED: *Reading time: 5 minutes*
Working time: 3 hours

INSTRUCTIONS: This paper consists of 10 multiple choice questions and 6 free-response questions
ANSWER all 16 questions
A multiple choice sheet and table of standard integrals are attached at the back of this paper
Start each question in a new booklet
Marks may not be given for untidy and poorly arranged work.
All relevant working should be shown.

THIS QUESTION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM

OUTCOMES ASSESSED:

E1	appreciates of the scope, usefulness, beauty and elegance of mathematics to solve a broad range of problems
E3	uses the relationship between algebraic and geometric representations of Complex numbers
E4	uses efficient techniques for the algebraic manipulation required in dealing with questions such as those involving conic sections and polynomials
E6	combines the ideas of algebra and calculus to determine the important features of the graphs of a wide variety of functions
E9	communicates abstract ideas and relationships using appropriate notation and logical argument

MC	
Q1	
Q2	
Q3	
Q4	
Q5	
Q6	

This assessment task constitutes 25% of the Higher School Certificate Course Assessment.

Section I

10 marks
Attempt Questions 1 – 10

Use the Multiple Choice answer sheet for Questions 1 – 10.

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1. The polynomial equation $P(z) = 0$ has one complex coefficient. Three of the roots of this equation are $z = 3 + i$, $z = 2 - i$ and $z = 0$.

   The minimum degree of $P(z)$ is

   (A) 2
   (B) 3
   (C) 4
   (D) 5

2. Given that $\int_1^4 f(x)dx = 6$, using the substitution $u = 5 - x$, what is the value of $\int_1^4 f(5-x)dx$?

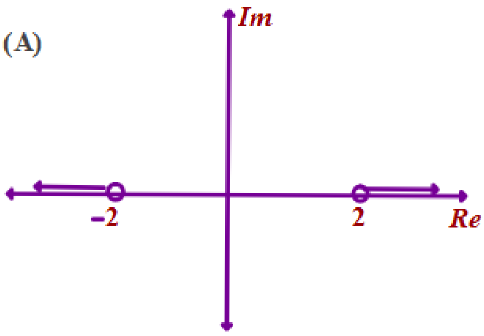
   (A) 6
   (B) 3
   (C) -1
   (D) -6

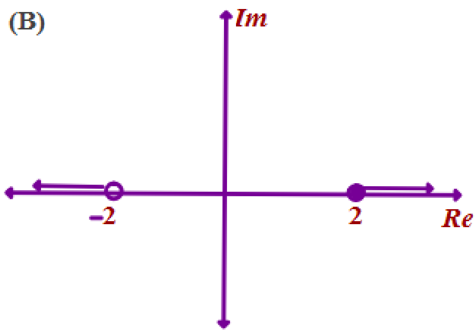
3. If ω is an imaginary cube root of unity, then $\left(1+\omega-\omega^2\right)^7$ is equal to

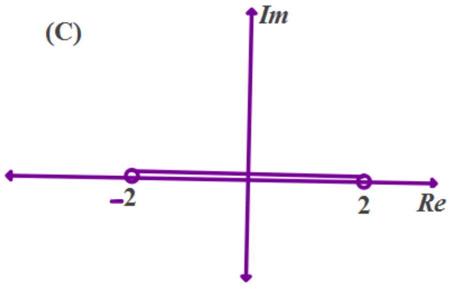
   (A) 128ω
   (B) -128ω
   (C) $128\ \omega^2$
   (D) $-128\ \omega^2$

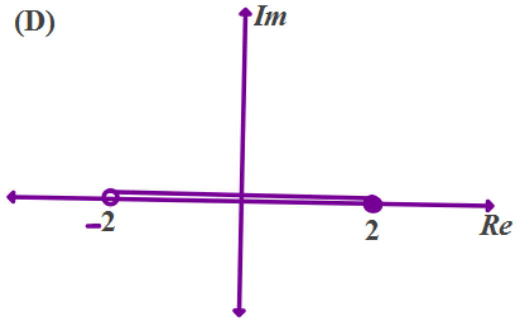
4. Which of the following diagram represents the locus of z satisfying the condition

$\arg\left(\dfrac{z-2}{z+2}\right) = 0$?









5 $P(z)$ is a polynomial of degree 5 with $P(z) = P(\bar{z})$.

Which of the following must be true about the polynomial equation $P(z) = 0$?

(A) All the roots can be real
(B) Only four of the roots can be real
(C) Only three of the roots can be real
(D) Only one of the roots can be real

6 What is the solution to the inequation $\dfrac{x(5-x)}{x-4} \geq -3$?

(A) $2 \leq x < 4$ or $x \geq 6$
(B) $1 \leq x < 4$ or $x \geq 5$
(C) $4 < x \leq 6$ or $x \leq 2$
(D) $4 > x \leq 5$ or $x \leq 1$

7 What are the values of p and q such that if $1 - i$ is a root of the equation

$z^3 + pz + q = 0$

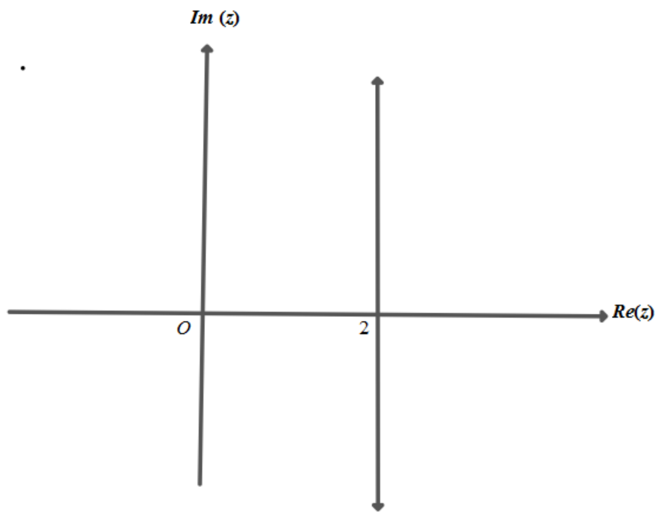
(A) $p = 2$ and $q = 4$
(B) $p = 2$ and $q = -4$
(C) $p = -2$ and $q = 4$
(D) $p = -2$ and $q = -4$

8 What is the value of $\frac{dy}{dx}$ at the point $(1,2)$ if $xy^2 + 2xy = 8$?

(A) $\frac{-5}{2}$ (B) $\frac{-4}{3}$

(C) -1 (D) $\frac{-1}{2}$

9

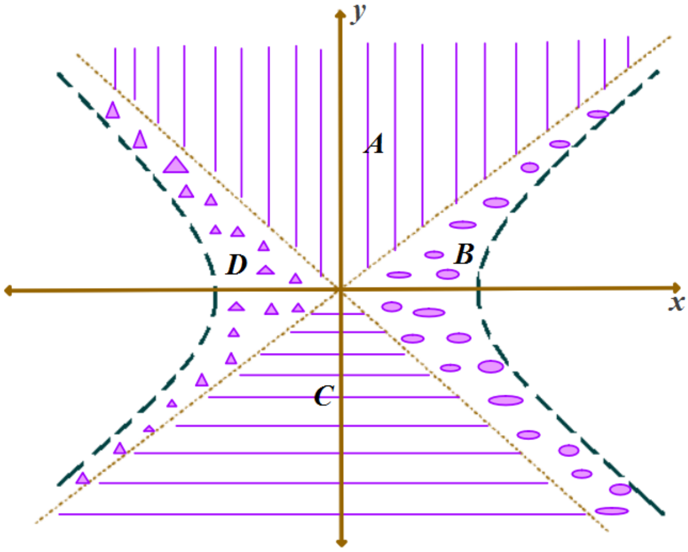


Which of the following is NOT a valid algebraic description of this locus?

(A) $\operatorname{Re} z = 2$ (B) $|z| = |z-4|$

(C) $\operatorname{Arg}(z-4) + \arg z = \pi$ (D) $z + \bar{z} = 4$

10. The endpoints of the chord of contact of tangents from the point T will lie on different branches of the hyperbola if T lies in the region



(A) A or B

(B) A or C

(C) C or D

(D) A or D

End of Section I

Section II

90 Marks
Attempt Questions 11 – 16

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11 − 16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet. **Marks**

a) Two complex numbers w and z are such that **2**

$$w + z = 6 + 2i$$

$$w - 3z = \frac{20}{2-i}$$

Find w and z, giving each answer in the form $x + iy$.

b) The points P and Q represent complex number p and q respectively, where

$$p = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4},$$

$$q = 2\cos\frac{\pi}{4} + 2i\sin\frac{\pi}{4}.$$

Find the moduli of the complex number a and b where

(i) $a = p + \overline{p}$ **1**

(ii) $b = q^2$ **1**

Question 11 continued on page 6

c) The complex number z is given by $z = \cos\theta + i\sin\theta$, **2**

Show that $|1+z|^2 = 2(1+\cos\theta)$.

d) On the Agrand diagram, sketch the locus of z described by **2**

$|z-2-3i| = Im\ (z) - 1$,

(clearly showing the main features and the Cartesian equation)

e) The complex number $z = x + iy$, where x and y are real, satisfies the parametric equation $z = 1 + 2i + t(3-4i)$ where t is a real parameter.

(i) Show that the Cartesian equation of the locus of the point P which represents z in an Argand diagram is given by $4x + 3y = 10$. **2**

(ii) Hence, find the minimum value of $|z|$. **2**

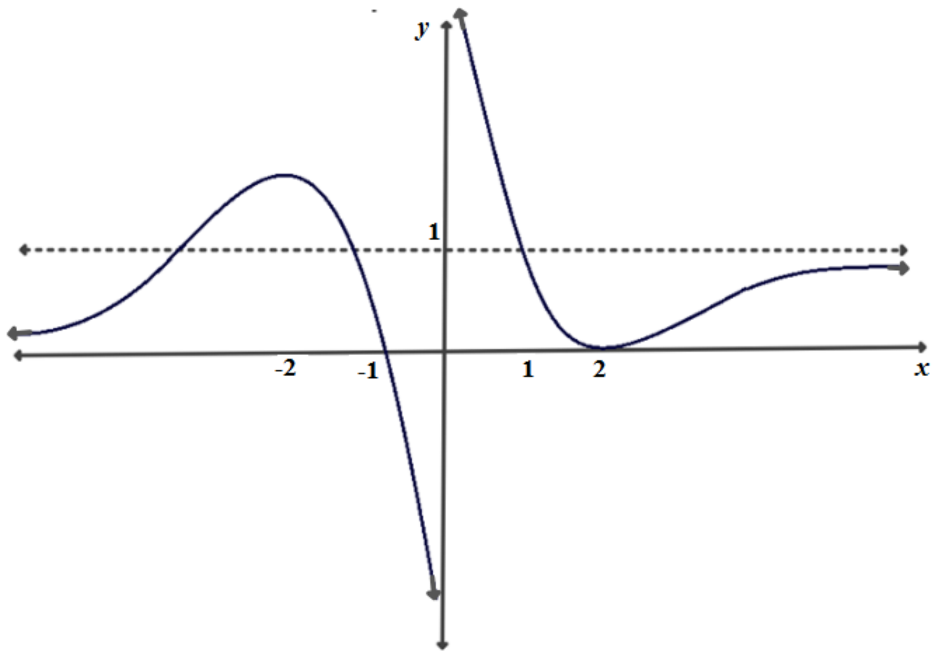
f) Solve the equation $z^5 + 16z = 0$, expressing each solution in the form $z = a + ib$ where a and b are real. **3**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet. **Marks**

a) The diagram below is a sketch of the function $y = f(x)$.

the lines $x = 0$, $y = 0$ and $y = 1$ are asymptotes.



Using the sketch of the function given above, sketch each of the graphs below.

In each case, clearly label any maxima or minima, intercepts and equations.

(i) $y = \dfrac{1}{f(x)}$ **2**

(ii) $y = (f(x))^2$ **2**

(iii) $y = \sin^{-1}(f(x))$ **2**

(iv) $y = \ln(f(x))$ **2**

Question 12 continued on page 8

Question 12 continued

b) Consider the graph of the function $f(x) = \frac{(x-2)^2}{x-1}$

(i) Determine the behaviour of $f(x)$ as $x \to \infty$ and $x \to -\infty$ and as x approaches other critical values. **2**

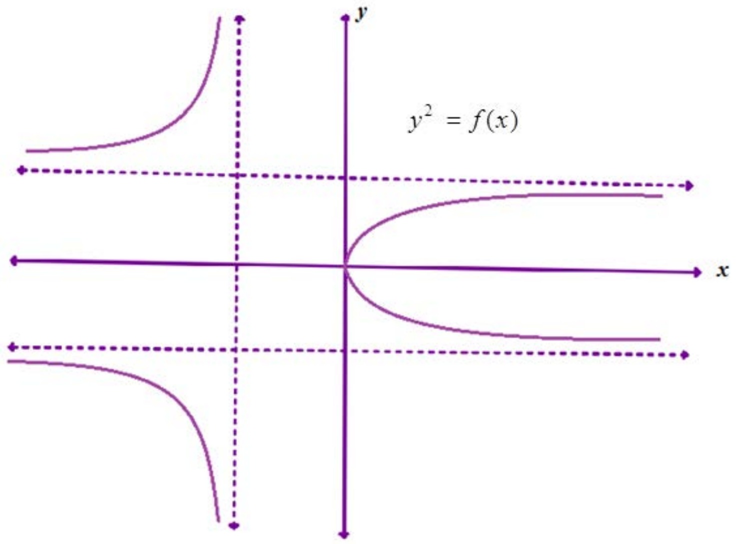
(ii) Hence, sketch the graph of $f(x) = \frac{(x-2)^2}{x-1}$ on the number plane, labelling clearly its Asymptotes, x and y intercept(s). **3**

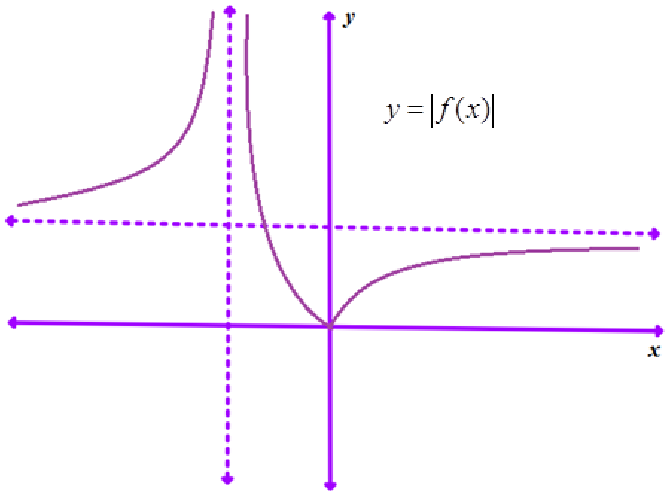
(iii) Hence, sketch the graph of $y = \sqrt{f(x)}$ on the same number plane. **2**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet. **Marks**

a) Given the graphs of $y^2 = f(x)$ and $y = |f(x)|$ as shown.





(i) Explain the shape and position of $y = f(x)$ in relation to the above graphs. **2**

(ii) Hence, sketch the graph of $y = f(x)$ **2**

(iii) Hence, sketch the graph of $y = f(|x|)$ **2**

Question 13 continued on page 10

b) Determine the value of p for which

$$\frac{x^2}{4+p} + \frac{y^2}{9+p} = 1 \text{ defines}$$

(i) an ellipse **1**

(ii) a hyperbola **2**

c) A hyperbola has the equation $\frac{x^2}{16} - \frac{y^2}{9} = 1$.

(i) Find the coordinates of the foci. **1**

(ii) Write the equations of the directrices. **1**

(iii) Hence, sketch the graph of this hyperbola, labelling clearly, its vertices, foci and asymptotes. **2**

d) Find the Cartesian equation of the complex number $z = x + iy$ if **2**

$$|z-2| + |z+2| = 10$$

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet. **Marks**

a) Given that $-\frac{1}{2}+\frac{\sqrt{3}}{2}i$ is a zero of the polynomial $P(x) = x^3+2x^2+2x+1$, **3**

factorise $P(x)$ over the field of real numbers

b) The equation $x^3+2x^2-20x-16=0$ has roots α, β and γ.

(i) Find the cubic equation with roots α^2, β^2 and γ^2. **2**

(ii) Hence, find the values of $\alpha^3+\beta^3+\gamma^3$. **2**

c) (i) Prove that for any polynomial $P(x)$, if k is a zero of multiplicity 2, then **2**

k is also a zero of $P'(x)$.

(ii) Show that $x = 1$ is a double root of $x^{2n}-nx^{n+1}+nx^{n-1}-1=0$ **2**

d) The notation $\sum_1^r k^2$ stands for $1^2 + 2^2 + 3^2 + ... + r^2$.

Given that $S_r = \sum_1^r k^2$, prove by mathematical induction or otherwise that **4**

$$nS_n - \sum_1^{n-1} S_r = \sum_1^n r^3 \quad \text{for all intergral values of } n > 1.$$

End of Question 14

Question 15 (15 marks) Use a SEPARATE writing booklet. **Marks**

a) (i) Show that $\cos(p+q)+\cos(p-q)=2\cos p\cos q$, and deduce that **2**

$$\cos\alpha+\cos\beta=2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2.}$$

$P(a\cos\alpha, b\sin\alpha)$ and $Q(a\cos\beta, b\sin\beta)$ lie on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

(ii) Show that the chord PQ has equation **3**

$$\frac{x}{a}\cos\frac{\alpha+\beta}{2}+\frac{y}{b}\sin\frac{\alpha+\beta}{2.}=\cos\frac{\alpha-\beta}{2}$$

(iii) Show that if PQ is a focal chord of the ellipse, then **3**

$$PQ = 2a\left\{1-e^2\cos^2\frac{\alpha+\beta}{2}\right\}.$$

b) $P(a\cos\alpha, b\sin\alpha)$, $Q(a\cos\beta, b\sin\beta)$ and $R(a\cos\theta, b\sin\theta)$ lie on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and PQ is focal chord, RT is the diameter through R, and $PQ \parallel RT$.

(i) Show this information on a sketch.

(ii) Deduce that T has coordinates $(-a\cos\theta, -b\sin\theta)$ and $RT = 2RO$. **2**

(iii) Show that $RT^2 = 4a^2\{1-e^2\sin^2\theta\}$. **2**

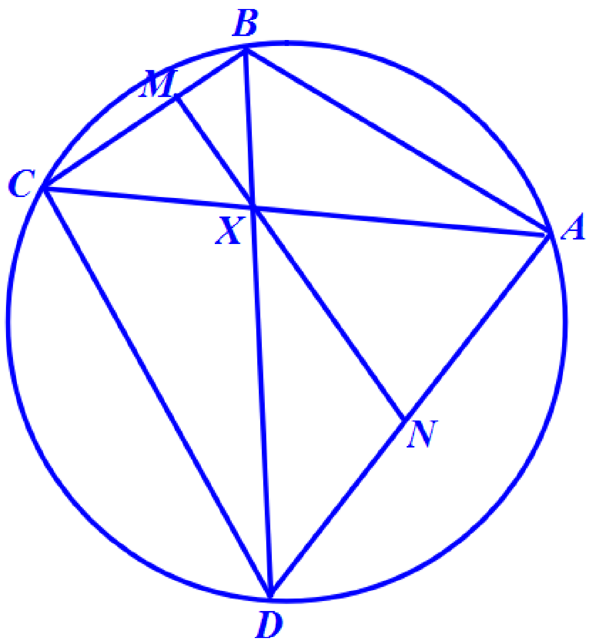
(iv) Show that $\tan\theta\tan\frac{\alpha+\beta}{2}=-1$. **1**

(v) Show that $RT^2 = 2a\,PQ$. **2**

Question 16 (15 marks) Use a SEPARATE writing booklet. **Marks**

a) Use the substitution $u = x - 1$ to find $\int \frac{x}{\sqrt{x-1}}\, dx$ **3**

b)



ABCD is a cyclic quadrilateral. The diagonals *AC* and *BD* intersect at right angles at *X*.

M is the midpoint of *BC*. *MX* produced meets *AD* at *N*.

(i) Copy the diagram showing the above information.

(ii) Show that $\angle MBX = \angle MXB$ **2**

(iii) Show that *MN* is perpendicular to *AD*. **3**

Question 16 continued on page 14

c) If $P(x)$ is divided by $(x-a)(x-b)$ a remainder of $R(x)$ is obtained. **3**

Show that the remainder is given by

$$R(x) = \left(\frac{P(a)-P(b)}{a-b}\right)x + \frac{aP(b)-bP(a)}{a-b}$$

d) The points P, Q and R represent three different non zero numbers p, q and r respectively on the Argand diagram. **4**

Show that these points must satisfy the condition $q^2 + r^2 + 2p^2 = 2(pq+pr)$ in order that ΔPQR forms a right angled triangle with $PQ = PR$

End of Paper

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

STUDENT NUMBER

Multiple Choice Answer Sheet

Completely fill the response oval representing the most correct answer.

1. A ⬭ B ⬭ C ⬭ D ⬭

2. A ⬭ B ⬭ C ⬭ D ⬭

3. A ⬭ B ⬭ C ⬭ D ⬭

4. A ⬭ B ⬭ C ⬭ D ⬭

5. A ⬭ B ⬭ C ⬭ D ⬭

6. A ⬭ B ⬭ C ⬭ D ⬭

7. A ⬭ B ⬭ C ⬭ D ⬭

8. A ⬭ B ⬭ C ⬭ D ⬭

9. A ⬭ B ⬭ C ⬭ D ⬭

10. A ⬭ B ⬭ C ⬭ D ⬭

2013 half-yearly Marking scheme

1. B

2. B

$U = 5 - x$

$Du = -dx$

When $x = 1$, $u = 4$

When $x = 4$, $u = 1$

$$\int_1^4 f(5-x)dx = -\int_4^1 f(u)du$$

$$= \int_1^4 f(u)du = 6$$

3. D

$$\left(1+\omega-\omega^2\right)^7 = \left(-\omega^2-\omega^2\right)^7$$

$$= \left(-2\omega^2\right)^7 = -128\,\omega^{14}$$

$$= -128\,\omega^2$$

4. A

5. A

6. C

$$\frac{x(5-x)}{x-4} \geq -3$$

$$x(5-x)(x-4) \geq -3(x-4)^2$$

$$x(5-x)(x-4) + 3(x-4)^2 \geq 0$$

$$(x-4)\{x(5-x)+3(x-4)\} \geq 0$$

$$(x-4)\{5x-x^2+3x-12\} \geq 0$$

$$(x-4)\{x^2+8x+12\} \leq 0$$

$$(x-4)(x+2)(x+6) \leq 0$$

$$4 < x \leq 6 \quad \text{or} \quad x \leq 2$$

7. C

$(1-i)^3 + p(1-i) + q = 0$

$-2 - 2i + p - pi + q = 0$

Comparing real and imaginary parts

$-2 + p + q = 0$

$-2 - p = 0$

$P = -2$ and $q = 4$

8. B

$$xy^2 + 2xy = 8$$

$$2xyy' + y^2 + 2xy' + 2y = 0$$

$$y'\{2xy+2x\} = -y^2 - 2y$$

$$y' = \frac{-y^2-2y}{2xy+2x}$$

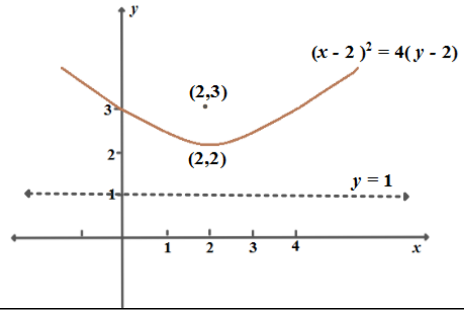
At (1,2) $\quad y' = \dfrac{-4}{3}$

9. C

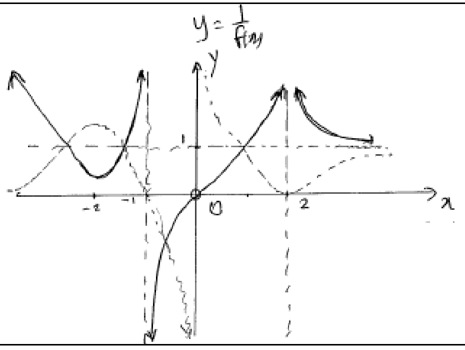
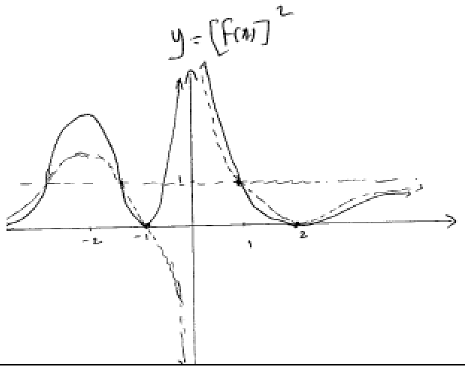
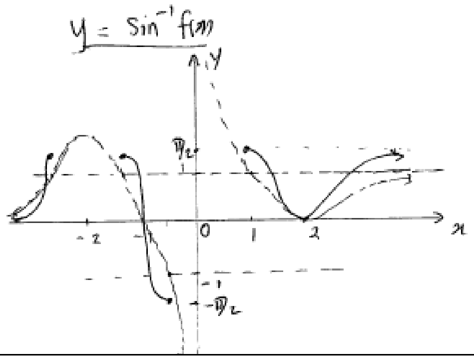
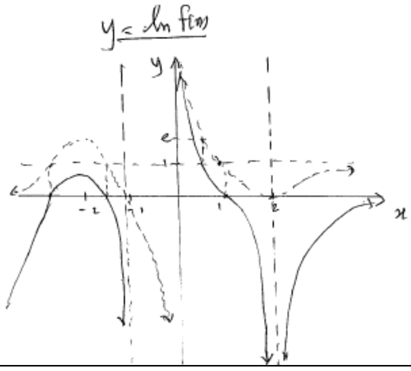
10. B

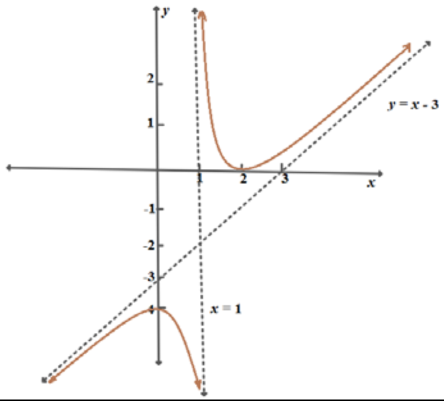
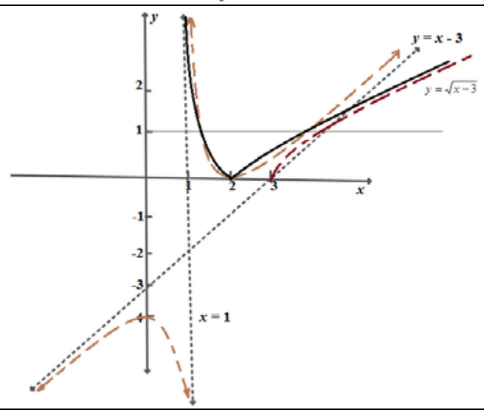
Question 11

a)	$w + z = 6 + 2i$ $w - 3z = \frac{20}{2-i} \times \frac{2+i}{2+i}$ $= 8 + 4i$ Subtracting, $4z = -2 - 2i$ $\therefore$ $z = -\frac{1}{2} - \frac{1}{2}i$ **1 mark** $\therefore$ $w = 6 + 2i + \frac{1}{2} + \frac{1}{2}i$ $= \frac{13}{2} + \frac{5}{2}i$ **1 mark**	1 mark: correctly evaluates the value of z 1 mark: uses the value of z to correctly evaluate *w*.	*Those students who approached it as simultaneous equations did well.*
b)(i) (ii)	$p = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$ $= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ $a = p + \overline{p}$ $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$ $\frac{2}{\sqrt{2}} = \frac{2}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \sqrt{2}$ $\lvert a\rvert = \sqrt{2}$ **1 mark** $b = q^2$ $= \left(2\cos\frac{\pi}{4} + 2i\sin\frac{\pi}{4}\right)^2$ $4 \operatorname{cis}\frac{\pi}{2}$ using De Moivre theorem $= 4i$ $\therefore$ $\lvert b\rvert = 4$	1 mark: correct answer 1 mark: correct answer	*Some students found a and b, but not* $\lvert a\rvert$ and $\lvert b\rvert$
c)	$z = \cos\theta + i\sin\theta$ $\lvert 1+z\rvert^2 = (1+z)(1+z)$ $= 1 + z + \overline{z} + z\overline{z}$ **1 mark** $= 1 + \cos\theta + i\sin\theta +$ $\cos\theta - i\sin\theta +$ $\cos^2\theta + \sin^2\theta$ $= 2 + 2\cos\theta$ $= 2(1 + \cos\theta)$ **1 mark** or $\lvert 1+z\rvert^2 = \lvert 1 + \cos\theta + i\sin\theta\rvert^2$ $= (1 + \cos\theta)^2 + (\sin\theta)^2 =$ $1 + \cos^2\theta + 2\cos\theta + \sin^2\theta = 2 + 2\cos\theta$	1 mark: applies the fact that $\lvert z\rvert^2 =$ $(1+z)(\overline{1+z})$ **1 mark:** correct proof	
d)		**1 mark:** correct shape with focus and directrix	*The students who unravelled the geometric*

		**1 mark:** correctly labels the vertex, y-intercept and the Cartesian equation.	*meaning of this statement could do this question a lot faster than those who approached it algebraically. It is important to have even scales and labels for focus, vertex, y-intercept and directrix.*
e)(i) (ii)	$x = 1 + 3t$, $y = 2 - 4t$ **1 mark** $t = \frac{x-1}{3}$ $\quad y = 2 - \frac{4(x-1)}{3}$ $3y = 6 - 4x + 4$ $4x + 3y = 10$ **1 mark** Minimum value will be the perpendicular distance from the origin to the line. $d = \frac{\lvert -10 \rvert}{\sqrt{4^2+3^2}}$ $= 2$	1 mark: setting up the parametric equation 1 mark: correctly verifies the solution Award **2 marks** for correct solution Only **1 mark:** for working towards the solution	*Well done!*
f)	$z^5 + 16z = 0$ $z(z^4 + 16) = 0$ $\therefore$ z = 0 or $z^4 = -16$ Z = 0 or z = $(-16)^{1/4}$ Z = 0 or $z = 2\left(\cos\frac{\pi + 2k\pi}{4} + \sin\frac{\pi + 2k\pi}{4}\right)$ ***1 mark*** where $k = 0, \pm 1, -2$ $z = 0,\ z = 2cis\frac{\pi}{4},\ 2cis\frac{3\pi}{4},$ $2cis\frac{-\pi}{4},\ 2cis\frac{-3\pi}{4}$ **1 mark** $z = 0 + 0i,\ \sqrt{2}(1+i),\ \sqrt{2}(-1+i),$ $\sqrt{2}(-1-i),\ \sqrt{2}(1-i)$ $z = 0 + 0i,\ \pm\sqrt{2}(1+i), \pm\sqrt{2}(1-i)$	1 mark: Applies De Moivre's theorem to find the solutios in general form. 1 mark: substitutes the values of k to give the solutions in r cis θ form. 1 mark: gives the solutions in a + ib form.	*Many students got into trouble by trying to solve* $z^4 + 16 = 0$ *as difference of perfect squares.*

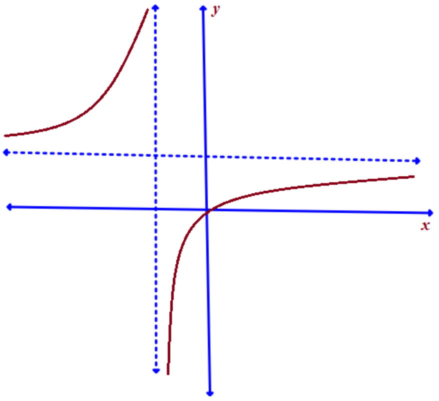
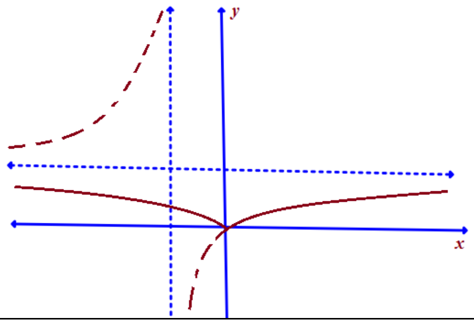
Question 12

a)(i)		2 mark: for the correct graph 1 mark: the graph is correct, but ignores the discontinuity at x = 0.	Majority of the students failed to note that at x = 0, f(x) and hence $\frac{1}{f(x)}$ is undefined. Discontinuous.
(ii)		2 marks: correct graph 1 mark: for correct shape without the important feature viz. when f(x)≤ 1, $f(x)^2 \leq f(x)$	
(iii)		2 marks: for correct graph 1 mark: for showing correct range or correct domain	Only a few attempted this question
(iv)		2 marks: for correct graph 1 mark: for showing the asymptote	
b)(i)	$f(x)=\frac{x^2-4x+4}{x-1}$ $\begin{array}{r} x-3 \\ x-1\,\overline{)\,x^2-4x+4} \\ x^2-x \\ \hline -3x+4 \\ -3x+3 \\ \hline 1 \end{array}$ $f(x)=\frac{(x-2)^2}{x-1} = x-3+\frac{1}{x-1}$	**1 mark:** for oblique asymptotes as $x \to \pm\infty$ **1 mark:** for vertical	Many students did not know how to get the oblique asymptote.

	$\lim\limits_{x\to\infty} x-3+\dfrac{1}{x-1} = (x-3)^{+}$ $Y = x - 3$ is an asymptote $\lim\limits_{x\to-\infty} x-3+\dfrac{1}{x-1} = (x-3)^{-}$ **1 mark** $x\to 1^{+},\quad f(x)\to\infty$ $x\to 1^{-},\quad f(x)\to-\infty$ **1 mark**	asymptotes $x\to 1^{\pm}$	
b)(ii)		**1 mark:** for shape **1 mark:** for asymptotes **1 mark:** for intercepts	
c)		**1 mark:** correct shape **1 mark:** shows $y=\sqrt{x-3}$ As the asymptote	In the graph of y=$\sqrt{f(x)}$ Many students failed to show that it approaches the graph of y=$\sqrt{x-3}$ as x tends to infinity.

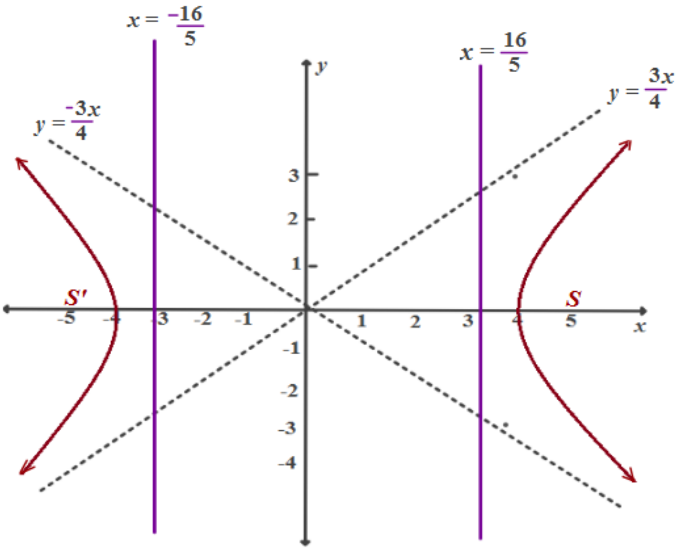
Question 13

a)	From the graph of $y^2 = f(x)$, the value of $f(x)$ between vertical asymptote and the y-axis is negative **1 mark** Therefore, part of the graph of y = |f(x)| must be obtained by reflecting about the x-axis. **1 mark**		*Many students found it very difficult to explain the concepts verbally, even though they had the correct sketch in (ii)*

(ii)		1 mark: for position of the sections of the graph of y = f(x) 1 mark: correct shape	*Generally well done!*
(iii)	Y = f|x| is symmetrical about the y-axis, Since, f|-x| = f|x| for x >0 The graph of y = f|x| is 	1 mark: correct position 1 mark: correct shape	*Well done!*
b)(i)	For an ellipse to exist, 4 + p > 0 and 9 + p > 0 Ie. P > -4 and p > -9 The combined region is p > -4 **1 mark**	1 mark: sets 4 + p > 0 and 9 + p > 0 And gives the correct solution.	*You needed to have both inequalities considered to get the mark.*
(ii)	For the hyperbola to exist Case 1: 4+p > 0 and 9 + p < 0. Ie. P > -4 and p < -9 There is no combined region, hence, the hyperbola does not exist for any value of p. Case 2: 4+p < 0 and 9 + p > 0. Ie. P < -4 and p > -9 Ie. -9 < p < -4.	1 mark: considering each of the cases and making the correct interpretation.	*Students need to realise that hyperbola can be of the form* $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ or $\frac{-x^2}{a^2}+\frac{y^2}{b^2}=1$
c) (i)	$9 = 16\,(e^2 - 1)$ $e^2 = \frac{25}{16}$ $e = \frac{5}{4}, \quad e > 0$ foci $(\pm ae, 0) = (\pm 5, 0)$	1 mark	
(ii)	Equations of the directrices are $x = 4 \times \frac{4}{5} = \frac{16}{5}$	1 mark	
(iii)	Vertices $(\pm 4, 0)$ Asymptotes $y = \pm\frac{b}{a}x = \pm\frac{3}{4}x$	1 mark: correct shape and labels foci and directrices 1 mark: correctly sketches and labels	

		vertices and asymptotes	

(iii)



13 (iv)	Using the property of the ellipse, SP + S'P = 2*a*, 2a = 10 $\therefore a = 5$ **1 mark** Foci (ae, 0) $\therefore 5e = 2$ $\therefore e = \frac{2}{5}$ $b^2 = a^2(1 - e^2)$ $= 25\left(1 - \left(\frac{2}{5}\right)^2\right) = 21$ $\therefore$ Cartesian equation is $\frac{x^2}{25} + \frac{y^2}{21} = 1$	1 mark: states the reasoning that SP + S'P = 2*a*, and evaluates *a* 1 mark: evaluates b and gives the Cartesian equation of the ellipse.	*Again, those who could see the geometrical property of the ellipse, solved the problem a lot faster than the students who followed algebraic method.*

Question 14

a)	$P(x) = x^3 + 2x^2 + 2x + 1$ The coefficients are real. $\therefore$ the roots come in conjugates. $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$ is a root, than	1 mark: states that since the coefficients are real, $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$ is a root.	Many failed to state why the complex conjugate is also a root. One student did not use the given

	$-\frac{1}{2}-\frac{\sqrt{3}}{2}i$ is also a root. Sum of roots = −2 $-\frac{1}{2}+\frac{\sqrt{3}}{2}i+ -\frac{1}{2}+\frac{\sqrt{3}}{2}i + \alpha = -2$ $\therefore \alpha = -1$ Sum of the two complex roots = 1 Product = 1 The quadratic with the two complex roots is $x^2 - x + 1$ $x^3+2x^2+2x+1 =$ $(x+1)(x^2-x+1)$	1 mark: correctly evaluates α the real root (can do by inspection too) 1 mark: correctly factorises over the real field.	root to solve the problem.
b)(i)	$x^3+2x^2-20x-16=0$ Let $\alpha^2 = y$ Then $\alpha = \sqrt{y}$ $(\sqrt{y})^3+2y-20\sqrt{y}-16=0$ **1 mark** $\sqrt{y}(y-20) = 16-2y$ Squaring both sides, $y(y^2-40y+400)=256-64y+4y^2$ $y^3-44y^2+464y-256=0$ is the cubic with α^2, β^2 and γ^2 as roots.	1 mark: substitutes $\alpha = \sqrt{y}$ into $x^3+2x^2-20x-16=0$ 1 mark: simplifies and correctly gives the correct cubic equation.	Most of the students had the correct method but made numerical errors when squaring and rearranging.
(ii)	$x^3+2x^2-20x-16=0$ has roots α, β and γ. $\alpha^3+2\alpha^2-20\alpha-16=0$ $\beta^3+2\beta^2-20\beta-16=0$ $\gamma^3+2\gamma^2-20\gamma-16=0$ Adding the three equations, $\sum\alpha^3+2\sum\alpha^2-20\sum\alpha-48=0$ $\sum\alpha^3+2\times 44-20\times -2-48=0$ $\sum\alpha^3 = -80$	1 mark: gives the correct values for $\sum\alpha$ and $\sum\alpha^2$ 1 mark: correctly substitutes $\sum\alpha$ and $\sum\alpha^2$ into the equation $\sum\alpha^3+2\sum\alpha^2-20\sum\alpha-48=0$ and evaluates $\sum\alpha^3$	
c) (i)	Using division transformation, $P(x) = (x-k)^2 Q(x)$ **1 mark** $P(k)=0$ $P'(x) = (x-k)^2 Q'(x)+2(x-k)Q(x)$ $= (x-k)[(x-k)Q'(x)+2Q(x)]$ $P'(k)=0$ Ie. $x = k$ is also a root of $P'(x) = 0$	1 mark: Uses division transformation and writes $P(x)$ correctly in the factored form 1 mark: Differentiates $P(x)$ correctly and demonstrates that $x = k$ satisfies $P'(x)$	One student used the fact that when it has a double root it touches the x-axis and therefore both P(x) and $P'(x)$ are zero.

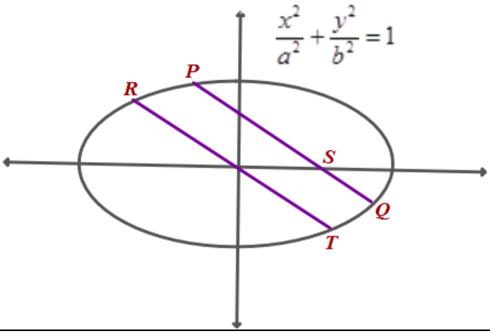
(ii)	$P(x)=x^{2n}-nx^{n+1}+nx^{n-1}-1$ $P(1)$ = $1-n+n-1=0$ Ie. $x=1$ is a root of $P(x)=0$ **1 mark** $P'(x)$ = $2nx^{2n-1}-n(n+1)x^{n}+n(n-1)x^{n-2}$ $P'(1)$ = $2n-n(n+1)+n(n-1)$ = $2n-n^2-n+n^2-n$ = 0 $P(1)$ = $P'(1)$ = 0 ie. $x=1$ is a double root of $P(x)$ = 0 **1 mark**	1 mark: shows that P(1) = 0 1 mark: Shows that $P'(1)$ = 0 And proves that x = 1 is a double root by giving the reasoning (since $P(1)=P'(1)$ = 0)	
d)	S_r = $\sum_{1}^{r} k^2$ = $1^2+2^2+\ldots+r^2$ The statement means $nS_n-\sum_{1}^{n-1}S_r=\sum_{1}^{n}r^3$, $n(1^2+2^2+3^2+.+n^2)-$ $(S_1+S_2+S_3+\ldots+S_{n-1})$ = $(1^3+2^3+3^3+.+n^3)$		
	Proof **Step 1** Let n = 2 (n > 1) $2(1^2+2^2)-(1^2)$ = (1^3+2^3) $2\times5-1=1+8$ $9=9$ True **Step 2** Assume statement is true for n = k, then , $kS_k-\sum_{1}^{k-1}S_r=\sum_{1}^{k}r^3$. ie. $k(1^2+2^2+3^2+.+k^2)-$ $(S_1+S_2+S_3+\ldots+S_{k-1})$ = $(1^3+2^3+3^3+.+k^3)$. (***) **Step 3** We need to prove the statement is true for n = k + 1. LHS is	1 mark: for proving the statement for n = 2. 1 mark: states that the statement is true for n = k. 1 mark: writes the n = k+1 statement and expands it correctly. 1 mark: uses the n = k result to prove the result.	

	$(k+1)\left(1^2+2^2+3^2+.+k^2+(k+1)^2\right)-$ $\left(S_1+S_2+S_3+...+S_{k-1}+S_k\right)$ $=k\left(1^2+2^2+3^2+.+k^2\right)+$ $\left(1^2+2^2+3^2+.+k^2\right)+(k+1)^3$ $-\quad\left(S_1+S_2+S_3+...+S_{k-1}\right)-S_k$ $=k\left(1^2+2^2+3^2+.+k^2\right)-$ $\left(S_1+S_2+S_3+...+S_{k-1}\right)+$ $\left(1^2+2^2+3^2+.+k^2\right)-S_k+(k+1)^3$ $=\sum_1^k r^3+S_k-S_k+(k+1)^3$ $=\sum_1^{k+1} r^3$ Ie. if the statement is true for n = k, then it is true for n = k+1. **Step 4** We have proved that statement is true for n = 2, hence, using the concept of mathematical induction, the statement is proved to be true.		

Question 15

a)(i)	$\cos(p+q)+\cos(p-q)=$ Cos p cos q – sin p sin q + cos p cos q + Sin p sin q = 2 cos p cos q **1 mark** Let p + q = α (1) and p – q = β (2) (1) + (2) gives p = $\dfrac{\alpha+\beta}{2}$ and	1 mark: correct proof 1 mark: correct proof (all steps of working must be shown)	Well done

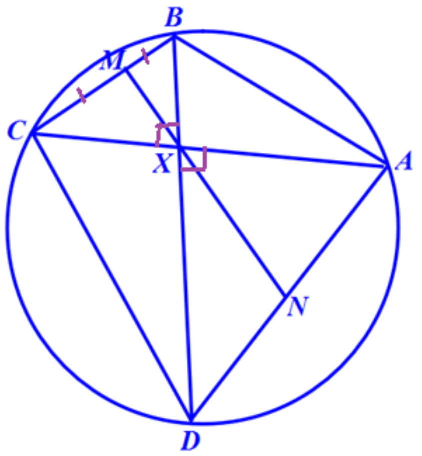
	(1) $-$ (2) gives $q = \dfrac{\alpha-\beta}{2.}$ Sub in the relation above, Cos α + cos β = $2\cos\dfrac{\alpha+\beta}{2}\cos\dfrac{\alpha-\beta}{2.}$		
(ii)	$P(a\cos\alpha, b\sin\alpha)$ and $Q(a\cos\beta, b\sin\beta)$ $m_{PQ} = \dfrac{b(\sin\alpha-\sin\beta)}{a(\cos\alpha-\cos\beta)}$ $= \dfrac{b(2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2})}{a(-2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2})}$ $= \dfrac{b(\cos\frac{\alpha+\beta}{2})}{-a(\sin\frac{\alpha+\beta}{2})}$ **1 mark** $\therefore$ Equation of the chord is of the form $b\cos\dfrac{\alpha+\beta}{2}x + a\sin\dfrac{\alpha+\beta}{2}y = k$ Sub $P(a\cos\alpha, b\sin\alpha)$, $ab\cos\dfrac{\alpha+\beta}{2}\cos\alpha + ab\sin\dfrac{\alpha+\beta}{2}\sin\alpha = k$ $k = ab\left(\cos\dfrac{\alpha+\beta}{2}\cos\alpha + \sin\dfrac{\alpha+\beta}{2}\sin\alpha\right)$ $k = ab\cos\left(\alpha - \dfrac{\alpha+\beta}{2}\right)$ $= ab\cos\left(\dfrac{\alpha-\beta}{2}\right)$ Chord has equation $b\cos\dfrac{\alpha+\beta}{2}x + a\sin\dfrac{\alpha+\beta}{2}y =$ $= ab\cos\left(\dfrac{\alpha-\beta}{2}\right)$ Divide by *ab*, $\dfrac{x}{a}\cos\dfrac{\alpha+\beta}{2} + \dfrac{y}{b}\sin\dfrac{\alpha+\beta}{2.} = \cos\dfrac{\alpha-\beta}{2}$	1 mark: correctly calculates the gradient of PQ 2 marks: correctly finds the equation of the straight line (chord PQ) Award 1 mark if significant progress towards getting the equation of the chord is made.	
(iii)	$P(a\cos\alpha, b\sin\alpha)$ $Q(a\cos\beta, b\sin\beta)$ PQ is a focal chord, then it satisfies $S(ae,0)$ or $S'(-ae,0)$	1 mark: Substitutes $S(ae,0)$ into the chord to get	Many students used the distance formula to get PQ even though it needed a lot

	Then $\frac{ae}{a}\cos\frac{\alpha+\beta}{2}=\cos\frac{\alpha-\beta}{2}$ ie. $\pm e\cos\frac{\alpha+\beta}{2}=\cos\frac{\alpha-\beta}{2}$ **1 mark** PQ = PS + SQ = e PM + e QM' (focus-directrix Formula) $= e\left(\frac{a}{e}-a\cos\alpha\right)+e\left(\frac{a}{e}-a\cos\beta\right)$ **1 mark** $= a(2-e(\cos\alpha+\cos\beta))$ $= a\left(2-e\left(2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}\right)\right)$ $2a\left(1-e\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}\right)$ $= 2a\left\{1-e^2\cos^2\frac{\alpha+\beta}{2}\right\}$ **1 mark**	$\pm e\cos\frac{\alpha+\beta}{2}=\cos\frac{\alpha-\beta}{2}$ 1 mark: Uses PS = e PM and PS' = ePM' and writes an expression for PQ 1 mark: Uses the result $\pm e\cos\frac{\alpha+\beta}{2}=\cos\frac{\alpha-\beta}{2}$ To get the desired result.	of algebraic work. Of course, you can see the merit of using a geometrical approach.
b)(i)			

(ii)	$P(a\cos\alpha, b\sin\alpha)$ $Q(a\cos\beta, b\sin\beta)$ $R(a\cos\theta, b\sin\theta)$ and let $T(a\cos\phi, b\sin\phi)$ RT satisfies (0, 0), sub. in $\frac{x}{a}\cos\frac{\theta+\phi}{2}+\frac{y}{b}\sin\frac{\theta+\phi}{2}=\cos\frac{\theta-\phi}{2}$, Then $\cos\frac{\theta-\phi}{2}=0$ $\therefore \theta-\phi=\pm\pi$	1 mark: proves that $\cos\frac{\theta-\phi}{2}=0$ and hence $\phi=\theta\pm\pi$ 1 mark: Applies $\phi=\theta\pm\pi$ to prove the coordinates of T and	*In this question you cannot show* (- θ) satisfies the equation or state verbally that since it is diametrically opposite T has parameter - $-\theta$. The question is "Deduce". Then you need to prove that $\therefore \phi=\theta\pm\pi$.

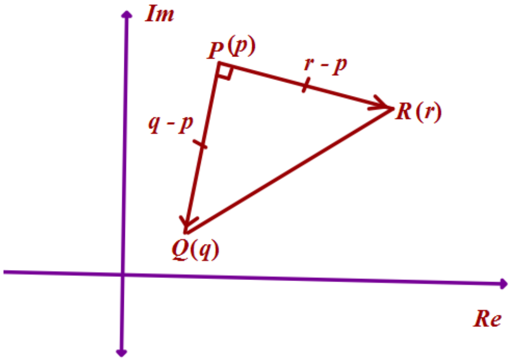
	$\therefore \phi = \theta \pm \pi$ **1 mark** $\therefore \cos\phi = -\cos\theta$, $\sin\phi = -\sin\theta$ Then $(-a\cos\theta, -b\sin\theta)$ Now, RO = TO $\Rightarrow$ RT = 2RO **1 mark**	RT = 2RO	
(iii)	$RO^2 = a^2\cos^2\theta + b^2\sin^2\theta$ $= a^2\cos^2\theta + a^2(1-e^2)\sin^2\theta$ $= a^2(1-e^2\sin^2\theta)$ **1 mark** Using (ii), $RT^2 = 4\,RO^2$ $= 4a^2(1-e^2\sin^2\theta)$	1 mark: applies distance formula and $b^2 = a^2(1-e^2)$ to get the expression for RO^2 1 mark: Uses (ii), to prove the result	
(iv)	$PQ \parallel RT$ $m_{PQ} = \dfrac{b(\cos\frac{\alpha+\beta}{2})}{-a(\sin\frac{\alpha+\beta}{2})}$ from Q15a)(ii) $m_{RT} = m_{RO} = \dfrac{b\sin\theta}{a\cos\theta} = \dfrac{b}{a}\tan\theta$ Equating gradients, $\dfrac{b}{-a}\cot\dfrac{\alpha+\beta}{2} = \dfrac{b}{a}\tan\theta$ $\therefore \tan\theta\tan\dfrac{\alpha+\beta}{2} = -1$	1 mark: equates gradients and proves the result	Some students just by looking at $\tan\theta\tan\dfrac{\alpha+\beta}{2} = -1$ and proceeded with the question stating that PQ $\perp$ RO. Your starting point here has to be $PQ \parallel RT$ and then gradients are equal.
(v)	From (a)(iii), PQ= $2a\left\{1-e^2\cos^2\dfrac{\alpha+\beta}{2}\right\}$ (*) From (b)(iii), $RT^2 = 4a^2(1-e^2\sin^2\theta)$ (**) We need to prove $\cos^2\dfrac{\alpha+\beta}{2} = \sin^2\theta$ From (iv), $\therefore \tan\theta\tan\dfrac{\alpha+\beta}{2} = -1$ $\therefore \tan^2\theta\tan^2\dfrac{\alpha+\beta}{2} = 1$ $\therefore \tan^2\theta\left(\sec^2\dfrac{\alpha+\beta}{2} - 1\right) = 1$ $\sec^2\dfrac{\alpha+\beta}{2} - 1 = \cot^2\theta$ $= \mathrm{cosec}^2\,\theta - 1$ $\sec^2\dfrac{\alpha+\beta}{2} = \mathrm{cosec}^2\,\theta$ $\cos^2\dfrac{\alpha+\beta}{2} = \sin^2\theta$ **1 mark** Comparing (*), (**), $RT^2 = 2a\,PQ.$	1 mark: proves the result $\cos^2\dfrac{\alpha+\beta}{2} = \sin^2\theta$ 1 mark: compares RT^2 and PQ and applies the above result to prove the relationship.	Many people did not attempt this question.

Question 16

a)	$\int \frac{x}{\sqrt{x-1}}\,dx \qquad u = x-1$ $du = dx$ $= \int \frac{u+1}{\sqrt{u}}\,du$ **1 mark** $= \int \sqrt{u} + \frac{1}{\sqrt{u}}\,du$ $= \frac{2}{3}u^{\frac{3}{2}} + 2u^{\frac{1}{2}} + C$ **1 mark** $= \frac{2}{3}u\sqrt{u} + 2\sqrt{u} + C$ $= \frac{2}{3}(x-1)\sqrt{x-1} + 2\sqrt{x-1} + C$ **1 mark**	**1 mark:** uses the substitution to convert the integrand in terms of u **1 mark:** integrates correctly 1 mark: expresses the answer in terms of x in radical sign	
b)(i)	 $\angle BXC = 90°$ (given) Ie. BC is the diameter of the circle through B, X, C and M is the centre of the circle. (angle in the semi-circle = $90°$) **1 mark** $\therefore MB = MX$ (radii of the circle are equal) $\therefore \angle MBX = \angle MXB$ (angles opposite equal sides are equal in ΔMBX **1 mark**	**1 mark:** proves that M is the centre of a circle with centre. **1 mark:** Proves that MX = MB and hence the result giving reason for every step.	
(ii)	$\angle CBD = \angle CAD$ (angles subtended by arc CD at the circumference are equal) $\therefore \angle MBX = \angle XAD$ ($\therefore \angle MBX = \angle CBD$ and $\angle XAD = \angle CAD$)		

	$\angle MXB = \angle DXN$ (vertically opposite angles are equal) $\therefore \angle XAD = \angle DXN$ ($\therefore \angle MBX = \angle MXB$ proven) $\angle XDN = \angle XAD = 90°$ ($\angle XAD = \angle DXN$) proven above $\therefore \angle XND = 90°$ ($\angle$ sum of $\Delta DXN = 180°$ $\therefore MN \perp AD$		

c)	Using divison transformation, $P(x) = (x-a)(x-b)\,Q(x) + R(x)$ Where max. degree of $R(x)$ is 1. Let $R(x) = mx + n$ $P(x) = (x-a)(x-b)\,Q(x) + mx + n$ Let $x = a$. $P(a) = ma + n \quad (1)$ Let $x = b$. $P(b) = mb + n \quad (2)$ (1) – (2) gives, $m\,(a-b) = P(a) - P(b)$ $\therefore m = \left(\dfrac{P(a)-P(b)}{a-b}\right)$ $aP(b) - bP(a) = mab + na - mab - nb$ $n(a-b) = aP(b) - bP(a)$ $\therefore n = \dfrac{aP(b)-bP(a)}{a-b}$ Sub. m and n in $R(x)$, $R(x) = \left(\dfrac{P(a)-P(b)}{a-b}\right)x + \dfrac{aP(b)-bP(a)}{a-b}$	1 mark: writes the division transformation and gives $R(x)$ as $mx + n$ 2 marks: correctly evaluates the values of m or n and hence $R(x)$ in the required form. 1 mark: evaluates either m or n.	

d)	 From the diagram, $(q-p) = i(r-p)$ Squaring both sides, $(q-p)^2 = -1(r-p)^2$ $q^2 - 2pq + p^2 = -1(r^2 - 2rp + p^2)$ Ie. $q^2 + p^2 + 2p^2 = 2(pq + pr)$	1 mark: represents the information on an Argand diagram. 1 mark: Demonstrates the understanding that $\lvert q-p \rvert = \lvert r-p \rvert$ and that $(q-p) = i(r-p)$ 2 marks: correctly expands and proves the result Award 1 mark (only): working towards the solution.	