Killara HIGH SCHOOL

Student Number

Extension 2 Mathematics

HSC Assessment task 1
4th December 2014

Total Marks – 40

General Instructions
- Reading time – 5 minutes
- Working tine 1 hours
- Write using blue or black pen
  Black pen is preferred
- Approved calculators may be used
- In Questions 6-8 show relevant mathematical reasoning and/or calculations
- Start a new booklet for each question

Section A
5 marks
- Attempt Questions 1 – 5
- Allow about 5 minutes for this section

Section B
35 marks
- Attempt Questions 6 – 8
- Allow about 55 minutes for this section

Question	Marks
1 - 5	/5
6	/11
7	/11
8	/13
Total	/40

THIS QUESTION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM

This assessment task constitutes 10% of the Higher School Certificate Course Assessment

Section A: (5 marks)
Attempt Questions 1 – 5
Allow about 5 minutes for this section
Use the multiple-choice answer sheet for questions 1 – 5 (detach from paper)

1. $2\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)$ is expressed in the form x + iy as:

(A) $\sqrt{3} - i$ (B) $\sqrt{3} + i$

(B) $-\sqrt{3} - i$ (D) $\sqrt{3 - i}$

2. If z and w are the roots of the equation $3x^2 + (2 - i)x + (4 + i) = 0$, then

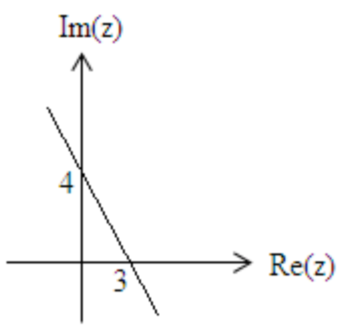
$\bar{z} + \bar{w}$ is:

(A) $\frac{2+i}{3}$ (B) $-\frac{2+i}{3}$

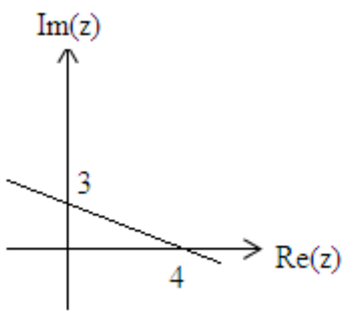
(C) $\frac{2-i}{3}$ (D) $\frac{-2+i}{3}$

3. The Argand diagram that represents the locus of 3Re(z) + 4Im(z) = 5 is:

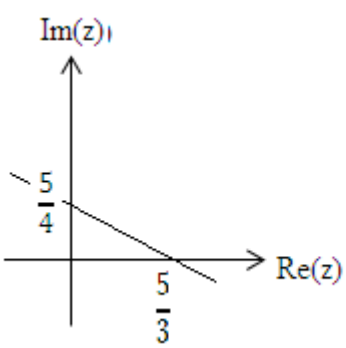
(A)



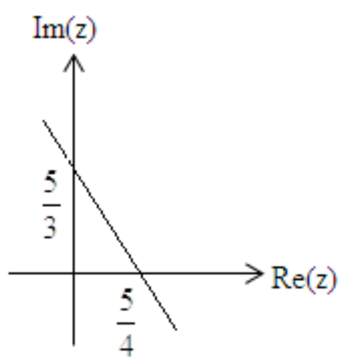
(B)



(C)



(D)



4. What are the values of n when $(1+i)^n + (1-i)^n = 0$. (Given k is an integer)

(A) $4 + 2k$ (B) $6 + 2k$

(C) $2 + 4k$ (D) $2 + 2k$

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5. If $z = a + bi$, where a and b are real, find $Re(\frac{1}{z^2})$

(A) $\frac{a^2-b^2}{(a^2-b^2)^2}$ (B) $\frac{a^2+b^2}{(a^2-b^2)^2}$

(C) $\frac{a^2-b^2}{a^2+b^2}$ (D) $\frac{a^2-b^2}{(a^2+b^2)^2}$

Section B (35 marks)
Attempt Questions 6 – 8
Allow about 55 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 6 - 8, your responses should include relevant mathematical reasoning and/or calculations.

Question 6 (11 marks) Use a SEPARATE writing booklet

a) Let $z = \frac{18+4i}{3-i}$

(i) Simplify $(18+4i)(\overline{3-i})$ 1

(ii) Express z in the form $a + bi$ where a and b are real. 1

(iii) Hence or otherwise find $|z|$ 1

b) Find the complex square roots of $-12 + 16i$, given your answers in the form $a + ib$, where a and b are real. 3

c) Let $z = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ and $w = \sqrt{3} + i$.

(i) Express w in modulus-argument form. 1

(ii) Hence or otherwise express $z^3 w$ in modulus-argument form. 2

d) The complex number $1+i$ is a root of $z^2 - (a-i)z + 5 + bi = 0$, where a and b are real numbers. Find the values of a and b. 2

Question 7 (11 marks) Use a SEPARATE writing booklet

a) Find a polynomial $P(x)$ with real coefficients having $3i$ *and* $1 + 2i$ as zeros. 2

b) (i) If w is a non-real sixth root of 1 of smallest positive argument, show that $w = cis\,\frac{\pi}{3}$. Hence find the six sixth roots of 1. Leave your answers in modulus-argument form and indicate these roots in an Argand diagram. 3

(ii) Find the two quadratic equations whose roots are

α) w and w^5 β) w^2 and w^4 2

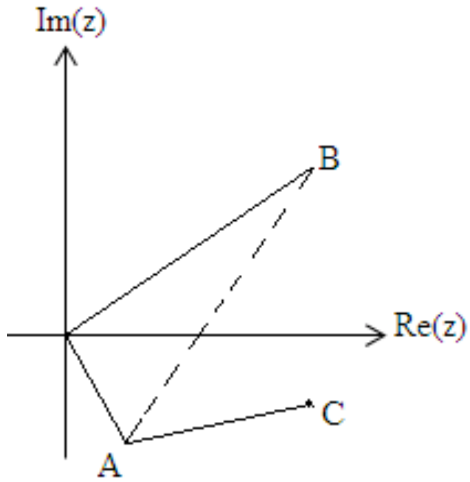
(iii) Hence express $P(x) = x^6 - 1$ as a product of quadratic factors with real coefficients. 1

c) If w is the complex cube root of unity, evaluate 3

$$(1 - 3w + w^2)(1 + w - 8w^2)$$

Question 8 (13 marks) Use a SEPARATE writing booklet

a) In the diagram the points A, B and C correspond to the complex numbers $z_1 = 1 - \sqrt{3}i$, $z_2 = 4 + 2\sqrt{3}i$ and $z_3 = x + iy$. Given that $AC = \frac{2}{3}AB$ and $\angle BAC = \frac{\pi}{3}$. Find the complex number z_3 which represents the point C. 4



b) Sketch the locus of each of the following. Draw separate diagrams.

(i) $\arg(z-1-2i)=\dfrac{\pi}{4}$. 1

(ii) $z\bar{z}-3(z+\bar{z})\leq 0$. 2

(iii) $\arg\left(\dfrac{z-1}{z+1}\right)=\dfrac{\pi}{3}$. 3

c) Find the locus of w if $w=\dfrac{z-2i}{1-z}$, given $|z|=2$ 3

END OF EXAMINATION

Student number : ______________________________

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SECTION A - MULTIPLE CHOICE ANSWER SHEET

If you think you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A	B	C	D

1	A	B	C	D
2	A	B	C	D
3	A	B	C	D
4	A	B	C	D
5	A	B	C	D

2014 EXT-Assessment 1 Solution

	1) A 2) B 3) C 4) C 5) D		
6. a)	(i)	$18 + 4i)\overline{(3-i)} = (18 + 4i)(3 + i)$ $= 54 + 18i + 12i - 4$ $= 50 + 30i$	**1 mark**
	(ii)	$z = \frac{50+30i}{10} = 5 + 3i$	**1 mark**
	(iii)	$\lvert z\rvert = \sqrt{5^2 + 3^2} = \sqrt{34}$	**1 mark**
b)	*Let* $z = a + bi$ $= \sqrt{-12+16i}$ $(a + bi)^2 = -12 + 16i$ *L.H.S.* $= a^2 - b^2 + 2abi$ *Equating real and imaginary part with R.H.S* $a^2 - b^2 = -12$ $2ab = 16 \qquad b = \frac{8}{a}$ $a^2 - \frac{64}{a^2} = -12$ $a^4 + 12a^2 - 64 = 0$ $(a^2 + 16)(a^2 - 4) = 0$ Since a is real, only $a^2 - 4 = 0 \quad \therefore a = \pm 2$ and $b = \pm 4$ $\therefore \alpha = 2 + 4i \quad and \quad \beta = -2 - 4i$		1 mark: gives the two simultaneous equations 1 mark: evaluates the value of *a* 1 mark: evaluates the value of *b* and hence the square roots.
c)	(i)	$w = 2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$	**1 mark**
	(ii)	$z^3w = 8(\cos\pi + i\sin\pi).2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$ $= 16\,(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6})$ $= 16(\cos\frac{5\pi}{6} - i\sin\frac{5\pi}{6})$ Or $-16(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$	**1 mark:** gives the correct expression for z^3 in mod-arg form **1 mark:** gives z^3w in mod-arg form with $-\pi \le \alpha \le \pi$ Many students failed to change $\frac{7\pi}{6}$ to $\frac{-5\pi}{6}$

b)	Sub. 1 + i into P(x) $(1+i)^2-(a-i)(1+i)+5+bi=0$ $(-a-1+5)+(2-a+1+b)i=0$ Equate real and imaginary part: $-a-1+5=0 \quad \rightarrow \quad a=4$ $b-a+3=0 \quad \rightarrow \quad b=1$ OR Let α be the other root. Then $1+i+\alpha = a-i$ $\alpha = a-1-2i$ Product $\alpha(1+i)=5+bi$ $(a-1-2i)(1+i)=5+bi$ **1 mark** $(a-1+2)+i(a-1-2)=5+bi$ Equate real and imaginary parts, $a+1=5$ then $a=4$ $4-1-2=b$ then $b=1$ **1 mark**	**1 mark:** substitutes and separates P(x) into real and imaginary parts **1 mark:** equates real and imaginary parts and solves for a and b. Many students substituted correctly and got $(-a-1+5)+(2-a+1+b)i=0$. But they failed to realise that the RHS was 0 + 0i and hence equate real and imaginary parts. They received 1 mark for this question.

Question 7

a)	i) P(x) has real coefficient, therefore its complex roots are conjugate. P(x) = $(x-3i)(x+3i)(x-(1-2i))(x-(1+2i))$ $=(x^2+9)(x^2-2x+5)$	**1 mark:** realises that for the coefficients to be real, the roots have to come in complex conjugates. Gives the four roots as $3i, \quad -3i, \quad 1-2i, \quad 1+2i$ **1 mark:** using the conjugates writes the quadratic factors and expresses P(x) as product of quadratic factors with real coefficients. Many students failed to translate that real coefficients means roots come in conjugates. If a quadratic with 3i and 1+2i as roots was given, they received no marks. Even after correctly giving the four roots, some students could not produce the quadratic factors. Use $x^2-(\alpha+\beta)x+\alpha\beta$ method.

<table>
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<td>b)</td>
<td>(i)</td>
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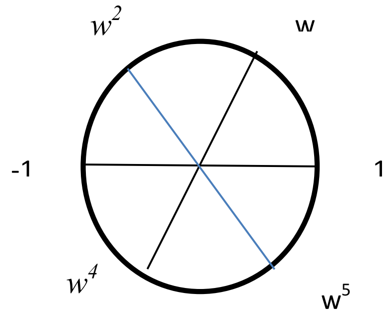
$z^6 = 1$
Let $z = r(cos\theta + isin\theta)$
$|z| = 1$
$cos6\theta + isin6\theta = 1 + 0i$
$cos6\theta = 1$
$\therefore \ \theta = \frac{2\pi k}{6}$ *where k =* $0, \pm 1, \pm 2$ *and 3*
The six sixth roots of 1 are:
$cis0 = 1$ *for* $k = 0$
$cis \pm \frac{\pi}{3}$ *for* $k = \pm 1$
$cis \pm \frac{2\pi}{3}$ *for* $k = \pm 2$
and $cis\ \pi = -1$ *for* $k = 3$


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1 mark: Develops the formula to find $\theta = \frac{2\pi k}{6}$ and substitutes $k = 1$ to get the smallest value of ω
1 mark: substitutes the k values to derive the other five solutions.
1 mark: Shows the solutions on the Argand diagram. Must have a circle with radius 1 and the arguments are labelled for each of the z values.
Many students just stated that $\theta = \frac{2\pi k}{6}$ and gave the six solutions with no substitution shown. You must also define what z is.
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<td>(ii)</td>
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α)
Quadratic equation with roots w and w^5
$(x - (cos\frac{\pi}{3} + isin\frac{\pi}{3}))\ (x - (cos\frac{\pi}{3} - isin\frac{\pi}{3}))$
$= x - 2cos\frac{\pi}{3} + 1$
$= x^2 - x + 1$
β)Quadratic equation with roots w^2 and w^4
$(x - (\ cos\frac{2\pi}{3} + isin\frac{2\pi}{3}))\ (x - (\ cos\frac{2\pi}{3} - isin\frac{2\pi}{3}))$
$= x - 2cos\frac{2\pi}{3} + 1$
$= x^2 + x + 1$
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1 mark for each question
Again, many students could not recognise that w^5 is $\overline{\omega}$. Same problem with w^2 and w^4.
This problem can be avoided by developing the habit of marking conjugates on the Argand diagram. Use $x^2 - (\alpha + \beta)x + \alpha\beta$ method to get the quadratics.
It is essential that you write simple numbers like $cos\frac{\pi}{3}$ as exact values.
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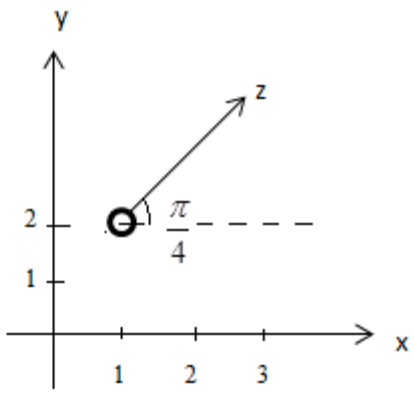
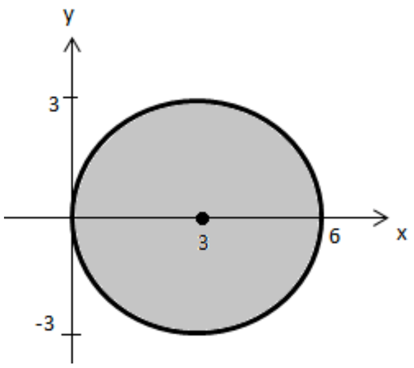
	(iii)	$\therefore\ z^6 = (x^2 - 1)(x^2 - x + 1)(x^2 + x + 1)$	**1 mark.** Very few students got this mark. In the end they forgot about the solution $zi = \cos\pi - i\sin\pi$,
c)		$w^3 - 1 = 0$ $= (w-1)(w^2 + w + 1)$ Since $w \neq 1$, $w^2 + w + 1 = 0$ $(1 - 3w + w^2)(1 + w - 8w^2)$ $= (1 + w + w^2 - 4w)(1 + w + w^2 - 9w^2)$ $= -4w \times -9w^2 = 36w^3 = 36 \quad (w^3 = 1)$	**3 marks:** correct solution from correct working **2 marks:** proves that $w^3 = 1$ and $w^2 + w + 1 = 0$ and significant progress towards the solution. **1 mark**: uses $w^3 = 1$ or $w^2 + w + 1 = 0$ and attempts to simplify. Many students are not deriving the results $w^3 = 1$ and $w^2 + w + 1 = 0$. It is worthwhile writing its implications immediately, such as $1 + w^2 = -w$ and $1 + w = -w^2$

Question 8

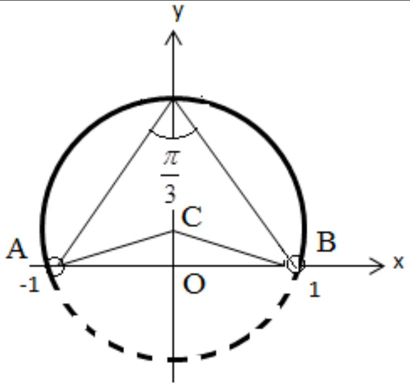
a)	$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$ $= z_2 - z_1$ $= 4 + 2\sqrt{3}\,i - 1 + \sqrt{3}\,i$ $= 3 + 3\sqrt{3}\,i$ **1 mark** $= 6\text{cis}\,\frac{\pi}{3}$ $\overrightarrow{AC} = \frac{2}{3}\overrightarrow{AB}\ \text{cis}(-\frac{\pi}{3})$ $= \frac{2}{3}\left(6cis\frac{\pi}{3}\right)cis\left(-\frac{\pi}{3}\right) = 4$ $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$ $\therefore\ \overrightarrow{OC} = \overrightarrow{AC} + \overrightarrow{OA}$ $= 4 + 1 - \sqrt{3}\,i$ $= 5 - \sqrt{3}\,i$ where $z_3 = \overrightarrow{OC}$	**1 mark:** for correct $z_2 - z_1$ 1 mark: correctly calculates $\frac{2}{3}(z_2 - z_1)$ 1 mark: calculates $\overrightarrow{AC}$ by multiplying $\frac{2}{3}(z_2 - z_1)$ by $\text{cis}(-\frac{\pi}{3})$. 1 mark: recognises that z_3 is $\overrightarrow{OC}$ and finds it by summation $\overrightarrow{AC} + \overrightarrow{OA}$

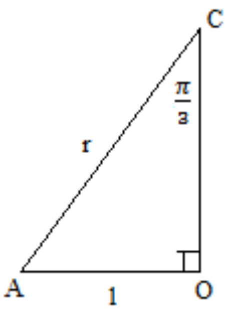
This question was done poorly by many students. Many silly errors were made such as writing cis($-\frac{\pi}{3}$) = - cis($\frac{\pi}{3}$).

Again, the other major problem was that students thought $\overrightarrow{AC} = z_3$.

b)	(i)		**1 mark:** Must have the horizontal line and the hollow circle at the vertex. This means that $z = 1 + 2i$
	(ii)	$z\bar{z} - 3(z + \bar{z}) \leq 0$ $x^2 + y^2 - 3\times 2x \leq 0$ $x^2 - 6x + 9 + y^2 \leq 9$ $(x-3)^2 + y^2 \leq 9$ 	1 mark: Correct inequation derived 1 mark: correct sketch Many students did not shade the inside of the circle. Must show the x-intercepts, centre and 3, -3 on the y-axis. Accuracy is a must, when you are sketching.

(iii)





$AC = \frac{1}{sin\frac{\pi}{3}} = \frac{2\sqrt{3}}{3}$ $\qquad$ $OC = \frac{1}{tan\frac{\pi}{3}} = \frac{\sqrt{3}}{3}$

Locus is the arc of the circle with centre $(0, \frac{\sqrt{3}}{3})$ and radius $r = \frac{2\sqrt{3}}{3}$

1 mark: correct sketch (must be major arc, not semi-circle, (1,0) and (-1, 0) must be excluded.)

1 mark: for calculating the centre of the circle.

1 mark: correct radius calculated and displayed on the sketch.

Many students did not calculate the centre and radius of the circle. You can draw many circles that pass through (-1,0) and (1,0). Hence, giving the centre and radius is a must to make the circle unique.

c)

$$w = \frac{z-2i}{1-z}$$

$$w(1-z) = z - 2i$$

$$w - wz = z - 2i$$

$$z(1 + w) = w + 2i$$

$$z = \frac{w + 2i}{1 + w}$$

$$|z| = \left|\frac{w+2i}{1+w}\right| \Rightarrow \quad 2 = \frac{|w+2i|}{|1+w|} \quad \Rightarrow$$

$2|1 + w| = |w + 2i|$ **1 mark**

$$2|(1 + x) + iy| = |x + (2 + y)i|$$

$$2\sqrt{(1+x)^2 + y^2} = \sqrt{x^2 + (2+y)^2}$$

1 mark

$$3x^2 + x^2 + 4 + 4y + y^2 = 1 + 8x + 4x^2 + 4y^2$$

$$\left(x + \frac{4}{3}\right)^2 + \left(y - \frac{2}{3}\right)^2 = \frac{20}{9}$$

$\therefore$ *locus is the circle with centre at* $\left(-\frac{4}{3}, \frac{2}{3}\right)$ *and radius is* $\frac{2\sqrt{5}}{3}$

1 mark: makes z subject of the formula and applies $|z| = 2$ to get the absolute value equation.

1 mark: correctly interprets that w such that its distance from ((0, -2) is double the distance from (-1, 0)

1 mark: correctly gives the locus and gives the geometrical significance.

Very poorly done. Those who took the algebraic method did not have much success.

Also, silly errors when squaring also caused some concern.