Killara HIGH SCHOOL

CONSERVA PROGREDERE

STUDENT NUMBER

MATHEMATICS

EXTENSION 2

ASSESSMENT TASK NO: 3

Date : June 12, 2014

TIME ALLOWED: *Reading time: 5 minutes*
Working time: 1 hour

INSTRUCTIONS: This paper consists of 3 questions
ANSWER all 3 questions and Multiple Choice questions
Start each question in a new booklet
Marks may not be given for untidy and poorly arranged work.
All relevant working should be shown.

THIS QUESTION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM

OUTCOMES ASSESSED:

E7	Uses the techniques of slicing and cylindrical shells to determine volumes
E8	Applies further techniques of integration, including partial fractions, integration by parts and recurrence formulae, to problems.
E9	Communicates abstract ideas and relationships using appropriate notation and logical argument.

	MARKS
QUESTION 1 Topic	/13
QUESTION 2 Topic	/11
QUESTION 3 Topic	/14
Multiple Choice:	/4
	/42

This assessment task constitutes 20% of the Higher School Certificate Course Assessment.

Section I

4 marks

Attempt Questions 1- 4

Use the multiple-choice answer sheet for Questions 1 – 4.

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1. The antiderivative of $\dfrac{e^{2x}}{2e^{2x}-1}$ (for $e^{2x} > \dfrac{1}{2}$) is

   (A) $4\log_e\left(2e^{2x}-1\right)$

   (B) $2\log_e\left(2e^{2x}-1\right)$

   (C) $\dfrac{1}{2}\log_e\left(2e^{2x}-1\right)$

   (D) $\dfrac{1}{4}\log_e\left(2e^{2x}-1\right)$

2. Using a suitable substitution, $\displaystyle\int_0^{\frac{\pi}{3}} \sin^3(x)\cos^4(x)dx$ can be expressed in terms of u as

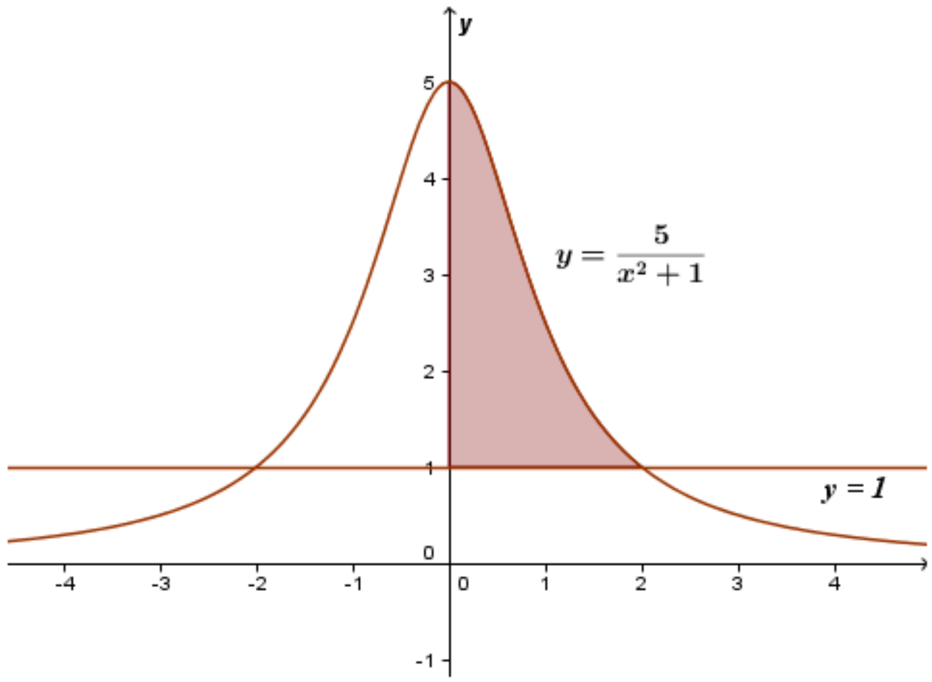
   (A) $\displaystyle\int_{\frac{1}{2}}^{1} \left(u^6-u^4\right)du$

   (B) $\displaystyle\int_{1}^{\frac{1}{2}} \left(u^6-u^4\right)du$

   (C) $\displaystyle\int_{0}^{\frac{\sqrt{3}}{2}} \left(u^6-u^4\right)du$

   (D) $\displaystyle\int_{0}^{\frac{\sqrt{3}}{2}} \left(u^4-u^6\right)du$

3. The graph of $f(x)=\dfrac{5}{x^2+1}$ for $x\geq 0$, is shown below. The shaded region is bounded by the graph of $f(x)$, the y - axis and the line with equation $y=1$.



The shaded region is rotated about the x-axis to form a solid of revolution.

The volume of the solid, in cubic units, is given by

(A) $\pi\displaystyle\int_0^2\left(\left(\frac{5}{x^2+1}\right)^2-1\right)dx$

(B) $\pi\displaystyle\int_0^2\left(\frac{5}{x^2+1}-1\right)^2 dx$

(C) $\pi\displaystyle\int_0^2\left(\frac{5}{x^2+1}\right)^2 dx$

(D) $\pi\displaystyle\int_0^2\left(\frac{5}{x^2+1}-1\right)dx$

4. The area bounded by the function $f(x)=\sqrt{x^2-9}$, the x - axis and the line with equation $x = 5$ is rotated about the y-axis to form a solid of revolution.

The volume of this solid in cubic units is given by:

(A) $\pi\int_0^4 \left(y^2-9\right)dy$

(B) $\pi\int_0^4 \left(y^2+9\right)dy$

(C) $\pi\int_0^4 \left(16-y^2\right)dy$

(D) $\pi\int_0^4 \left(5-\sqrt{y^2+9}\right)^2 dy$

Section II

Attempt Question 1- 3

Answer each question on a SEPARATE writing booklet.

Your response should include relevant mathematical reasoning and/or calculations.

Question 1 (13 marks) Use a SEPARATE writing booklet.

a) (i) Differentiate $\dfrac{\log_e x}{x}$. **1**

(ii) Hence, evaluate $\displaystyle\int_e^{e^2} \frac{1-\log_e x}{x\log_e x}\,dx$ in simplest form. **3**

b) Use a suitable substitution to evaluate $\displaystyle\int_3^8 \frac{2x}{(x-1)+\sqrt{x+1}}\,dx$ **4**

c) (i) Find the real numbers A, B and C such that

$$\frac{2x^2+7x-1}{(x-2)(x^2+x+1)} = \frac{A}{x-2}+\frac{Bx+C}{x^2+x+1}$$ **2**

(ii) Hence, find:

$$\int \frac{2x^2+7x-1}{(x-2)(x^2+x+1)}\,dx$$ **3**

End of Question 1

Question 2 (11 marks) Use a SEPARATE writing booklet.

a) Using the substitution $t = \tan\dfrac{x}{2}$, or otherwise , **3**

evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{2\sin x + \cos x + 3}\,dx.$

b) Let $I_n = \displaystyle\int_0^1 \left(1 - x^2\right)^n dx$ and $J_n = \displaystyle\int_0^1 x^2\left(1 - x^2\right)^n dx$

(i) Applying integration by parts to I_n, show that $I_n = 2n\,J_{n-1}$. **3**

(ii) Hence, show that $I_n = \dfrac{2n}{2n+1} I_{n-1}$. **2**

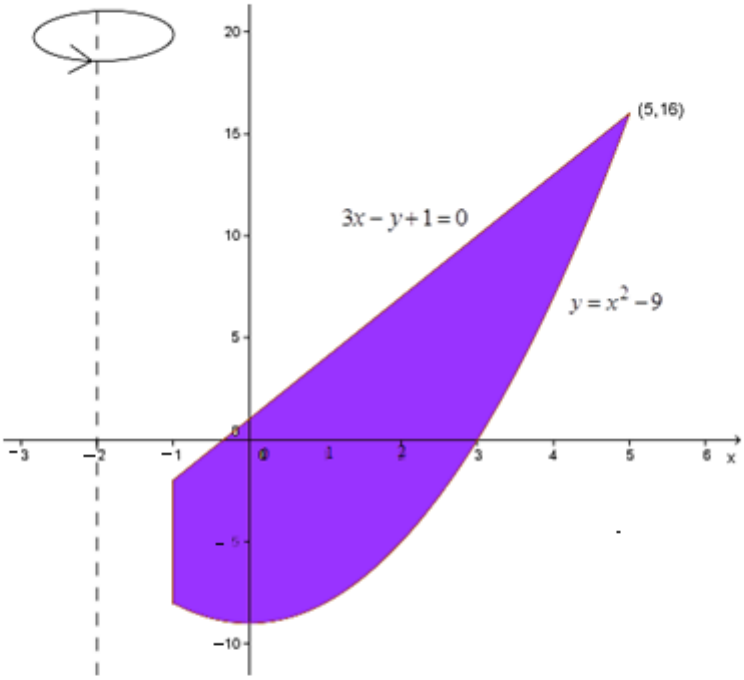
(iii) Show that $J_n = I_n - I_{n+1}$, and hence deduce that $J_n = \dfrac{1}{2n+3} I_n$. **2**

(iv) Hence, write down a reduction formula for J_n in terms of J_{n-1}. **1**

End of Question 2

Question 3 (14 marks) Use a SEPARATE writing booklet.

a) The region bounded by the curve $y = x^2 - 9$, and the line $3x - y + 1 = 0$ and the line $x = -1$ is rotated about the line $x = -2$ to form a solid.



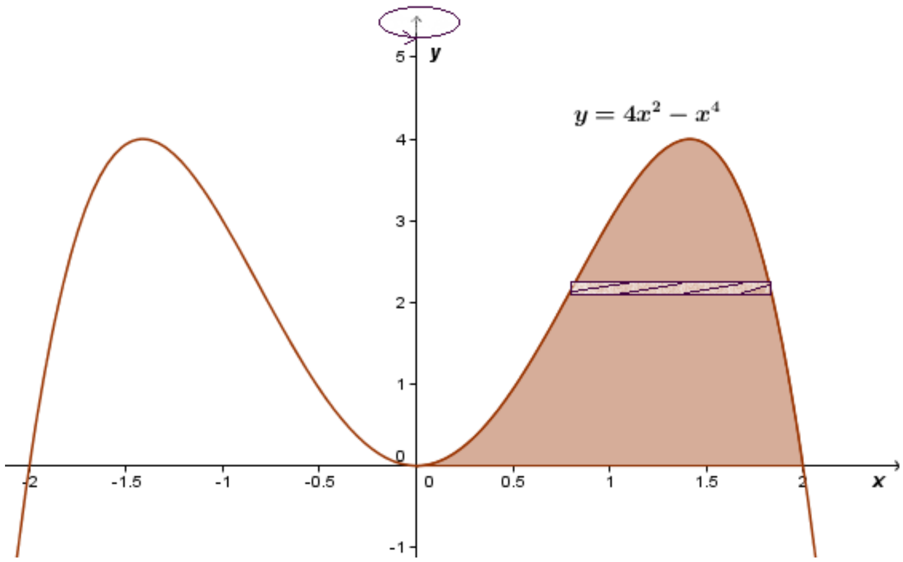
(i) Using method of cylindrical shells, show that the volume of an elemental shell is given by **3**

$$\delta V = 2\pi(x+2)(10+3x-x^2)\delta x$$

(ii) Find the volume of the solid formed. **2**

Question 3 continues on next page

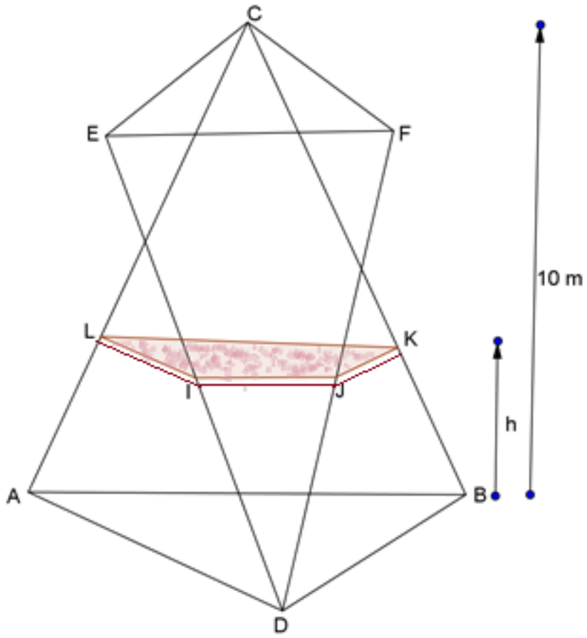
b) The shaded region shown below represents the area bounded by the curve $y = 4x^2 - x^4$, the x-axis for $0 \le x \le 2$.

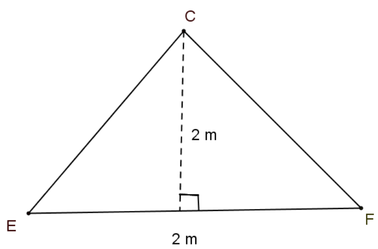


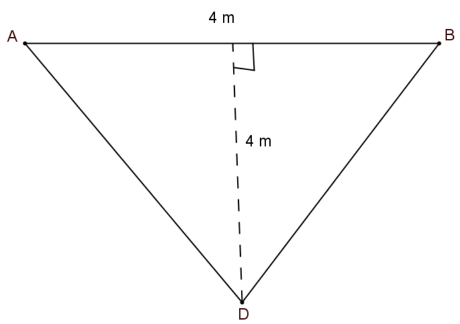
By taking slices perpendicular to the axis of rotation, use the method of slicing to find the volume of the solid obtained by rotating the shaded region about the y-axis? **3**

Question 3 continues on next page

c) A large sculpture has a triangular base (ABD) and top (EFC) as shown. The height of the sculpture is 10 metres. Each cross- section parallel to the base is an isosceles trapezium, and all other dimensions are shown in the diagram below. A typical cross-section $IJKL$ is shown in the diagram.







(i) Show, with working, that the cross-sectional area of the slice at **4**

h metres above the base, is given by

$$A = 8 - \frac{4h}{5} + \frac{h^2}{50}$$

(ii) Hence by considering the typical slice $IJKL$ of thickness δh, **2**

find the volume of the sculpture.

End of paper

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STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

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Multiple Choice Answer Sheet

Completely fill the response oval representing the most correct answer.

1. A ◯ B ◯ C ◯ D ◯
2. A ◯ B ◯ C ◯ D ◯
3. A ◯ B ◯ C ◯ D ◯
4. A ◯ B ◯ C ◯ D ◯

Marking scheme

MC	1. D 2. B 3. A 4. C	
Q1 (i)	$y=\dfrac{\log_e x}{x}$ $y'=\dfrac{x.\frac{1}{x}-\log_e x.1}{x^2}$ $=\dfrac{1-\log_e x}{x^2}$	1 mark: correctly differentiates
(ii)	$\displaystyle\int_e^{e^2}\frac{1-\log_e x}{x\log_e x}dx = \int_e^{e^2}\frac{1-\log_e x}{x^2}.\frac{x}{\log_e x}dx$ $=\displaystyle\int_e^{e^2}\frac{\frac{1-\log_e x}{x^2}}{\frac{\log_e x}{x}}dx$ $=\left[\log_e\left(\dfrac{\log_e x}{x}\right)\right]_e^{e^2}$ $=\log_e\left(\dfrac{\log_e e^2}{e^2}\right)-\log_e\left(\dfrac{\log_e e}{e}\right)$ $=\log_e\left(\dfrac{2\log_e e}{e^2}\right)-\log_e\left(\dfrac{1}{e}\right)$ $=\log_e\left(\dfrac{2}{e}\right)$	1 mark: writes in $\dfrac{f'(x)}{f(x)}$ form. 1 mark: integrates correctly 1 mark: correctly evaluates the integral.
b)	$\displaystyle\int_3^8\frac{2x}{(x-1)+\sqrt{x+1}}dx$ Let $u^2=x+1$ $2udu=dx$ $x=3,\ u=2;\ x=8,\ u=3$ **1 mark** $\displaystyle\int_3^8\frac{2x}{(x-1)+\sqrt{x+1}}dx = \int_2^3\frac{2(u^2-1)2udu}{(u^2-2)+u}$ $=4\displaystyle\int_2^3\frac{u^3-u}{u^2+u-2}du$ **1 mark** $=4\displaystyle\int_2^3 u-1+\frac{2u-2}{u^2+u-2}du$ $=4\displaystyle\int_2^3 u-1+\frac{2}{(u+2)}du$ **1 mark**	1 mark: uses the correct substitution and converts the terminals. 1 mark: correctly converts the integral in terms of u. simplifies the integrand into simpler terms

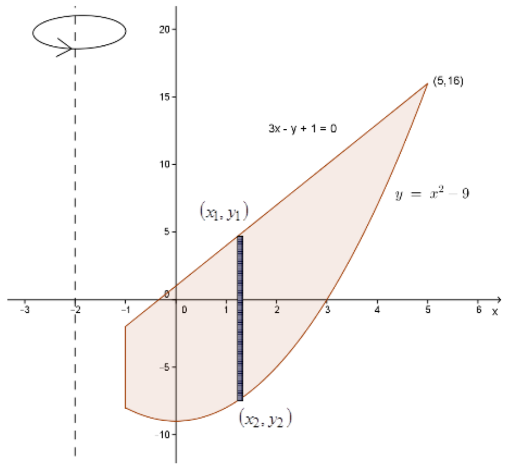
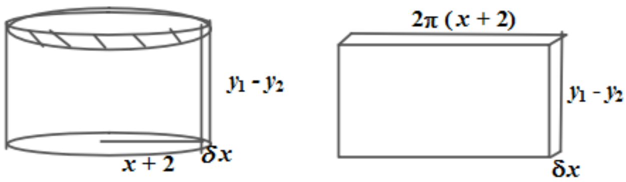
	$= \; 4\left[\frac{u^2}{2} - u + 2\log_e(u+2)\right]_2^3$ $= \; 4\left[\left(\frac{9}{2} - 3 + 2\log_e 5\right)\right] - \left(\frac{4}{2} - 2 + 2\log_e 4\right)$ $= \; 6 + 8\log_e\left(\frac{5}{4}\right)$ **1 mark**	1 mark: correct integration and evauation.
c)(i)	$\frac{2x^2+7x-1}{(x-2)(2x^2+x+1)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+x+1}$ $A(x^2+x+1) + (Bx+C)(x-2) = 2x^2+7x-1$ $x = 2, \quad 7A = 21$ **1 mark** $A = 3$ Coefficient of x^2, $2 = A + B$ $B = -1$ Constant, $-1 = A - 2C$ $2C = 4$ $\quad \therefore C = 2$ **1 mark**	1 mark: writes the equivalent polynomial to $2x^2+7x-1$ in terms of A, B and C and attempts to evaluate A, B or C. 2 mark: Solves for the correct values of A, B and C.
(ii)	$\int \frac{2x^2+7x-1}{(x-2)(2x^2+x+1)}dx = \int \frac{3}{x-2} + \frac{-x+2}{x^2+x+1}dx$ $= \; 3\ln\lvert x-2\rvert + \int \frac{-x+2}{x^2+x+1}dx$ **1 mark** $= \; 3\ln\lvert x-2\rvert - \frac{1}{2}\int \frac{2x+1-5}{x^2+x+1}dx$ $=$ $3\ln\lvert x-2\rvert - \frac{1}{2}\int \frac{2x+1}{x^2+x+1}dx + \frac{5}{2}\int \frac{1}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}dx$ **1 mark** $= \; 3\ln\lvert x-2\rvert - \frac{1}{2}\ln\lvert x^2+x+1\rvert + \frac{5}{2}.\frac{2}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}} + \text{C}$ $= \; 3\ln\lvert x-2\rvert - \frac{1}{2}n\lvert x^2+x+1\rvert + \frac{5}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}} + \text{C}$ **1 mark**	1 mark: expresses the integrand in $\frac{A}{x-2} + \frac{Bx+C}{x^2+x+1}$ form and integrates $\frac{A}{x-2}$ correctly. 1 mark: correctly separates $\int \frac{-x+2}{x^2+x+1}dx$ to integrable form. 1 mark: correctly integrates.

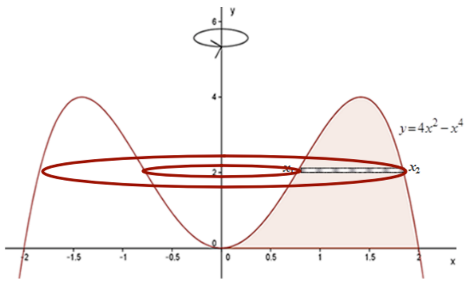
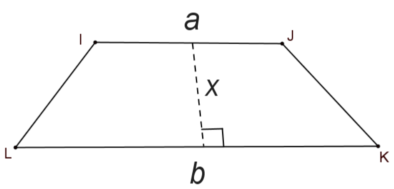
Question 2

a)	$t=\tan\frac{x}{2}$, When $x=0,\ t=0;\quad x=\frac{\pi}{2},\ t=1\quad dx=\frac{2dt}{1+t^2}$ **1 mark** $\int_0^{\frac{\pi}{2}}\frac{1}{2\sin x+\cos x+3}dx.=$ $\int_0^1\frac{1}{2.\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}+3}\frac{2dt}{1+t^2}$ $=\int_0^1\frac{\frac{2dt}{1+t^2}}{\frac{4t+1-t^2+3+3t^2}{1+t^2}}$ $=\int_0^1\frac{2dt}{2t^2+4t+4}$ $=\int_0^1\frac{dt}{t^2+2t+2}$ **1 mark** $=\int_0^1\frac{dt}{(t+1)^2+1}$ $=\left[\tan^{-1}(t+1)\right]_0^1=\tan^{-1}(2)-\tan^{-1}(1)$ $=\tan^{-1}(2)-\frac{\pi}{4}$ or $\tan^{-1}\left(\frac{1}{3}\right)$ **1 mark**	1 mark: converts the limits and dx in terms of t 1 mark: uses the correct substitutions for sinx and cosx and simplifies to give $\int_0^1\frac{dt}{t^2+2t+2}$ 1 mark: correctly integrates and evaluates.
b) (i)	$I_n=\int_0^1(1-x^2)^n dx$ $=\int_0^1(1-x^2)^n.1\,dx$ $=\left[(1-x^2)^n.x\right]_0^1-\int_0^1 n(1-x^2)^{n-1}.-2x.x\,dx$ **2 marks** $=0+2n\int_0^1 x^2\,(1-x^2)^{n-1}\,dx$ $=2n\,J_{n-1}$ **1 mark**	1 mark (each part): applies integration by parts. 1 mark: deduces the correct result.

(ii)	$I_n = 2n\,J_{n-1}$ $= 2n\int_0^1 x^2\left(1-x^2\right)^{n-1}dx$ $= 2n\int_0^1 \left[1-\left(1-x^2\right)\right]\left(1-x^2\right)^{n-1}dx$ **1 mark** $= 2n\int_0^1 \left(1-x^2\right)^{n-1}dx \;-\; 2n\int_0^1 \left(1-x^2\right)^{n}dx$ $= 2n\,I_{n-1} - 2n\,I_n$ $\left(2n+1\right)I_n = 2n\,I_{n-1}$ $I_n = \dfrac{2n}{2n+1}I_{n-1}$ **1 mark**	1 mark: expresses x^2 as $1-\left(1-x^2\right)$ and rewrites the integrand into difference of two terms. 1 mark: uses the definition of I_n and I_{n-1} to give the required result.
(iii)	$J_n = \int_0^1 x^2\left(1-x^2\right)^n dx$ $\int_0^1 \left[1-\left(1-x^2\right)\right]\left(1-x^2\right)^n dx$ $= \int_0^1 \left(1-x^2\right)^n dx \;-\; \int_0^1 \left(1-x^2\right)^{n+1}dx$ $\therefore J_n = I_n - I_{n+1}$ **1 mark** $I_n - \dfrac{2(n+1)}{2(n+1)+1}I_n$ using the result from (ii) $= I_n\left[\dfrac{2n+3-2n-2}{2n+3}\right]$ $= I_n\left(\dfrac{1}{2n+3}\right)$ **1 mark**	1 mark: 1 mark: expresses x^2 as $1-\left(1-x^2\right)$ to get the required result 1 mark: uses the result from (ii) for I_{n+1} and simplifies to get the result
(iv)	$J_n = \int_0^1 x^2\left(1-x^2\right)^n dx$ $= \left(\dfrac{1}{2n+3}\right)I_n$ from (iii) $= \left(\dfrac{1}{2n+3}\right)2n\,J_{n-1}$ using (i) $= \dfrac{2n}{2n+3}J_{n-1}.$	1 mark: uses the results from (i) and (iii) to give the relationship.

Question 3

a) (i)	 $x^2 - 9 = 3x + 1$ $x^2 - 3x - 10 = 0$ $(x-5)(x+2) = 0$ $X = 5, \ -2$  **1 mark** Volume of the slice is $\delta V = 2\pi(x+2)(y_1 - y_2)\delta x$ **1 mark** $= 2\pi(x+2)(3x + 1 - x^2 + 9)\delta x$ $= 2\pi(x+2)(10 + 3x - x^2)\delta x$ **1 mark**	1 mark: Draws the sketch of the cylindrical shell with correct dimensions 1 mark: Develops the volume of the cylindrical shell δV by finding the volume of the rectangular prism (diagram or explanation required) 1 mark: converts $y_1 - y_2$ in terms of x and proves the required expression for δV
(ii)	Summing up the volumes of all such typical shells between $x = -1$ to $x = 5$ and then let $\delta x \to 0$ gives the volume of the solid. Thus, $V = 2\pi \int_{-1}^{5} (x+2)(10 + 3x - x^2)dx$ **1 mark** $= 2\pi \int_{-1}^{5} (10x + 3x^2 - x^3 + 20 + 6x - 2x^2)dx$ $= 2\pi \int_{-1}^{5} (20 + 16x + x^2 - x^3)dx$ $= 2\pi \left[20x + 8x^2 + \frac{x^3}{3} - \frac{x^4}{4} \right]_{-1}^{5}$	(ii) 1 mark: writes the expression for V as the sum of δV s and let $\delta x \to 0$ And gives the form $2\pi \int_{-1}^{5} (x+2)(10 + 3x - x^2)dx$ 1 mark: correctly integrates to give the correct answer.

	$= 2\pi\left[100+200+\frac{125}{3}-\frac{625}{4}\right]-\left[20+8+\frac{1}{3}-\frac{1}{4}\right]$ $= 396\pi$ unit^3. **1 mark**	
b)	 Area of the Annulus $\delta A = \pi\left(x_2{}^2 - x_1{}^2\right)$ Volume of the slice $\delta V = \pi\left(x_2{}^2 - x_1{}^2\right)\delta y$ **1 mark** $y = 4x^2 - x^4$ *ie.* $x^4 - 4x^2 + y = 0$ $x^2 = \frac{4 \pm \sqrt{16-4y}}{2} = 2 \pm \sqrt{4-y}$ $x_1{}^2 = 2 - \sqrt{4-y}$ $x_2{}^2 = 2 + \sqrt{4-y}$ $\therefore x_2{}^2 - x_1{}^2 = 2\sqrt{4-y}$ $\therefore \delta V = \pi\left(2\sqrt{4-y}\right)\delta y$ **1 mark** Volume of the solid = $V = 2\pi\int_0^4 \sqrt{4-y}\ \delta y$ $= -2\pi.\frac{2}{3}\left[(4-y)^{\frac{3}{2}}\right]_0^4$ $= \frac{-4\pi}{3}(0-8) = \frac{32\pi}{3}$ unit^3 **1 mark**	1 mark: gives the cross-section area of the perpendicular slice. 1 mark: gives the volume of the slice in terms of y. 1 mark: calculates the volume
c) (i)	 When $h = 0$, $a = 0$ $h = 10$, $a = 2$	4 marks: Expresses a, b and x in terms of h and calculates the area of the cross-section $IJKL$.

	$a = mh + p$ $h = 0, \quad a = 0$ $a = mh$ $= \frac{2-0}{10-0}h \qquad \therefore a = \frac{h}{5}$ Similarly, When $h = 0$, $b = 4$ $h = 10$, $b = 0$ $\therefore b = \frac{0-4}{10-0}h + 4 \qquad b = \frac{-2}{5}h + 4$ Also, When $h = 0$, $x = 4$ $h = 10$, $x = 2$ $\therefore x = \frac{2-4}{10-0}h + 4 \qquad x = \frac{-1}{5}h + 4$ Area of $IJKL = \frac{x}{2}(a+b)$ $= \frac{1}{2}\left(\frac{-1}{5}h + 4\right)\left(\frac{h}{5} + \frac{-2}{5}h + 4\right)$ $= \frac{h^2}{50} - \frac{2h}{5} - \frac{2h}{5} + 8$ $= \frac{h^2}{50} - \frac{4h}{5} + 8$	Subtract 1 mark for errors in each of the calculations. Award 1 mark: If an attempt to find the area of cross-section with reasonable mathematical working.
(ii)	Volume of the cross-section $\delta V = Ah$ $= \left(\frac{h^2}{50} - \frac{4h}{5} + 8\right)\delta h$ Summing up the volumes of all such typical slices between $h = 0$ to $h = 10$ and then let $\delta h \to 0$ gives the volume of the solid. Thus volume of the solid $V = \int_0^{10} \left(\frac{h^2}{50} - \frac{4h}{5} + 8\right) dh$ **1 mark** $\left[\frac{h^3}{150} - \frac{4h^2}{10} + 8h\right]_0^{10}$ $= \frac{140}{3} - 0 = \frac{140}{3}$ unit^3 **1 mark**	1 mark: develops the correct expression for V. 1 mark: Calculates the correct volume.