Killara HIGH SCHOOL

CONSERVA PROGREDERE

STUDENT NUMBER

MATHEMATICS EXTENSION 2

ASSESSMENT TASK NO 2

Date 20 March 2014

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen Black pen is preferred.
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- In Questions 11 – 16, show relevant mathematical reasoning and/ or calculations

M.C.	/10
Q11	/15
Q12	/15
Q13	/15
Q14	/16
Q15	/14
Q16	/15

Total marks – 100

<u>Section 1</u> Pages 2 - 5

10 marks

- Attempt Questions 1 – 10
- Allowed about 15 minutes for this section

<u>Section 2</u> Pages 6 - 13

90 marks

- Attempt Questions 11- 16
- Allow about 2 hours and 45 minutes for this section

This assessment task constitutes 30% of the Higher School Certificate Course Assessment.

THIS QUESTION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM

Multiple choice questions (10 marks). **Marks**

Allow 15 minutes for this section

Use the multiple-choice answer sheet for Q1 to 10

1. $\sqrt{3} + i$ expressed in modulus argument form is:

(A) $2\,cis\,\dfrac{\pi}{4}$

(B) $2\,cis\,\dfrac{\pi}{6}$

(C) $\sqrt{2}\,cis\,\dfrac{\pi}{3}$

(D) $2\,cis\,\pi$

2. The parametric equations of the hyperbola $\dfrac{(x+1)^2}{4} - \dfrac{(y-3)^2}{2} = 1$ are:

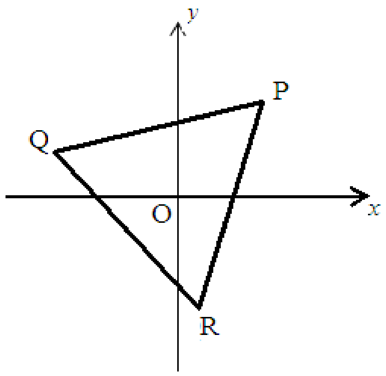
(A) $x = -1 + 2\sec\theta,\ y = 3 + \sqrt{2}\ \tan\theta$

(B) $x = 1 + 2\sec\theta,\ y = 3 - \sqrt{2}\ \tan\theta$

(C) $x = 2\sec(\theta - 1),\ y = \sqrt{2}\ \tan(\theta + 3)$

(D) $x = 3 + 2\sec\theta,\ y = -1 + \sqrt{2}\ \tan\theta$

3. P represent the complex number $p = 1 + 2i$ on an Argand diagram. Given that PQR is an equilateral triangle whose centre is at the origin O. Find the complex number q, represented by the point Q.



(A) $\dfrac{-(1+2\sqrt{3}) + i\ (\sqrt{3}-2)}{2}$

(B) $\dfrac{(1+2\sqrt{3}) - i\ (\sqrt{3}-2)}{2}$

(C) $\dfrac{-(1+3\sqrt{2}) + i\ (\sqrt{3}-3i)}{2}$

(D) $\dfrac{-(1+2\sqrt{3}) + i\ (\sqrt{3}-2i)}{4}$

4. Which pair of equations gives the asymptotes of $9x^2 - 16y^2 = 144$

(A) $y = \pm\frac{4x}{3}$

(B) $y = \pm\frac{4}{3}$

(C) $y = \pm\frac{9x}{16}$

(D) $y = \pm\frac{3x}{4}$

5. Factorize $x^4 + x^2 - 12$ over the complex field.

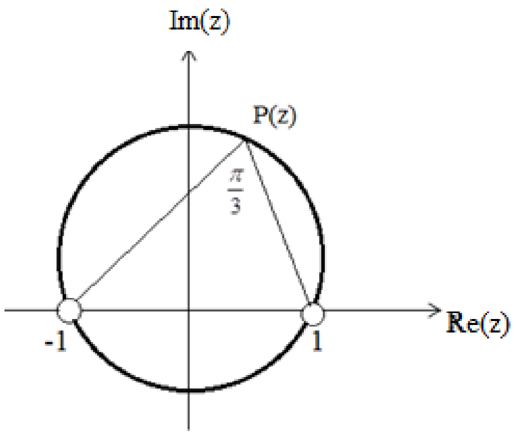
(A) $(x^2+4)(x-i\sqrt{3})(x+i\sqrt{3})$

(B) $(x-2i)(x-\sqrt{3})(x+\sqrt{3})$

(C) $(x+2i)(x-2i)(x+\sqrt{3})(x-\sqrt{3})$

(D) $(x+2)(x-2)(x+i\sqrt{3})(x-i\sqrt{3})$

6. The locus of z is represented by the circle in the Argand diagram below. The centre and radius of the circle are:



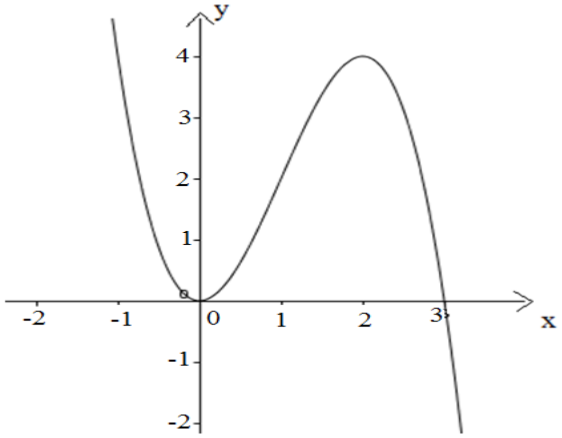
(A) $C\left(0, \frac{\sqrt{3}}{2}\right)$ *and* $r = \frac{\sqrt{3}}{2}$

(B) $C\left(0, \frac{\sqrt{3}}{3}\right)$ *and* $r = \frac{\sqrt{3}}{2}$

(C) $C\left(0, \frac{\sqrt{3}}{3}\right)$ *and* $r = \frac{2\sqrt{3}}{3}$

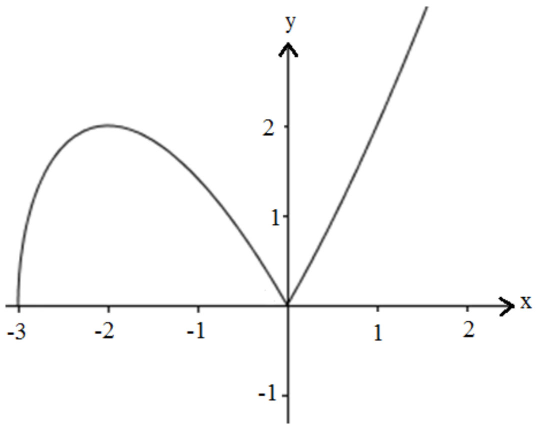
(D) $C\left(0, \sqrt{3}\right)$ *and* $r = \frac{2\sqrt{3}}{3}$

7. The sketch of $y = f(x)$ is shown.

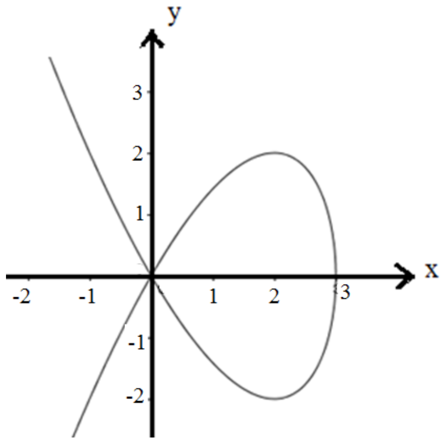


The graph which represents $y^2 = f(x)$ is:

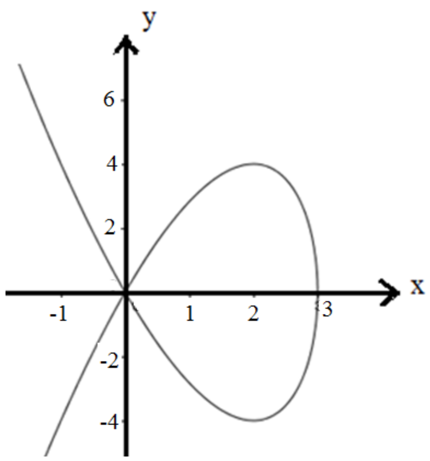
(A)



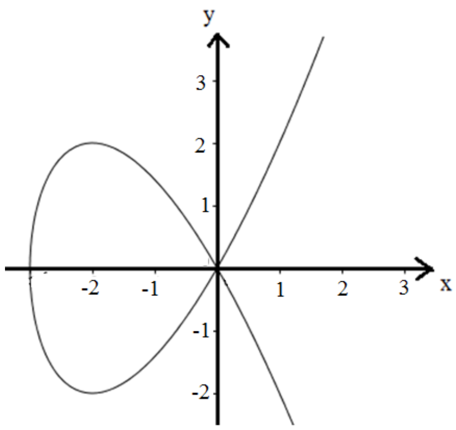
(B)



(C)



(D)



8. The length of the latus rectum of the hyperbola $\frac{x^2}{25}-\frac{y^2}{4}=1$ is:

(A) $\frac{8}{25}$

(B) $-\frac{8}{25}$

(C) $\frac{4}{25}$

(D) $\frac{8}{5}$

9. Given $4x^3 + 3x - 2$ has a rational zero, find all other zeros.

(A) $\frac{1}{2}$ *and* $\pm\frac{1+\sqrt{15i}}{4}$

(B) $\frac{1}{2}$ *and* $\frac{-1\pm\sqrt{15i}}{2}$

(C) $\frac{1}{2}$ *and* $\frac{-1\pm i\sqrt{15}}{4}$

(D) $\frac{1}{2}$ *and* $\frac{1\pm i\sqrt{15}}{4}$

10. If α,β *and* λ are the roots of the equation $x^3+2x^2-5x-6=0$, the value of $(\alpha-1)(\beta-1)(\lambda-1)$ is:

(A) 12

(B) - 6

(C) 15

(D) 8

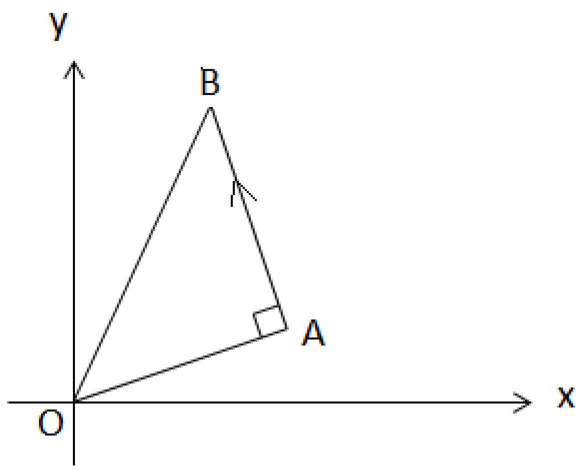
Question 11 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Evaluate $|3-2i|$ **1**

(b) Express in the form $x+iy$, where x and y are real

$$(3+4i)\overline{(4-i)}$$ **2**

(c)



In the Argand diagram ΔOAB is isosceles and right angled at A. $\vec{AB}$ represents the complex number α

(i) What complex number corresponds to the vertex A? **1**

(ii) What complex number correspond to the vertex B? **1**

(iii) Show that the area of ΔOAB is $\frac{1}{2}\alpha\overline{\alpha}$. **1**

Question 11 continues on page 7

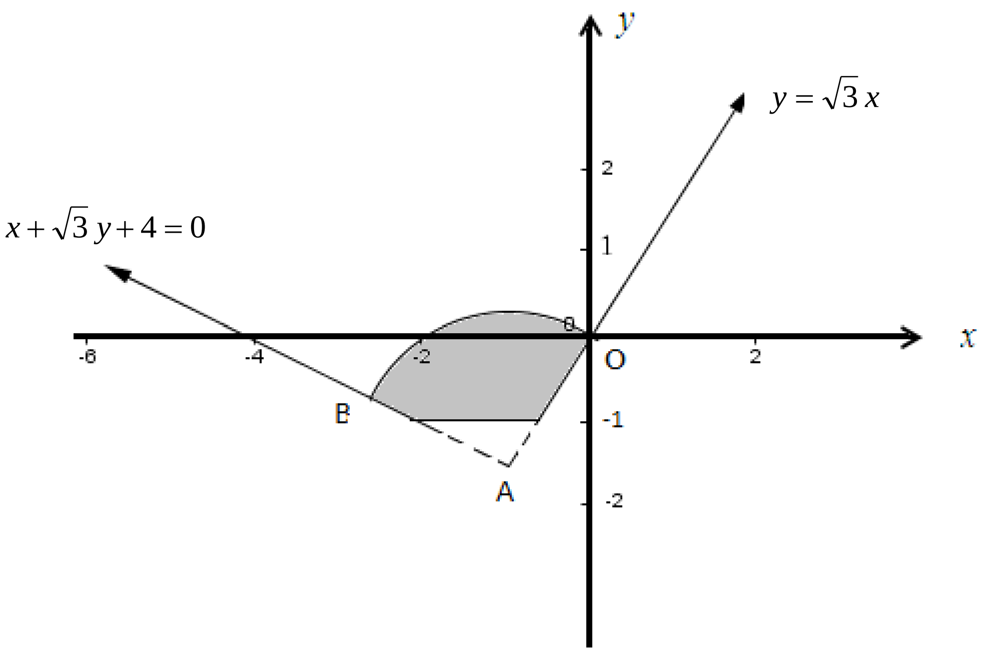
(d) The complex number z is given by:

$$z = 1 + \frac{1+i}{1-i}$$

(i) Express z in modulus-argument form. **2**

(ii) Hence find z^5 in the form $x + iy$, where x and y are real. **2**

(e) In the Argand diagram below the shaded region is a part of a quarter of a circle whose centre is A and its radius is AO. Find the conditions that should be satisfied by the complex numbers Z which represents the shaded region. **5**



End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) The equation $2x^3 - 2x^2 + 1 = 0$ has roots α, β and γ. **3**

Find the equation with roots α^3, β^3 and γ^3.

(b) The graph of function $f(x) = x\sqrt{9 - x^2}$ is given below (on page 9).

α) Find the co-ordinates of the stationary points and x-intercepts. **3**

Then copy or trace the graph into your writing booklet and mark all the important features.

β) Use the graph of $f(x)$ to sketch each of the following on a separate number plane

(i) $y = f(-x)$ **1**

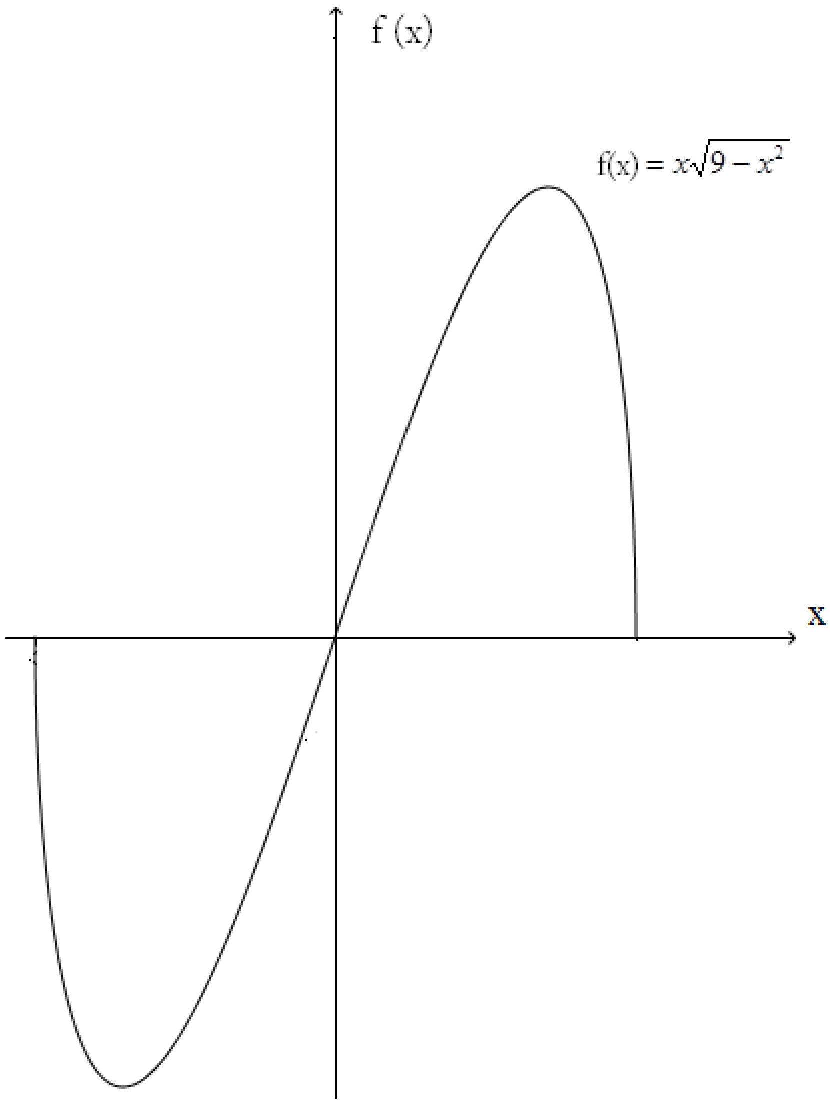
(ii) $y = \dfrac{1}{f(x)}$ **2**

(iii) $y = |f(x)|$ **1**

(iv) $y = \left(f(x)\right)^2$ **2**

(v) $y = e^{f(x)}$ **3**

Question 12 continues on page 9



End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Find the equation of the locus of a point *P(x, y)* such that its distance from the point *S(ae, 0)* to the line $x = \frac{a}{e}$ is in the ratio $e : 1$ where $e < 1$. **3**

(b) For the hyperbola $5x^2 - 4y^2 = 20$:

(i) Find the equations of the asymptotes **1**

(ii) If the point $P(2\sec\theta, \sqrt{5}\tan\theta)$ lies on this hyperbola prove that the equation of the tangent to the hyperbola at P is $\frac{x\sec\theta}{2} - \frac{y\tan\theta}{\sqrt{5}} = 1$. **3**

(iii) If the tangent cuts the asymptotes at L and M, prove that LP = PM. **4**

(iv) Prove that the area of triangle OLM is independent of the position of P, where O is the origin. **4**

End of Question 13.

Question 14 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) The polynomial P(z) has the equation $P(z) = z^4 - 5z^3 + 7z^2 + 3x - 10$ **2**

Given that $z - 2 + i$ is a factor of $P(z)$, express $P(z)$ as a product of factors with real coefficients.

(b) Let $f(t) = t^3 + ct + d$, where c and d are constants.

Suppose that the equation $f(t) = 0$ has three distinct real roots, $t_{1,}\ t_2$ *and* t_3.

(i) Find $t_1 + t_2 + t_3$. **1**

(ii) Show that $t_1^{\ 2} + t_2^{\ 2} + t_3^{\ 2} = -2c.$ **2**

(iii) Since the roots are real and distinct, the graph of $y = f(t)$ has two turning points, at $t = u$ and $t = v$, and $f(u).f(v) < 0$. **4**

Show that $27d^2 + 4c^3 < 0.$

(c)

(i) Suppose that k is a double root of the polynomial equation $f(x) = 0$. Show that $f\ '(k) = 0$. **2**

(ii) The polynomial $P(x) = ax^7 + bx^6 + 1$ is divisible by $(x-1)^2$. Find the coefficient a and b. **2**

(iii) Let $E(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$. Prove $E(x) = 0$ has no double root **2**

End of Question 14

Question 15 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) The ellipse E, is given in terms of the complex number z by: **2**

$|z+3|+|z-3|=10$. Find the Cartesian equation of E.

(b) Find the integral $\int \frac{x-1}{\sqrt{2x+3}}\,dx$ **2**

(c) (i) Show that the tangent to the rectangular hyperbola $xy=c^2$ at the point **2**

$T\left(ct,\frac{c}{t}\right)$ has equation $x+t^2y=2ct$.

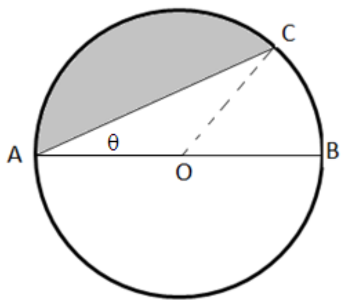
(ii) The tangents to the rectangular hyperbola $xy=c^2$ at the points $P\left(cp,\frac{c}{p}\right)$

and $Q\left(cq,\frac{c}{q}\right)$, where $pq=1$, intersect at R.

α) Find the equation of the locus of R. **3**

β) State any restriction on the values of **x** for this locus. **1**

(d)



In the diagram, AB is the diameter of the circle, and $\angle$BAC = θ.

(i) If the area of the shaded region is one third of the area of the circle,

show that $\sin 2\theta=\frac{\pi}{3}-2\theta$ **3**

(ii) Show graphically that this equation has only one solution. **2**

End of Question 15

Question 16 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Given that $\sin(x+y)+\sin(x-y)=2\sin x\cos y$ **3**

and $\cos(x+y)+\cos(x-y)=2\cos x\cos y$ and by using the fact that

A + B + C = π for triangle ABC:

Show that $\dfrac{\sin A+\sin B}{\cos A+\cos B}=\cot\dfrac{C}{2}$

(b) Let $f(x)=\begin{cases}\dfrac{\sin x}{x} & \textit{for} \quad 0 < x \le \dfrac{\pi}{2} \\ 1 & \textit{for} \quad x=0\end{cases}$

(i) Find the derivative of $f(x)$ *for* $0<x<\dfrac{\pi}{2}$ and prove that $f'(x)$ is negative in this interval. **2**

(ii) Sketch the graph of y = *f*(*x*) for $0<x<\dfrac{\pi}{2}$ and deduce that $\sin x \ge \dfrac{2x}{\pi}$ in this interval **2**

(c) Let $\omega=\cos\dfrac{2\pi}{7}+i\sin\dfrac{2\pi}{7}$. The complex number $\alpha=\omega+\omega^2+\omega^4$ is a root of the quadratic equation $x^2+ax+b=0$, where a and b are real.

(i) Prove that $1+\omega+\omega^2+........+\omega^6=0$. **2**

(ii) The second root of the quadratic equation is β. Express β in terms of positive powers of ω. Justify your answer. **1**

(iii) Find the values of the coefficients a and b. **2**

(iv) Deduce that $-\sin\dfrac{\pi}{7}+\sin\dfrac{2\pi}{7}+\sin\dfrac{3\pi}{7}=\dfrac{\sqrt{7}}{2}$. **3**

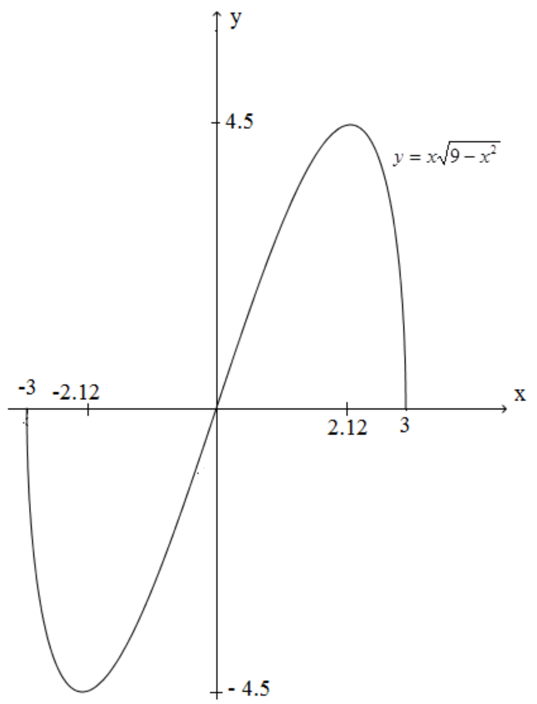
END OF EXAMINATON

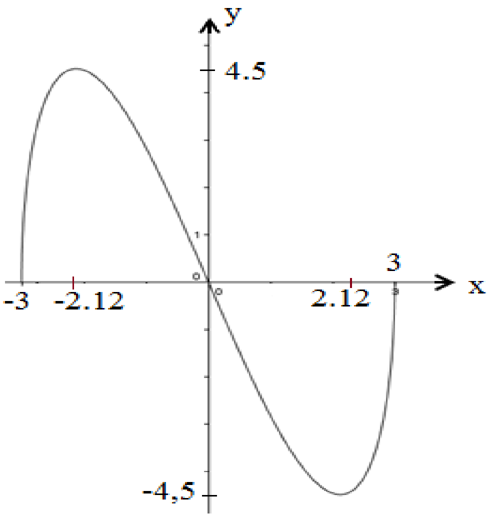
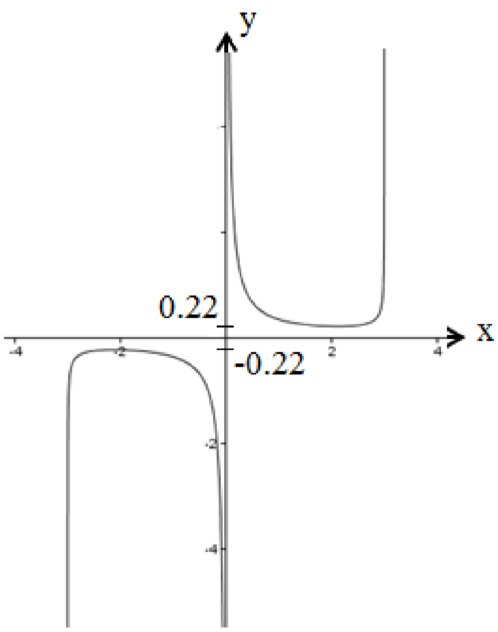
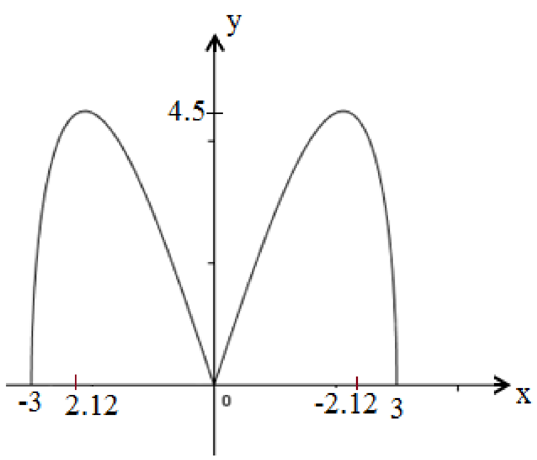
Extention 2 Mathematics-2014 Half yearly solution

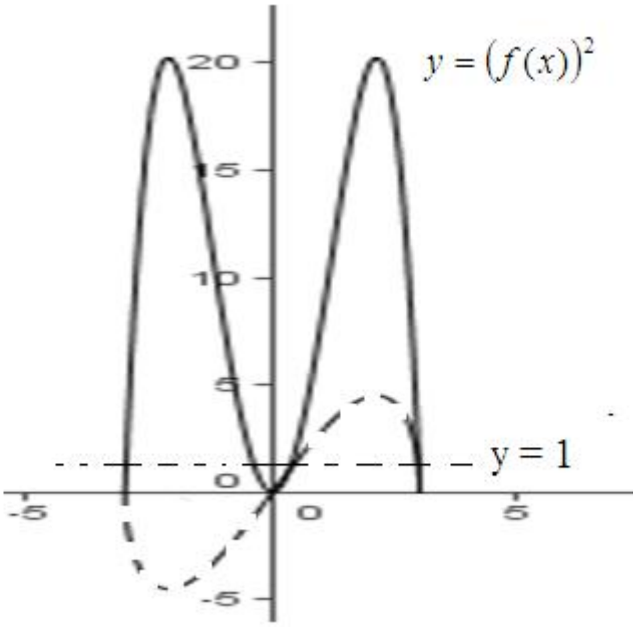
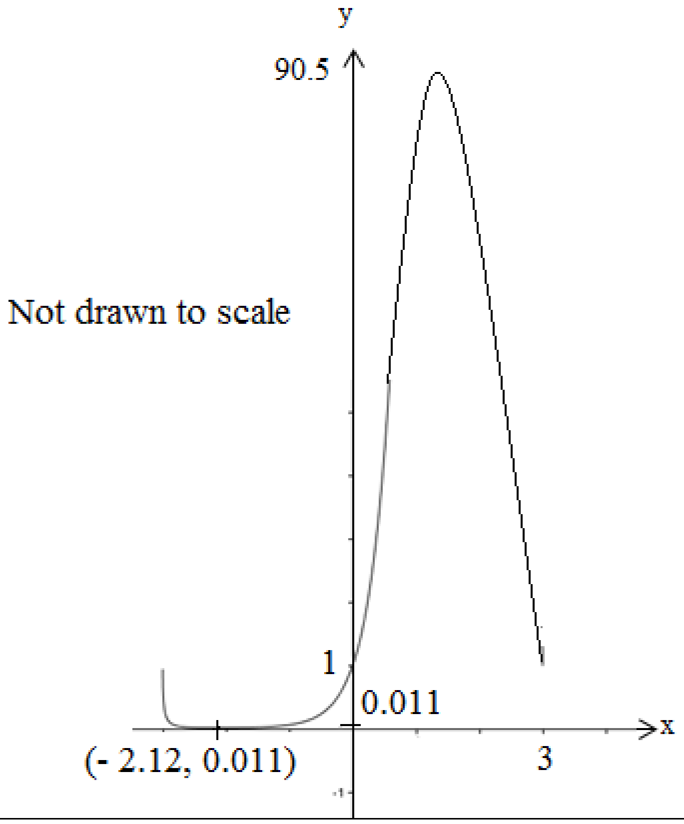
1. B **2.** A **3.** A **4.** D **5.** C **6.** B **7.** B **8**. D **9.** C **10.** D	1 mark each	
Question 11 (15 marks)		
a) $\lvert 3-2i\rvert = \sqrt{9+4} = \sqrt{13}$	1	
b) $(3+4i)\overline{(4-i)} = (3+4i)(4+i)$ $= 12+3i+16i-4$ $= 8+19i$	1 1	
c) (i) $\overrightarrow{AO} = i\,\overrightarrow{AB}$ $= i\alpha$ $\overrightarrow{OA} = -i\alpha$ correspond to vertex A.	1	
(ii) $\overrightarrow{OB} = \overrightarrow{AB} - \overrightarrow{OA}$ $= \alpha - i\alpha$ $= \alpha(1-i)$ $\quad$ *corresponds to vertex B*	1	
(iii) $A = \frac{1}{2} \times OA \times AB$ $= \frac{1}{2}\lvert\alpha\rvert^2 \quad (\Delta OAB \text{ is isosceles}, OA = AB)$ $= \frac{1}{2}\alpha.\overrightarrow{\alpha}$	1	
d) (i)	1	

$1+\dfrac{1+i}{1-i}=\dfrac{1-i+1+i}{1-i}$ $=\dfrac{2(1+i)}{(1-i)(1+i)}$ $=1+i$ $\therefore \quad z=\sqrt{2}\,cis\dfrac{\pi}{4}$	1 1	
(ii) $z^5=(\sqrt{2})^5 cis\dfrac{5\pi}{4}$ $=4\sqrt{2}\left(-\dfrac{1}{\sqrt{2}}-\dfrac{1}{\sqrt{2}}i\right)$ $=-4-4i$	 1	
e) Point of intersection: sub. $y=\sqrt{3}x$ into $x+\sqrt{3}y+4=0$ $x+\sqrt{3}.\sqrt{3}x+4=0$ $4x=-4$ $x=-1$ and $y=-\sqrt{3}$	1 finding correct answer for point of intersection	
Radius = $\sqrt{3+1}=2$	1 Correct radius	
$m_1=\tan^{-1}(\sqrt{3})$ $=\dfrac{\pi}{3}$ $m_2=\tan^{-1}\left(-\dfrac{1}{\sqrt{3}}\right)$ $=-\dfrac{5\pi}{6}$	1 correct gradients	
$\therefore$ The conditions which satisfy the complex number z in the region are: im z ≥ -1, $\quad \left\|z+1+\sqrt{3}\right\| \leq 2$ and $\dfrac{\pi}{3} \leq \arg z \leq \dfrac{5\pi}{6}$	2 for three correct conditions	

Question 12

(a) $Let\ y = x^3\ ,\ then\quad x = \sqrt[3]{y}$ Sub. x into $2x^3 - 2x^2 + 1 = 0$ $2\left(\sqrt[3]{y}\right)^3 - 2y^{2/3} + 1 = 0$ $2y + 1 = 2y^{2/3}$ $(2y+1)^3 = 8y^2$ $8y^3 + 12y^2 + 6y + 1 = 8y^2$ $8y^3 + 4y^2 + 6y + 1 = 0$ $\therefore\ The\ equation\ with\ roots\ \alpha^3, \beta^3\ and\ \gamma^3\ is$ $8x^3 + 4x^2 + 6y + 1 = 0$	1 correct substitution of x 1 simplifying 1 correct answer	
b)  (i)	1 both x-intercepts 2 Correct values of x & y coordinates of max & min points.	

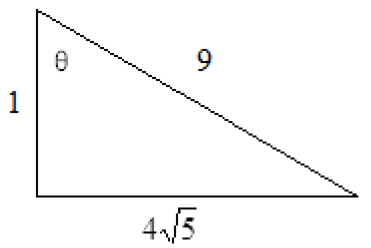
 (ii)	1 correct shape & locations of max & min points	
 (iii)	1 correct shape 1 correct approximation values of max & min points	
 (iv)	1 correct shape & values of max points	

	1 correct shape 1 The graph must be below original graph for y <1	
(v) 	1 correct shape 1 correct approximation values of max & min points 1 y-coordinates of both end points of the graph is at 1	

Question 13 (15 marks)

a) $PS = e.PM$ $PS^2 = e^2.PM$ $(x-ae)^2+y^2 = e^2.\left(x-\frac{a}{e}\right)^2$ $x^2-2aex+a^2e^2+y^2 = e^2x^2-2eax^2+a^2$ $(1-e^2)x^2+y^2 = a^2(1-e^2)$ $\frac{x^2}{a^2}+\frac{y^2}{a^2(e^2-1)}=1$ $\frac{x^2}{a}+\frac{y^2}{b}=1 \quad where\ \ b^2 = a^2.(1-e^2)$	1 1 1	
(b) (i) $y=\pm\frac{\sqrt{5}}{2}x$	1	
(ii) $10x-8y\frac{dy}{dx}=0$ $\frac{dy}{dx}=\frac{5x}{4y}$ $At\ \ P(2\sec\theta,\sqrt{5}\tan\theta) \quad \frac{dy}{dx}=\frac{5\times2\sec\theta}{4\times\sqrt{5}\tan\theta}$ $=\frac{\sqrt{5}\sec\theta}{2\tan\theta}$	1 correct differentiation to find m.	
Equation of the tangent at P is $y-\sqrt{5}\tan\theta=\frac{\sqrt{5}\sec\theta}{2\tan\theta}(x-2\sec\theta)$ $2y\tan\theta-2\sqrt{5}\tan^2\theta=\sqrt{5}x\sec\theta-2\sqrt{5}\sec^2\theta$ $\sqrt{5}x\sec\theta-2y\tan\theta=2\sqrt{5}(\sec^2\theta-\tan^2\theta)$ $\therefore \quad \frac{x\sec\theta}{2}-\frac{y\tan\theta}{\sqrt{5}}=1 \quad (1)$	1 correct substitution 1 correct answer	
(iii) For points of intersection of the tangent and the		

asymptotes: Sub. $y = \pm\frac{\sqrt{5}}{2}x$ into (1) $\frac{x\sec\theta}{2} - \frac{\sqrt{5}x\tan\theta}{2\sqrt{5}} = 1$ $\sqrt{5}x\ (\sec\theta - \tan\theta) = 2\sqrt{5}$ $x_1 = \frac{2}{\sec\theta - \tan\theta}$ *similarly* $x_2 = \frac{2}{\sec\theta + \tan\theta}$	1 finding x-coordinates	
Sub. x_1 into (1) $\frac{\sec\theta}{\sec\theta - \tan\theta} - \frac{y\tan\theta}{\sqrt{5}} = 1$ $\sqrt{5}\sec\theta - y\tan\theta(\sec\theta - \tan\theta) = \sqrt{5}(\sec\theta - \tan\theta)$ $y_1 = \frac{\sqrt{5}}{\sec\theta - \tan\theta}$ *similarly* $y_2 = \frac{-\sqrt{5}}{\sec\theta + \tan\theta}$	1 Finding y coordinates	
The mid-point of LM is: $M_x = \frac{2}{2(\sec\theta - \tan\theta)} + \frac{2}{2(\sec\theta + \tan\theta)}$ $= \frac{2(\sec\theta + \tan\theta) + 2(\sec\theta - \tan\theta)}{2(\sec^2\theta - \tan^2\theta)}$ $= 2\sec\theta$ $M_y = \frac{\sqrt{5}}{2(\sec\theta - \tan\theta)} - \frac{\sqrt{5}}{2(\sec\theta + \tan\theta)}$ $= \frac{\sqrt{5}(\sec\theta + \tan\theta) - \sqrt{5}(\sec\theta - \tan\theta)}{2(\sec^2\theta - \tan^2\theta)}$ $= \sqrt{5}\tan\theta$ The mid-point M is the coordinate of P $\therefore$ PM = LP	2 finding the mid-point coordinates of LM and conclusion	
(iii)		

$OL = \sqrt{\frac{4}{(\sec\theta+\tan\theta)^2}+\frac{5}{(\sec\theta+\tan\theta)^2}}$ $= \frac{3}{\sec\theta+\tan\theta}$ Similarly $OM = \frac{3}{\sec\theta-\tan\theta}$ $m_{OL} = \frac{\sqrt{5}}{2}$ *and* $m_{OM} = \frac{-\sqrt{5}}{2}$ $\tan\theta = \left\| \frac{\frac{\sqrt{5}}{2}--\frac{\sqrt{5}}{2}}{1-\left(\frac{\sqrt{5}}{2}\right)^2} \right\|$ $= \frac{\sqrt{5}}{\frac{1}{4}} = 4\sqrt{5}$  $Area\ \Delta OLM = \frac{1}{2}OM \times OL \times \sin\theta$ $= \frac{1}{2}\times\frac{3}{\sec\theta-\tan\theta}\times\frac{3}{\sec\theta+\tan\theta}\times\frac{4\sqrt{5}}{9}$ $= 2\sqrt{5}$ *units square* Which is independent of the position of P.	1 Finding the length OL & OM 1 finding the angle θ 1 finding sinθ 1 finding the area and conclusion	

<u>Question 14 (15 marks)</u>

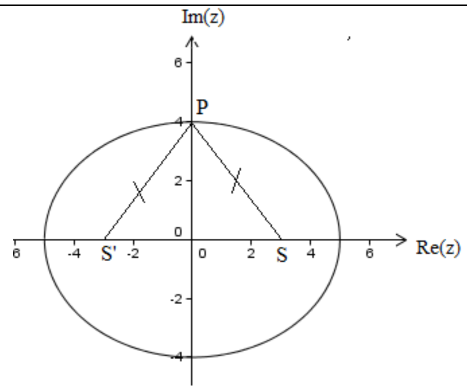
(a) P(z) has real coefficient, its complex roots are in conjugate. $\therefore$ 2 + i is also a root.	1 state 2 + i is a root	
$(z - 2 + i)(z - 2 - i) = z^2 - 4z + 5$	1 Finding the factors	
By polynomial division or by inspection 2 and -1 are the other 2 roots.		
$\therefore\ P(z) = (z^2 - 4z + 5)(z^2 - z - 2)$ $= (z^2 - 4z + 5)(z - 2)(z + 1)$	1 Correct answer	
(b) (i) $t_1 + t_2 + t_3 = 0$ (sum of roots)	1	
(ii) $t_1{}^2 + t_2{}^2 + t_3{}^2 = (t_1 + t_2 + t_3)^2 - 2(t_1t_2 + t_1t_2 + t_2t_3)$ $= 0 - 2 \times c$ $= -2c$	2	
(iii) $\dfrac{df(t)}{dt} = 3t^2 + c = 0$ $\Rightarrow\ t = \pm\sqrt{-\dfrac{c}{3}}$	1 finding u and v	
Let $\Rightarrow\ u = \sqrt{-\dfrac{c}{3}}$, $v = -\sqrt{-\dfrac{c}{3}}$ $and\ \ v = -u$		
$f(u) \,.\, f(v) = (u^3 + cu + d)(-u^3 - cv + d)$ $= \left(u\ (u^2 + c) + d\right)\left(-u\ (u^2 + c) + d\right)$ (1)	1 finding f(u). f(v)	
Since $3u^2 + c = 0$, sub $u^2 = -\dfrac{c}{3}$ into (1)		
$f(u) \,.\, f(v) = \left(u\ (-\dfrac{c}{3} + c) + d\right)\left(-u\ (-\dfrac{2}{3} + c) + d\right)$ $= \left(u.\dfrac{2c}{3} + d\right).\left(-u.\dfrac{2c}{3} + d\right)$	1 Showing sub. of $u^2 = -\dfrac{c}{3}$	
$= d^2 - \dfrac{4u^2c^2}{9}$ Sub. $u = \sqrt{-\dfrac{c}{3}}$	1 showing the	

$= \; d^2 + \dfrac{4c^3}{27} \; < \; 0$ Since f(u) . f(v) < 0 $\therefore \quad 27d^2 + 4c^3 < 0$ (c) (i) If k is a double root of *f(x)* then $(x-k)^2$ is a factor of f*(x)* Let f(x) = $(x - k)^2$ Q(x) f '(x) = 2(x – k) Q(x) + $(x - k)^2$ Q'(x) Sub. x = k into f '(x) f '(k) = 2(k – k) Q(x) + $(k - k)^2$ Q(x) = 0 as required. (ii) Since P(x) is divisible by $(x - 1)^2$, $(x - 1)^2$ is a double root of f(x). $\therefore$ P(1) = P'(1) = 0 P'(x) = $7ax^6 + 6bx^5$ P(1) = a + b + 1 = 0 $\Rightarrow \; a = -b - 1$ P'(1) = 7a + 6b = 0 (1), sub a into (1) 7(- b – 1) + 6b = 0 $\therefore$ b = - 7 and a = 6 (ii) Suppose k is a double root of E(x) then E(k) = E'(k) = 0 E'(x) = $1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{5}$ Sub. x = k into E(x) and E'(x)	substitution of u to obtain the result. 1 finding f '(x) 1 Using P(x) and P'(x) to find a & b 1 correct answer for a & b 1 Using factor & remainder theorems to show the result.	

$\Rightarrow \quad 1+k+\frac{k^2}{2}+\frac{k^3}{5} = 1+k+\frac{k^2}{2}+\frac{k^3}{5}+\frac{k^4}{24}$ $\Rightarrow \quad \frac{k^4}{24} = 0$ $\therefore \; k = 0$ Sub. k = 0 into E(x) $\Rightarrow$ E(0) = 1 $\therefore$ k is not a root of E(x), by factor theorem hence E(x) has no double root.	1 Conclusion with explanation	

Question 15 (15 marks)

(a)		

		
The graph of the complex number z is drawn above. PS + PS’ = 2a (property of ellipse) $= 10$ $\therefore \quad a = 5$	1 correct value of a	
By selecting P at the position shown, $OP^2 = PS^2 - OS^2$ $= 25 - 9 = 16$ $\therefore \quad b = 4$	1 correct value of b and equation of ellipse	
The Cartesian equation of the ellipse is $\frac{x^2}{25} + \frac{y^2}{16} = 1$		
(b) $Let \quad u = 2x + 3 \quad \Rightarrow \quad x = \frac{u-3}{2}$ $and \quad du = 2\ dx$ $\int \frac{x-1}{\sqrt{2x+3}}\, dx = \frac{1}{4}\int \frac{u-5}{\sqrt{u}}\, du$	1 correct substitution	
$= \frac{1}{4}\int u^{\frac{1}{2}} - 5u^{-\frac{1}{2}}\ du$ $= \frac{1}{4}\left(\frac{2}{3}u^{\frac{3}{2}} - 10u^{\frac{1}{2}}\right)$ $= \frac{1}{6}(2x+3)^{\frac{3}{2}} - \frac{5}{2}(2x+3)^{\frac{1}{2}} + C$	1 correct integration and answer in term of x	
(c) (i) $y = \frac{c^2}{x} \qquad \frac{dy}{dx} = -\frac{c^2}{x^2}$		

At $x = ct$ $\quad \dfrac{dy}{dx} = -\dfrac{c^2}{c^2t^2} = -\dfrac{1}{t^2}$	1 finding 'm'	
Equation of the tangent at T is: $y - \dfrac{c}{t} = -\dfrac{1}{t^2}(x - ct)$ $yt^2 - ct = -x + ct$ $or \quad x + t^2 y = 2ct$	1 Correct equation of tangent	
(ii) Similarly the equation of the tangent at Q and P are $x + p^2 y = 2cp \qquad (1)$ $-$ $x + q^2 y = 2cq$ $y(p^2 - q^2) = 2c(p - q)$ $y = \dfrac{2c}{p+q}$ sub into (1)	1 finding y co-ordinate	
$\Rightarrow \quad x = 2cp - \dfrac{2cp^2}{p+q}$ $= -\dfrac{2cpq}{p+q}$	1 finding x coordinate	
$Since \;\; pq = 1 \qquad x = \dfrac{2c}{p+q}$ $\therefore \quad R\left(\dfrac{2c}{p+q}, \dfrac{2c}{p+q}\right)$	1 finding locus of R	
$Hence\ the\ locus\ of\ R\ is\ the\ line\ y = x$ Where $-c < x < 0$ and $0 < x < c$, since pq = 1.	1 Correct restrictions on the values of x.	
(c) (i) Area of the circle = πr^2 Area of the segment = $\dfrac{1}{2}r^2(\pi - 2\theta) - \dfrac{1}{2}r^2\sin(\pi - 2\theta)$	2	

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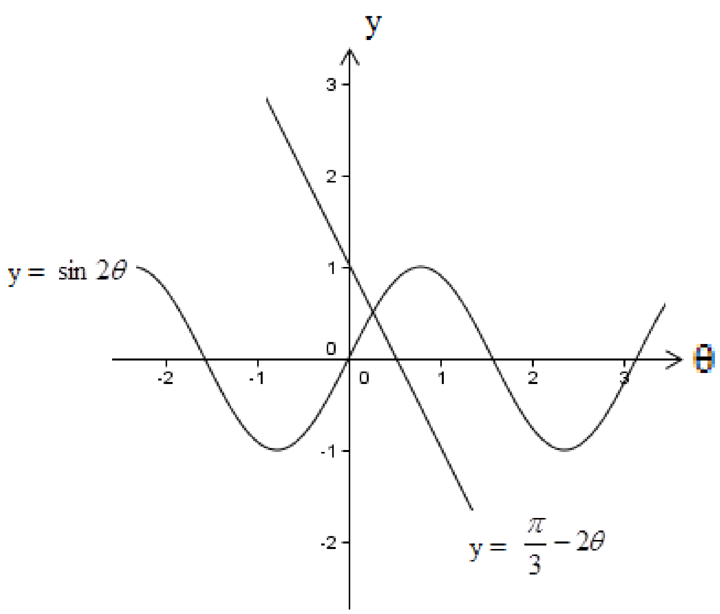
$$\frac{1}{2}r^2(\pi-2\theta)-\frac{1}{2}r^2\sin(\pi-2\theta)=\frac{1}{3}\pi r^2$$

$$\frac{1}{2}r^2\pi-r^2\theta)-\frac{1}{2}r^2\sin 2\theta=\frac{1}{3}\pi r^2$$

$$\frac{1}{6}r^2\pi-r^2\theta=\frac{1}{2}r^2\sin 2\theta$$

$$\therefore\quad \sin 2\theta=\frac{\pi}{3}-2\theta$$

(ii)



Both graphs shown only have one point of intersection.

$\therefore\quad \sin 2\theta=\frac{\pi}{3}-2\theta$ has only one solution.

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correct use of both formulae

1
correct answer

2
Correct graphs and explanation

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<u>Question 16</u> (15 marks)

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(a)
Let A = x + y and
B = x – y

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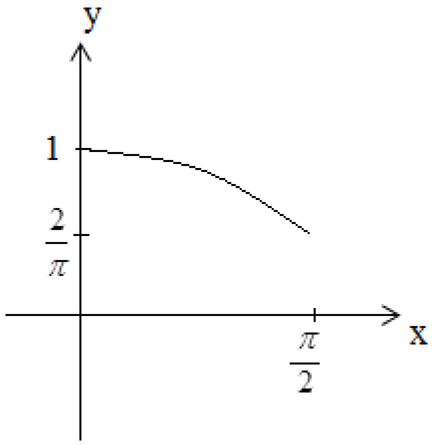
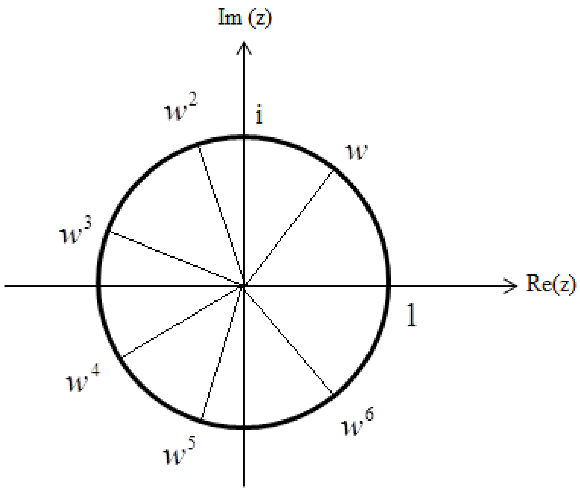
$\therefore\quad A + B = 2x \quad \Rightarrow \quad x = \frac{A+B}{2}$ $A - B = 2y \quad \Rightarrow \quad y = \frac{A-B}{2}$ $\therefore\ \sin A + \sin B = 2\sin\frac{A+B}{2}\ \cos\frac{A-B}{2}$ *and* $\cos A + \cos B = 2\cos\frac{A+B}{2}\ \cos\frac{A-B}{2}$	expressing x and y in term of A & B	
$\frac{SinA + \sin B}{\cos A + \cos B} = \frac{2\sin\frac{A+B}{2}\ \cos\frac{A-B}{2}}{2\cos\frac{A+B}{2}\ \cos\frac{A-B}{2}} \qquad (1)$ $= \tan\frac{A+B}{2}$	1 correct simplifying	
Now $A + B + C = \pi$ $\therefore\quad A + B = \pi - c$ $\therefore\quad \tan\frac{A+B}{2} = \tan\frac{\pi - c}{2}$ $R.H.S = \tan\left(\frac{\pi}{2} - \frac{c}{2}\right)$ $= \cot\frac{c}{2}$ $= L.H.S \qquad$ *from* (1)	1 Correct answer	
(b) (i) $f'(x) = \frac{x\cos x - \sin x}{x^2}$ $= \frac{x\cos x - \cos x\tan x}{x^2}$ $= \frac{\cos x\ (x - \tan x)}{x^2}$	1 Correct differentiation	

$x^2 > 0$, $\cos x > 0$ *and* $\tan x > x$ *for* $0 < x < \frac{\pi}{2}$ $\therefore\ x - \tan x < 0$, *hence* $f'(x) < 0$	1 correct reason	
(ii) f(x) is a decreasing function for $0 < x < \frac{\pi}{2}$ as proven above. $f\left(\frac{\pi}{2}\right) = \frac{2}{\pi}$ *and* $f(0) = 1$		
 	1 correct graph	
From the graph $f(x) \geq \frac{2}{\pi}$ *for* $0 \leq x \leq \frac{\pi}{2}$ *ie* $\frac{\sin x}{x} \geq \frac{2}{\pi}$ $\therefore \sin x \geq \frac{2x}{\pi}$	1 correct reason	
(c) (i) $w = cis\frac{2\pi}{7}$ $\therefore\ w^7 = \left(cis\frac{2\pi}{7}\right)^7$ $= cis 2\pi$ $= 1$	1 expressing w^7 as cis 2π	

$\therefore\ w \text{ is a root of } x^7 - 1 = 0, \text{ that is of}$ $(x-1)(x^6 + x^5 + \ldots\ldots\ldots\ldots + 1) = 0$ $\text{Since } w \neq 1, \text{ then } w \text{ satisfy } x^6 + x^5 + \ldots\ldots + 1 = 0$ $\therefore\ 1 + w + w^2 + \ldots\ldots + w^6 = 0$ (ii) Since the quadratic equation $x^2 + ax + b = 0$ has real coefficients therefore complex roots are conjugate. Hence $\overline{w}$ is also a root. $\therefore\ \beta = \overline{\alpha} = \overline{w} + \overline{w^2} + \overline{w^4}$  $\text{Since } \overline{w} = w^6,\ \overline{w^2} = w^5,\ \overline{w^4} = w^3 \quad \text{from the diagram}$ $\beta = w^3 + w^5 + w^6$ (iii) $\alpha + \beta = -a$ $\Rightarrow a = -(w + w^2 + w^4 + w^3 + w^5 + w^6)$ = - - 1 (using (i)) = 1	1 explanation 1 correct answer with explanation. 1 correct answer for a	

$\alpha\beta = b$ $= (w + w^2 + w^4).(w^3 + w^5 + w^6)$ $= w^4(1 + w + w^3).(1 + w^2 + w^3)$ $= w^4\ (1 + w^2 + w^3 + w + w^3 + w^4 + w^3 + w^5 + w^6$ $= w^4(1 + w + w^2 + ... + w^6 + 2w^3)$ $= w^4(0 + 2w^3)$ $= 2w^7 = 2$ $\therefore \qquad b = 2$ (iv) Hence the quadratic equation is $x^2 + x + 2 = 0$ $x = \dfrac{-1 \pm \sqrt{1 - 4 \times 1 \times 2}}{2}$ $= \dfrac{-1 \pm i\sqrt{7}}{2}$ $\therefore \operatorname{Im}\alpha = \pm\dfrac{\sqrt{7}}{2}$ *Now,* $\operatorname{Im}\alpha = \operatorname{Im}(w + w^2 + w^4)$ $= \sin\dfrac{2\pi}{7} + \sin\dfrac{4\pi}{7} + \sin\dfrac{8\pi}{7}$ $= \sin\dfrac{2\pi}{7} + \sin\left(\pi - \dfrac{3\pi}{7}\right) + \sin\left(\pi + \dfrac{\pi}{7}\right)$ $= -\sin\dfrac{\pi}{7} + \sin\dfrac{2\pi}{7} + \sin\dfrac{3\pi}{7}$ $= \dfrac{\sqrt{7}}{2}$	1 correct answer for b 1 Finding the roots of $x^2 + x + 2 = 0$ 1 expressing im α 1 simplifying to achieve the result	