Killara HIGH SCHOOL

Student Number

Extension 2 Mathematics

HSC Assessment task 1
10th December 2015

General Instructions
- Reading time – 5 minutes
- Working tine 1 hour
- Write using blue or black pen
  Black pen is preferred
- Approved calculators may be used
- In Questions 5-7 show relevant mathematical reasoning and/or calculations
- Start a new booklet for each question

Total Marks – 45

Section A
4 marks
- Attempt Questions 1 – 4
- Allow about 5 minutes for this section

Section B
41 marks
- Attempt Questions 5 – 7
- Allow about 55 minutes for this section

Question	Marks
1 - 4	/4
5	/15
6	/14
7	/12
Total	/45

THIS QUESTION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM

This assessment task constitutes 10% of the Higher School Certificate Course Assessment

Section I

4 marks
Attempt Questions 1 – 4
Allow about 5 minutes for this section
Use the multiple-choice answer sheet for questions 1 – 4 (Detach from paper)

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1 If $x^3 - 11x^2 + 40x - k = (x-4)^2 . P(x)$, what is the value of k ?

(A) 16

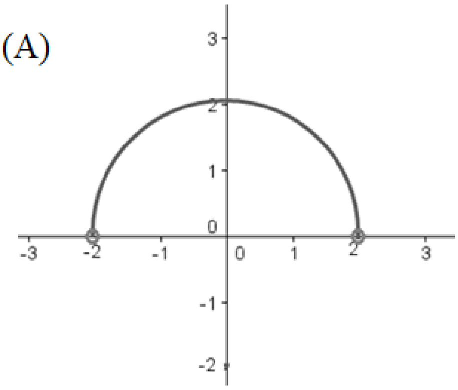
(B) 32

(C) 48

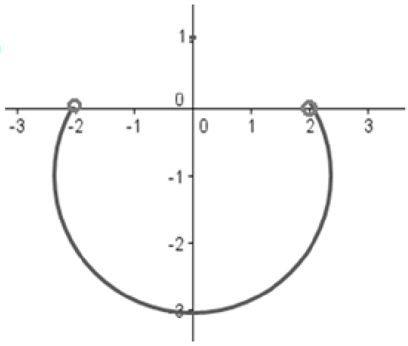
(D) 64

2 $\arg(z+2) - \arg(z-2) = \frac{\pi}{4}$ is best described by:

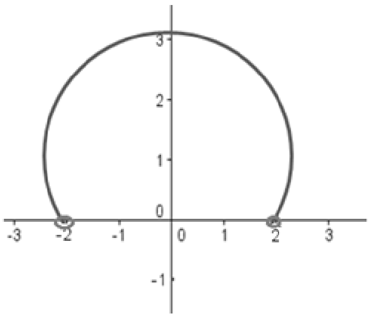
(A)



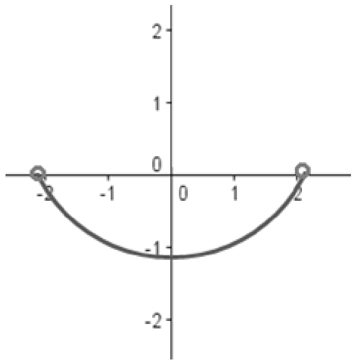
(B)



(C)



(D)



Please Turn Over

3 The locus of the complex number z is the line $4x-3y-12=0$. What is the minimum value of $|z|$?

(A) $\frac{12}{5}$

(B) 3

(C) 4

(D) 5

4 All the solutions of the equation $z^4=(z-1)^4$ are:

(A) $z=\frac{1}{2}, \quad z=\frac{i}{-1+i}, \quad z=\frac{i}{1+i}$

(B) $z=\frac{1}{2}, \quad z=\frac{i}{-1+i}, \quad z=\frac{-i}{1+i}$

(C) $z=\frac{1}{2}, \quad z=\frac{2i}{-1+i}, \quad z=\frac{2i}{1+i}$

(D) $z=\frac{1}{2}, \quad z=\frac{2i}{1+i}, \quad z=\frac{-2i}{1+i}$

End of multiple choice

Section II

41marks
Attempt Questions 5 – 7
Allow about 55 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 5 – 7, your responses should include relevant mathematical reasoning and/or calculations.

Question 5(16 marks) Use a SEPARATE writing booklet

(a) If $z=3-i$ and $\omega=2+4i$.

(i) Express $z\overline{\omega}$ in $x+iy$ form. **2**

(ii) Find $|z|^2$ **1**

(b) (i) Find the square root of $8+6i$ **2**

(ii) Hence or otherwise, find the roots of $4z^2-4iz-9-6i=0$. **2**

(c) $2-i$ is a root of the polynomial $P(x)=x^3+4x^2-27x+40$. **3**
Factorise $P(x)$ over the real field.

(d) If α, β, γ are roots of the equation $x^3-2x+5=0$, find the cubic polynomial that has roots:

(i) $2-\alpha,\ 2-\beta,\ 2-\gamma$. **2**

(ii) $\alpha^2+\beta^2,\ \beta^2+\gamma^2,\ \gamma^2+\alpha^2$. **3**

End of Question 5

Question 6 (14 marks) Use a SEPARATE writing booklet

(a) Express $\dfrac{2\sqrt{3}+i}{\sqrt{3}-i}$ in the form $x+iy$, where x and y are real. **2**

(b) In the Argand Diagram, the point P representing the complex number z moves so that $|z-(1+i)|=1$.

(i) Sketch the locus of P **2**

(ii) Shade the region where $|z-(1+i)|\le 1$ and $0<\arg(z-i)<\dfrac{\pi}{4}$. **3**

(c) If $z_1 = i\sqrt{2}$ and $z_2 = \dfrac{2}{1-i}$,

(i) Express z_1 and z_2 in modulus-argument form. **2**

(ii) If $z_1 = \omega z_2$, find ω in modulus-argument form. **1**

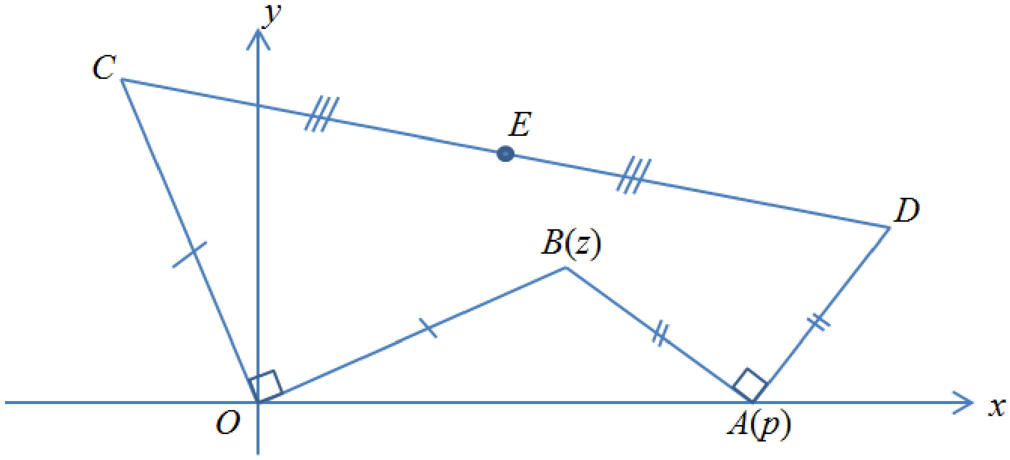
(iii) On the Argand diagram, plot the points P, Q and R, where P represents z_1 and Q represents z_2 and R represents (z_1+z_2). **2**

(iv) Show that $\operatorname{Arg}(z_1+z_2) = \dfrac{3\pi}{8}$ and hence find the exact value of $\tan\dfrac{3\pi}{8}$. **2**

End of Question 6

Question 7 (12 marks) Use a SEPARATE writing booklet

(a) In the Argand Diagram O, A and B represent the origin, the number p and the complex number z respectively.
By rotating B about A at $90°$ in a clockwise direction we get the point D and by rotating B about O at $90°$ in an anti-clockwise direction, we get the point C. Let E be the midpoint of CD.



(i) Find the complex number that represents the point D **2**

(ii) Show that is ΔOEA is right angled isosceles triangle with right angle at E. **3**

(b) (i) Find the three cube roots of unity **2**

(ii) If ω is one of the complex cube root of unity, prove that ω^2 is the other and hence show that $1+\omega+\omega^2 = 0$. **2**

(iii) Prove that if n is a positive integer, then $1+\omega^n+\omega^{2n} = 3$ or 0 depending on whether n is or is not a multiple of 3. **3**

END OF EXAMINATION

Student number : ______________________________

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SECTION A - MULTIPLE CHOICE ANSWER SHEET

If you think you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ○	B ~~●~~	C ●	D ○

1	A ○	B ○	C ○	D ○
2	A ○	B ○	C ○	D ○
3	A ○	B ○	C ○	D ○
4	A ○	B ○	C ○	D ○

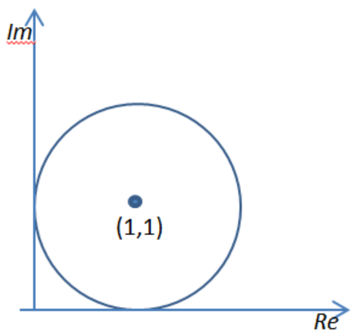
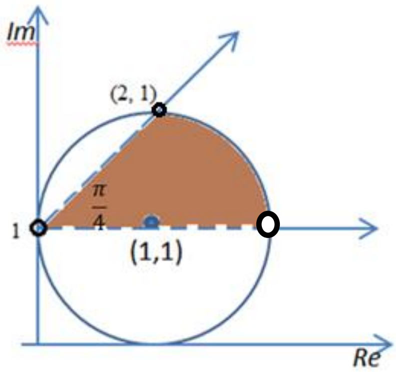
2014 EXT-Assessment 1 Solution

	1. C	2. B	3. A	4. A

		Solution	Marking
		4. $z^4=(z-1)^4$ $\dfrac{(z-1)^4}{z^4}=1 \qquad \therefore \left(\dfrac{z-1}{z}\right)^4=1$ $\therefore \dfrac{z-1}{z}=\pm 1 \qquad \dfrac{z-1}{z}=\pm i$ $z-1=\pm z \qquad z-1=\pm zi$ $2z=1 \qquad (1\pm i)z-1$ $\therefore z=\dfrac{1}{2} \qquad \therefore z=\dfrac{1}{1\pm i}$	Generally well done
5. a)	(i)	$z=3-i \quad \omega=2+4i$. $z\overline{\omega}=(3-i)(2-4i)$ $=6-4+i(-2-12)$ $=2-14i$	**1 mark:** for correct $\overline{\omega}$ 1 Mark: correctly multiplies $z\overline{\omega}$
	(ii)	$\lvert z\rvert^2 = z\bar{z} = 9+1=10$	**1 mark:** correct answer
b) (i)		*Let* $z=a+bi$ $=\sqrt{8+6i}$ $(a+bi)^2=8+6i$ *L.H.S.* $=a^2-b^2+2abi$ *Equating real and imaginary part with R.H.S* $a^2-b^2=8$ $ab=3$ $\therefore$ a = 3and b = 1 $\sqrt{8+6i}=\sqrt{(3+i)^2}$ Roots are $3+i$ *and* $-3-i$	Generally well done 1 mark: gives the two simultaneous equations 1 mark: solves for *a* and *b* and gives the two roots
b(ii)		$4z^2-4iz-9-6i=0$. $z=\dfrac{4i\pm\sqrt{-16+4\times 4(9+6i)}}{8}$ $=\dfrac{4i\pm 4\sqrt{-1+(9+6i)}}{8}$ $=\dfrac{i\pm\sqrt{8+6i}}{2}$	**1 mark:** applies the quadratic formula **1 mark:** uses the result from (i) to simplify the roots

	$= \dfrac{i \pm (3+i)}{2}$ $= \dfrac{3+2i}{2}, \quad \dfrac{-3}{2}$	
c)	$P(x) = x^3 + 4x^2 - 27x + 40$. As the coefficients of the polynomial equation are real, if $2-i$ is a root, then $2+i$ is also a root. Let the roots be $2-i$, $2+i$ and α. Sum of roots = -4 $2-i+2+i+\alpha = -4$ Hence, $\alpha = -8$ The quadratic with $2-i, \quad 2+i$ as roots is $x^2 - 4x + 5$ $P(x) = (x^2 - 4x + 5)(x+8)$	**1 mark:** explains the reason for conjugate roots **1 mark:** finds the third root **1 mark:** expresses the polynomial a product of real factors A few students used division method to obtain the other root, which is a less efficient method)
(d) (i)	Let $y = 2-x$ $x = 2-y$ $x^3 - 2x + 5 = 0$ $(2-y)^3 - 2(2-y) + 5 = 0$ $8 - 12x + 6x^2 - x^3 - 4 + 2x + 5 = 0$ $x^3 - 6x^2 + 10x - 9 = 0$	**1 mark:** substitutes $x = 2-y$ into the equation **1 mark: simplifies to** give the correct equation
(ii)	$\alpha^2 + \beta^2 + \gamma^2$ $= (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$ $= 0 - 2(-2) = 4$ **1 mark** $\alpha^2 + \beta^2, \quad \beta^2 + \gamma^2, \quad \alpha^2 + \gamma^2 =$ $4-\gamma^2, \quad 4-\alpha^2, \quad 4-\beta^2$ Let $y = 4 - x^2$ Hence, $x^2 = 4 - y \quad \Rightarrow x = \sqrt{4-y}$ **1 mark** $x^3 - 2x + 5 = 0$ $\left(\sqrt{4-y}\right)^3 - 2\sqrt{4-y} + 5 = 0$ $\sqrt{4-y}(4-y-2) = -5$ $\sqrt{4-y}(2-y) = -5$ $(4-y)(4-4y+y^2) = 25$ $16 - 16y + 4y^2 - 4y + 4y^2 - y^3 = 25$ $x^3 - 8x^2 + 20x + 9 = 0$	**3 mark:** gives the correct equation from correct working **2 marks:** correctly expresses the roots as $4-\gamma^2, \quad 4-\alpha^2, \quad 4-\beta^2$, substitutes $x = \sqrt{4-y}$ and attempts to solve the equation after substitution **1 mark:** any one of the steps correct. (reasonable attempt) Some students used the sum and product of roots to work out the coefficient of the required polynomial. This is less efficient and time consuming, and numerical errors were made as a result.

Question 6

a)	$\frac{2\sqrt{3}+i}{\sqrt{3}-i}$ $\frac{(2\sqrt{3}+i)(\sqrt{3}+i)}{(\sqrt{3}-i)(\sqrt{3}+i)}$ $\frac{6+2\sqrt{3}i+\sqrt{3}i-1}{3+1}$ $\frac{5+3\sqrt{3}i}{4}$ $\frac{5}{4}+\frac{3\sqrt{3}i}{4}$	**1 mark:** multiplies by the conjugate of the denominator. **1 mark:** correctly expresses the answer in the $x+iy$ form
b)(i)		2 mark: Circle with centre (1, 1) and radius 1 1 mark: shows the understanding that the locus is a circle and draws a circle.
(ii)		**1 mark:** shaded the inside of the circle. **1 mark:** Marks the lines that correspond to the arguments 0 to $\frac{\pi}{4}$ and shades the region in between. 1 mark: Labels and excludes the point (0,1) and l (2,1)
c)(i)	$z_1 = i\sqrt{2}$, $z_2 = \frac{2}{1-i} = \frac{2(1+i)}{(1-i)(1+i)}$ $= \frac{2+2i}{2} = 1+i$	**1 mark each:** correctly represents z_1 and z_2 in mod-arg form

Q6 both (a) & (b) were generally well done. Except in (ii) students forgot to exclude the points shown in the diagram.

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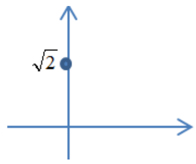

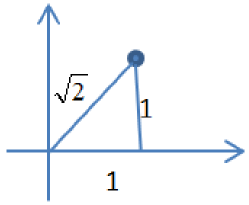

$z_1 = \sqrt{2}cis\,\frac{\pi}{2}$ $\quad$ $z_2 = \sqrt{2}cis\,\frac{\pi}{4}$
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<tr>
<td>(ii)</td>
<td>
$z_1 = \omega z_2$
$\omega = \frac{z_1}{z_2} = \frac{\sqrt{2}cis\,\frac{\pi}{2}}{\sqrt{2}cis\,\frac{\pi}{4}}$
$= \frac{\sqrt{2}}{\sqrt{2}}cis\left(\frac{\pi}{2} - \frac{\pi}{4}\right)$
$= cis\,\frac{\pi}{4}$
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<td>1 mark: correctly applies De-Moivre theorem</td>
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<td>(iii)</td>
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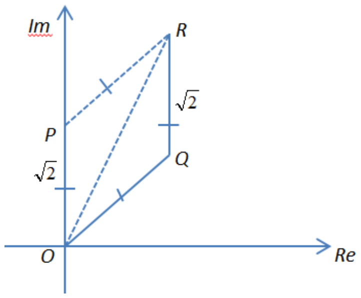

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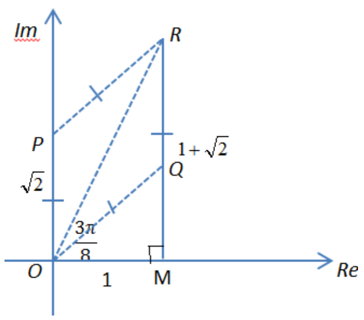
$\angle POQ = \angle POx - \angle QOx$
$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$
$\arg(z_1 + z_2) = \angle ROx$
OR bisects $\angle POQ$.
Hence, $\angle ROQ = \frac{\pi}{4} \div 2 = \frac{\pi}{8}$
$\arg(z_1 + z_2) = \angle ROx = \frac{\pi}{4} + \frac{\pi}{8} = \frac{3\pi}{8}$


In ΔROQ, $\tan\frac{3\pi}{8} = \frac{\sqrt{2}+1}{1} = \sqrt{2}+1$
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1 mark: Calculates $\arg(z_1 + z_2)$ correctly

1 mark: Evaluates $\tan\frac{3\pi}{8}$ correctly.
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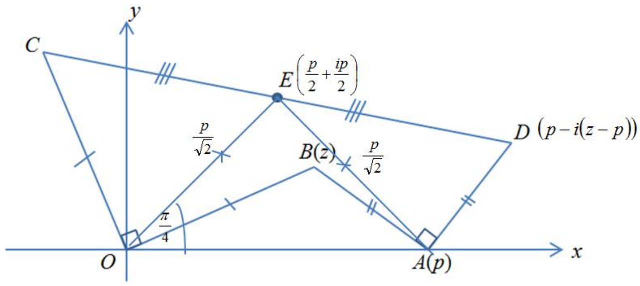
<u>**Question 7**</u>

a)	.  $\overrightarrow{AB} \ = \ z-p$ $\overrightarrow{AD} \ = \ -i(z-p)$ **1 mark** So, $\overrightarrow{OD} \ = \ \overrightarrow{OA} + \overrightarrow{AD}$ $= \ p+-i(z-p)$ $= \ p-iz+ip$ $= \ p-i(z-p)$ **1 mark**	. Majority of students had difficulty in vector operations (Q7 (a)). They also used incorrect notation for complex **1 mark:** expresses $\overrightarrow{AD}$ as $-i.\overrightarrow{AB}$ **1 mark:** expresses $\overrightarrow{OD} \ = \ \overrightarrow{OA} + \overrightarrow{AD}$
(ii)	Since, E is the midpoint of *CD*, it has coordinates $\dfrac{iz-i(z-p)+p}{2}$ $= \dfrac{p+ip}{2}$ $= \dfrac{p}{2}+\dfrac{pi}{2}$ **1 mark** Distance OE = $\sqrt{\left(\frac{p}{2}\right)^2+\left(\frac{p}{2}\right)^2} \ = \ \frac{p}{\sqrt{2}}$ Distance EA = $\sqrt{\left(p-\frac{p}{2}\right)^2+\left(\frac{p}{2}\right)^2} \ = \ \frac{p}{\sqrt{2}}$ OE = EA ΔOEA is isosceles **1 mark** $\angle EOA = \arg(OE) \ = \ \tan^{-1}\left(\dfrac{\frac{p}{2}}{\frac{p}{2}}\right)$ $= \ \tan^{-1}1 \ = \ \dfrac{\pi}{4}$ $\angle EOA = \angle EAO \ = \ \dfrac{\pi}{4}$ (being isosceles) $\angle OEA = \pi-\left(\dfrac{\pi}{4}+\dfrac{\pi}{4}\right) \ = \ \dfrac{\pi}{2}$ **1 mark**	**1 mark:** finds the coordinates of E. **1 mark:** proves that ΔOEA is isosceles **1 mark:** proves that ΔOEA is right-angled.

b) **(i)**	$z^3 = 1$ $z^3 - 1 = 0$ $(z-1)(z^2+z+1) = 0$ $z = 1, \quad z = \dfrac{-1 \pm \sqrt{1-4}}{2}$ $= \dfrac{-1 \pm \sqrt{3}i}{2}$	**2marks:** gives all the three roots with correct working **1 mark:** progresses towards the correct answer with minor error
(ii)	Let $\omega = \dfrac{-1+\sqrt{3}i}{2}$ $\omega^2 = \left(\dfrac{-1+\sqrt{3}i}{2}\right)^2$ $= \dfrac{1-2\sqrt{3}i-3}{4}$ $= \dfrac{-1-\sqrt{3}i}{2}$ which is the other root. Hence the roots are 1, ω and ω^2. $1+\omega+\omega^2 = 1+ \dfrac{-1+\sqrt{3}i}{2} + \dfrac{-1-\sqrt{3}i}{2} = 0$ **OR** $z^3 - 1 = 0$ If ω is a root, $\omega^3 = 1$ Consider $(\omega^2)^3 - 1$ $= (\omega^3)^2 - 1 = 1 - 1 = 0$ Ie. ω^2 satisfies the equation $z^3 - 1 = 0$ and hence is a root. **1 mark** Sum of roots $1+\omega+\omega^2 = 0$ as the coefficient of z^2 in the equation $z^3 - 1 = 0$ is zero. **1 mark**	**1 mark:** correctly proves that ω^2 is the other root. **1 mark:** proves that $1+\omega+\omega^2 = 0$
(iii)	If n is a multiple of 3, then $n = 3k$. $1+\omega^{3k}+\omega^{6k} = 1+(\omega^3)^k+(\omega^3)^{2k}$ $= 1+1^k+1^{2k}$ $= 3$ **1 mark** If n is not a multiple of 3, then $n = 3k+1$ or $n = 3k+2$ If $n = 3k+1$, $1+\omega^{3k+1}+\omega^{6k+2} = 1+\omega(\omega^3)^k+\omega^2(\omega^3)^{2k}$ $= 1+\omega+\omega^2$ $= 0$ If $n = 3k+2$, $1+\omega^{3k+2}+\omega^{6k+4} = 1+\omega^2(\omega^3)^k+\omega^4(\omega^3)^{2k}$	**1 mark:** proves that when $n = 3k$, $1+\omega^{3k}+\omega^{6k} = 3$ **1 mark:** proves that when $n = 3k+1$, $1+\omega^{3k}+\omega^{6k} = 0$ **1 mark:** proves that when $n = 3k+2$, $1+\omega^{3k}+\omega^{6k} = 0$

Majority of students scored only 1/3 marks for this section. They only showed correctly the case when n = 3k.

	$= 1+\omega^2+\omega\omega^3$ $= 1+\omega^2+\omega$ $= 0$	