Killara HIGH SCHOOL

Student Number

Mathematics Extension 2

HSC Assessment task 3
28th May 2015

General Instructions
- Reading time – 5 minutes
- Working tine 1 hours
- Write using blue or black pen
  Black pen is preferred
- Approved calculators may be used
- In Questions 6 to 8 show relevant mathematical reasoning and/or calculations
- Start a new booklet for each question

Total Marks –

Section A
5 marks
- Attempt Questions 1 – 5
- Allow about 5 minutes for this section

Section B
45 marks
- Attempt Questions 6 to 8
- Allow about 55 minutes for this section

Question	Marks
1-5	/5
6	/15
7	/14
8	/16
Total	/50

THIS QUESTION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM

This assessment task constitutes 20% of the Higher School Certificate Course Assessment

Section A: (5 marks)
Attempt Questions 1 – 5
Allow about 5 minutes for this section
Use the multiple-choice answer sheet for questions 1 – 5 (detach from paper)

1. The polynomial $z^4 + z^2 - 12 = 0$ will have:

   A. Two real solutions only

   B. Two real and two complex solutions

   C. Four complex solutions

   D. No solutions

2. A suitable substitution for the integral given by $\int \frac{x^2}{\sqrt{4-x^2}}\,dx$ is

   A. $x = \cos\theta$

   B. $u = x^2$

   C. $u = \sin^2 x$

   D. $x = 2\sin\theta$

3. The value of $\int_0^1 \frac{2x}{1+2x}\,dx$ is

   A. $1 - \frac{1}{2}\ln 3$

   B. $\frac{1}{2}\ln 3 - 1$

   C. $1 + \frac{1}{2}\ln 3$

   D. $1 - \frac{1}{2}\ln 2$

4. In how many ways can n students be placed in 2 distinct rooms so that neither room will be empty?

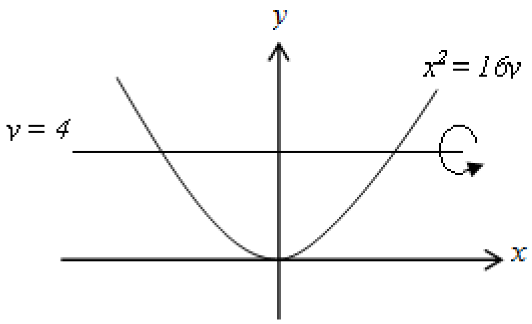
A. $2^n - 1$

B. $n^2 - 1$

C. $2^n - 2$

D. $n^2 - 2$

5. The volume V of the solid generated when the region bounded by the line $y = 4$ and the curve $x^2 = 16y$ is rotated about the line $y = 4$, using the method of slicing is given by:



A. $V = \pi \int_{-8}^{8} \left(16 - \frac{x^4}{256}\right) dx$

B. $V = \pi \int_{-8}^{8} \left(4 - \left(4 - \frac{x^4}{256}\right)\right)^2 dx$

C. $V = \pi \int_{-8}^{8} \left(16 - \left(4 - \frac{x^4}{256}\right)^2\right) dx$

D. $V = \pi \int_{-8}^{8} \left(4 - \frac{x^2}{16}\right)^2 dx$

Section B (45 marks)
Attempt Questions 6 - 8
Allow about 55 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 6 to 8, your responses should include relevant mathematical reasoning and/or calculations.

Question 6 (15 marks) Use a SEPARATE writing booklet

a) Find $\int \frac{2}{x^2+9}\,dx$. **1**

b) Evaluate $\int_e^4 \frac{\ln x}{x^2}\,dx$. **3**

c) Find $\int \frac{\sin^3 x}{\cos^2 x}\,dx$. **3**

d) i) Write $\frac{4x+3}{(x^2+1)(x+2)}$ in the form $\frac{ax+b}{x^2+1} - \frac{1}{x+2}$. **2**

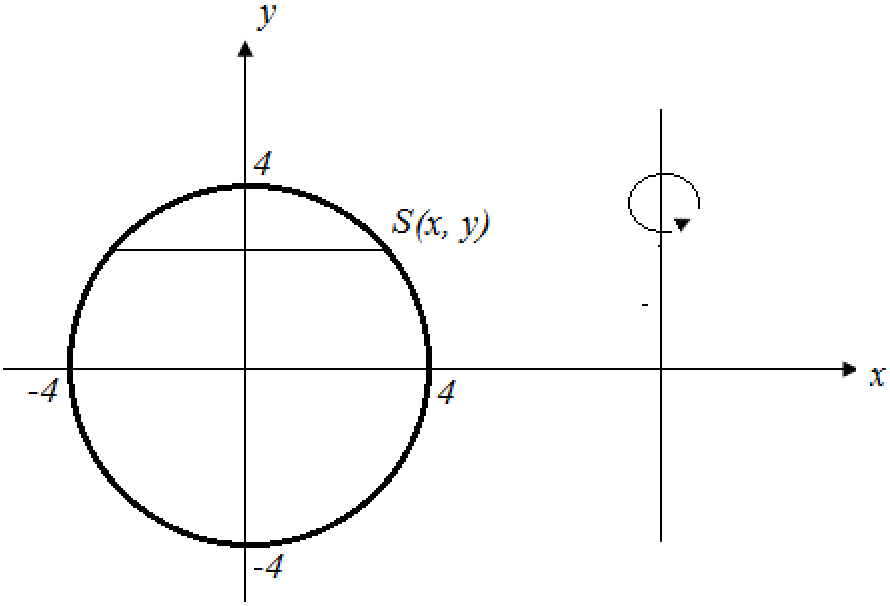
ii) Hence find $\int \frac{4x+3}{(x^2+1)(x+2)}\,dx$ **2**

e) Using the substitution $t = \tan\frac{\theta}{2}$ to find **4**

$$\int \frac{d\theta}{1+\sin\theta+\cos\theta}$$

Question 7 (14 marks) Use a SEPARATE writing booklet

a)

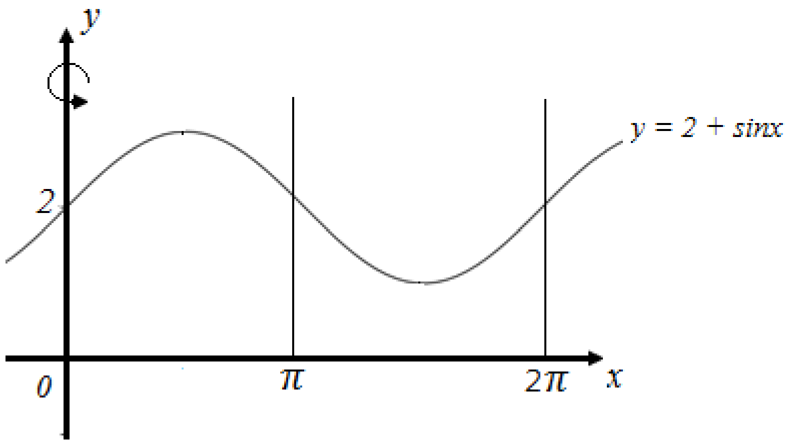


The circle $x^2 + y^2 = 16$ is rotated about the line x = 9 to form a ring. When the circle is rotated, the line segment *S* at a height y sweeps out an annulus.

i) Show that the area of the annulus is equal to $36\pi\sqrt{16-y^2}$. **2**

ii) Hence find the volume of the ring. **2**

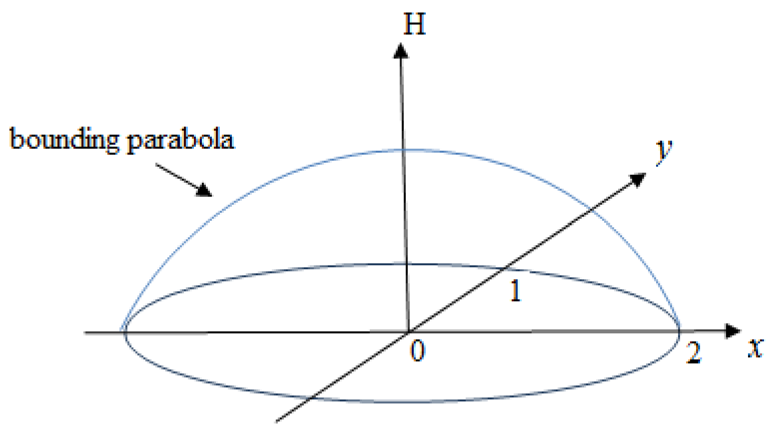
b) A mould for a circular fish pond is made by rotating the region bounded by the curve $y = 2 + \sin x$ and the x axis between $x = \pi$ and $x = 2\pi$ through one complete revolution about the y-axis. All measurements are in metres. Use the method of cylindrical shells to find the capacity of the fish pond to the nearest litres **5**



c. The base of a solid is an ellipse with semi-axes of lengths 2 metres and 1 metre.

Vertical cross-sections taken at right angles to the major axis of the elliptical base is bounded by parabolic arcs, the axis of the parabola being vertical and passing through the major axis of the ellipse.

The cross-section taken vertically through the centre of the ellipse along the major axis is also bounded by a parabolic arc with axis through the centre of the ellipse. The maximum height of the solid is 3 metres.
The shape of the solid is shown in the diagram below.



i) Show that the equation of the bounding parabola is $H = \frac{12-3x^2}{4}$. **2**

ii) Find the volume of the solid. **3**

Question 8 (16 marks) Use a SEPARATE writing booklet

a) i) Using a suitable substitution, show that

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$ **1**

ii) A function *f(x)* hs the property that *f(x) +f(a – x) = f(a).*

Using part (i), or otherwise, show that $\int_0^a f(x)\,dx = \frac{a}{2} f(a)$ **2**

b) The curve $y = \cos x$ and $y = \tan x$ intersect at a point P whose x-coordinate is α.

i) Show that the curve intersect at right angle at P. **2**

ii) Show that $\sec^2 \alpha = \frac{1+\sqrt{5}}{2}$ **3**

c) Let $I_n = \int_0^{\frac{\pi}{4}} \tan^n x\,dx$ and let $J_n = (-1)^n I_{2n}$ for $n = 0, 1, 2, 3\ldots\ldots$

i) Show that $I_n + I_{n+2} = \frac{1}{n+1}$. **2**

ii) Deduce that $J_n - J_{n-1} = \frac{(-1)^n}{2n-1}$ **1**

iii) Show that $J_m = \frac{\pi}{4} + \sum_{n=1}^{m} \frac{(-1)^n}{2n-1}$. **2**

iv) Use the substitution u = tan x to show that

$$I_n = \int_0^1 \frac{u^n}{1+u^2}\,du$$ **1**

v) Deduce that $0 \le I_n \le \frac{1}{n+1}$ and conclude that $J_n \to 0$ as $n \to \infty$. **2**

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\quad n\neq -1;\quad x\neq 0,\text{ if } n<0$$

$$\int \frac{1}{x}dx = \ln x,\quad \boldsymbol{x}>0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\quad a\neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\quad \boldsymbol{a}\neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\quad a\neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\quad a\neq 0$$

$$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax,\quad a\neq 0$$

$$\int \frac{1}{\boldsymbol{a}^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\quad a\neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a},\quad \boldsymbol{a}>0,\quad -a<x<a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}dx = \ln\left(x+\sqrt{x^2-a^2}\right),\quad x>a>0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

Student number : ______________________________

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SECTION A - MULTIPLE CHOICE ANSWER SHEET

If you think you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ⬭ B ⬬ C ⬬ D ⬭

1 A ⬭ B ⬭ C ⬭ D ⬭

2 A ⬭ B ⬭ C ⬭ D ⬭

3 A ⬭ B ⬭ C ⬭ D ⬭

4 A ⬭ B ⬭ C ⬭ D ⬭

5 A ⬭ B ⬭ C ⬭ D ⬭

Ex2 Task 3-2015 solution

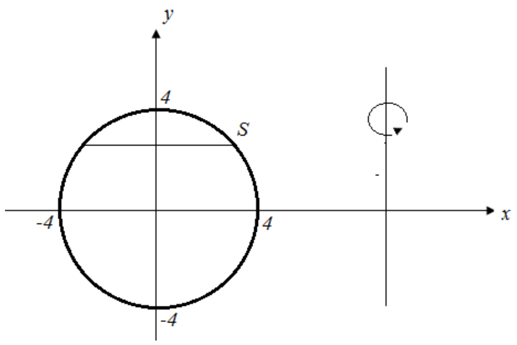
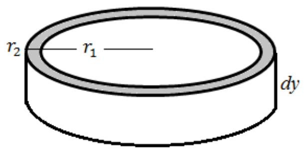
Q1 B **Q2 D** **Q3 A** **Q4 C** **Q5 D**

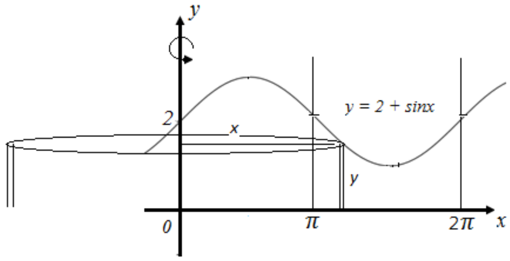
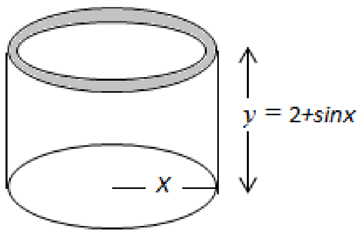
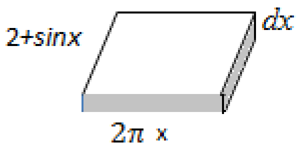
Q6 (15 marks)

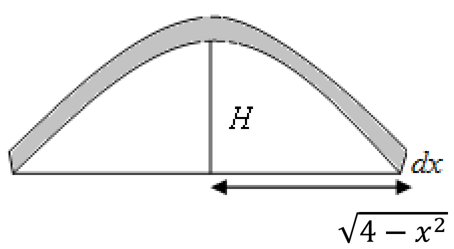
a)	$\frac{2}{3}\tan^{-1}\frac{x}{3}+C$	1 mark: correct answer	
b)	$\int_0^4 \frac{\ln x}{x^2}\,dx = \left[-\frac{1}{x}\ln x\right]_0^4 + \int_0^4 \frac{1}{x^2}\,dx$ $= -\frac{1}{e}+\frac{\ln 4}{4}-\frac{1}{4}+\frac{1}{e}$ $= \frac{1}{4}(\ln 4 - 1)$	1 mark: correctly applies integration by substitution 1 mark: correct integration 1 mark: correct evaluation	$\frac{\ln x}{x}d(\ln x)$ method is creating so much problem with students not scoring full marks General problems with managing negative sign
c)	$\int \frac{\sin^3 x}{\cos^2 x}\,dx = \int \frac{\sin x\,(1-\cos^2 x)}{\cos^2 x}\,dx$ $= \int \frac{\sin x}{\cos^2 x} - \sin x\,dx$ $= \frac{1}{\cos x}+\cos x$ $= \sec x + \cos x + C$	1 mark: breaks $\sin^3 x$ into $\sin x\,(1-\cos^2 x)$ 1 mark: correctly applies integration by substitution 1 mark: correct integration	Again, General problems with managing negative sign Many students had $\int \frac{\sin x}{\cos^2 x} - 1\,dx$ in step 2
d) (i)	(x+2)(ax+b) - (x^2+1) = 4x+3 *Let x = 0* 2b - 1 = 3, b = 2 Let x = 1 3(a + b) - 2 = 7 a + b = 3 $\Rightarrow$ a = 1	1 mark: correctly calculates the values of *a* and *b*.	Well done
(ii)	$\therefore \int \frac{4x+3}{(x^2+1)(x+2)}\,dx = \int \frac{x+2}{x^2+1} - \frac{1}{x+2}\,dx$ $= \int \frac{x}{x^2+1}\,dx + \int \frac{2}{x^2+1}\,dx - \int \frac{1}{x+2}\,dx$ $= \frac{1}{2}\ln(x^2+1) + 2\tan^{-1}x - \ln(x+2) + C$	1 mark: expresses the integrand into three integrable separate terms 1 mark: correct integration	Well done

e)	Let $t = \tan\frac{\theta}{2}$ $dt = \frac{1}{2}\sec^2 d\theta$ $\frac{2dt}{1+t^2} = d\theta$ $I = \int \frac{\frac{2}{1+t^2}}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}\, dt \quad 1\,mark$ $= \int \frac{2}{1+t^2+2t+1-t^2}\, dt$ $= \int \frac{2}{2(1+t)}\, dt \quad 1mark$ $= \ln(1+t)+C \quad 1\,mark$ $\therefore I = \ln\left(1+\tan\frac{\theta}{2}\right)+C \quad 1mark$	1 mark: correctly converts the expression in terms of t 1 mark: correctly simplifies 1 mark: correct integration 1 mark: correctly transforms back to θ.	Well done

Q7 (14 marks)

a) **(i)**	  $r_1 = 9 - x \quad and \quad r_2 = 9 + x$ $x = \sqrt{16-y^2}$ $\Rightarrow \quad r_1 = 9 - \sqrt{16-y^2} \quad and$ $r_2 = 9 + \sqrt{16-y^2}$ $A = \pi\left[\left(9+\sqrt{16-y^2}\right)^2 - \left(9-\sqrt{16-y^2}\right)^2\right]$ $= \pi\left(9+\sqrt{16-y^2}+9-\sqrt{16-y^2}\right)$ $\times\left(9+\sqrt{16-y^2}-9+\sqrt{16-y^2}\right)$ $= \pi\left(18\times 2\sqrt{16-y^2}\right)$ $= 36\pi\sqrt{16-y^2}$	1 mark: correct r_1, r_2 and explains area = $\pi\left[(9+r_1)^2 - (9+r_2)^2\right]$ 1 mark: changes the area expression in terms of y to prove the result	Not drawing a diagram and labelling has left many students confused, wasting time on simplifying massive expressions, consequently losing marks. *Always draw the typical slice, you are not saving time by not doing so.*

(ii)	$\partial V = A(y)\,\partial y$ $= 36\pi\sqrt{16-y^2}\,\partial y$ $V = \lim\limits_{\partial y \to 0} \sum\limits_{-4}^{4} 36\pi\sqrt{16-y^2}\,\partial y$ $= 36\pi \int\limits_{-4}^{4} \sqrt{16-y^2}\,dy$ $= 36\pi \times \frac{1}{2}\pi \times 4^2$ $= 288\pi^2$ *unit s cube*	1 mark: develops the correct volume expression with working 1 mark: correctly integrates and evaluates	Many students could not see that $\int\limits_{-4}^{4} \sqrt{16-y^2}\,dy$ was the area of semi-circle and resorted to integration by substitution.
b)	   $dV = 2\pi \times x \times (2+\sin x)dx$ $V = \lim\limits_{dx \to 0} \sum\limits_{\pi}^{2\pi} 2\pi\, x \times (2+\sin x)\ dx$ $= 2\pi \int\limits_{\pi}^{2\pi} 2x\,dx + 2\pi \int\limits_{\pi}^{2\pi} (x\sin x)\,dx$ $= 4\pi\left(\frac{x^2}{2}\right)_{\pi}^{2\pi} + 2\pi[-x\cos x]_{\pi}^{2\pi} +$ $2\pi \int\limits_{\pi}^{2\pi} (\cos x)\,dx$ $= 2\pi\left(4\pi^2 - \pi^2\right) - 2\pi(3\pi) + 2\pi[\sin x]_{\pi}^{2\pi}$ $= 6\pi^3 - 6\pi + 0\,m^3$ $= 6000\left(\pi^3 - 6\pi\right)$ *litres*	1 mark: develops the expression for volume with working and diagrams 1 mark: correct expression for volume 2 mark: correct integration Award 1 mark for minor errors in integration 1 mark: gives the volume in litres	Again, lack of diagram made some students to think that the radius was $x-\pi$. Comprehension of the question is important here: Many used limits 0 to π Did not convert to capacity units.

(i)	The bounding parabola is of the form $H = k(x-2)(x+2)$ When $H = 3$, $x = 0$ $\Rightarrow\ 3 = -4k$ $k = -\frac{3}{4}$ $\therefore\ H = -\frac{3}{4}(x-2)(x+2)$ $= \frac{12-3x^2}{4}$ (0, 3) 	(i) 2 marks: correct proof 1 mark: uses a general quadratic form and attempts to find k.	It is astonishing to see how many students cannot find the equation of a parabola, given the x- and y-intercepts.
(ii)	Using Simpson's rule, area of the cross section is: $A = \frac{\sqrt{4-x^2}}{6}\left[0 + 4\left(\frac{12-3x^2}{4}\right) + 0\right]$ $= \frac{(4-x^2)^{\frac{3}{2}}}{2}$ $dV = \frac{(4-x^2)^{\frac{3}{2}}}{2}.dx$ $V = \lim\limits_{x \to 0} \sum_{-2}^{2} \frac{(4-x^2)^{\frac{3}{2}}}{2}\ dx$ $\therefore\ I = \int_{-2}^{2} \frac{(4-x^2)^{\frac{3}{2}}}{2}\ dx$ $= \int_{0}^{2} (4-x^2)^{\frac{3}{2}}\ dx$ since $(4-x^2)^{\frac{3}{2}}$ is an even function	1 mark: $\delta V = \frac{4y}{3} H\delta x$ 1 mark: obtains the expression for volume $V = \int_{0}^{2} (4-x^2)^{\frac{3}{2}}\ dx$ 1 mark: correctly integrates and gives the volume.	Those students who applied the Simpson's Rule were successful in getting 2 mark. In this question many showed lack of understanding that $\delta V = \delta A \times \delta x$ $\int_{0}^{2} H\ dx$ does not give you volume

	$= \left[x(4-x^2)^{\frac{3}{2}} \right]_0^2 - 3\int_0^2 x^2 \ (4-x^2)^{\frac{1}{2}} \ dx$ $= -3\int_0^2 (4-x^2-4)(4-x^2)^{\frac{1}{2}} \ dx$ $= -3\int_0^2 (4-x^2)^{\frac{3}{2}} + 12\int_0^2 \sqrt{4-x^2} \ dx$ Since $\sqrt{4-x^2} = \pi$ square units (is the area of $\frac{1}{4}$ circle with r = 2) $\therefore \quad I = (-3I + 12 \times \pi)$ $4I = 12\pi$ $I = 3\pi$ $\therefore \quad V = 3\pi$ cubic units.		

Q8 (16 marks)

a)	**i)** Let u = a – x x = a – u When x = 0, u = a x = a, u = 0 du = - dx $\int_0^a f(x) \ dx = -\int_a^0 f(u) \ du$ $= \int_0^a f(a-x) \ du$ **(ii)** Given that *f(x) + f(a – x) = f(a)* *Using (i) b* $\int_0^a f(x) \ dx + \int_0^a f(a-x) \ dx = 2\int_0^a f(x) \ dx$ **(1)**	(i) 1 mark: for the correct proof (ii) 1 mark: uses result (i) to explain result (1) 1 mark: proves $\int_0^a f(a) \ dx =$ $a \ f(a)$ and hence the result	Well done (ii) most students did this question well.

	$2f(x)=\int_0^a f(a)\,dx$ $=\int_0^a f(a)\,dx$ $=\left[x\,f(a)\right]_0^a$ $=a\;f(a)-0f(a)$ $=a\,f(a)$ $\therefore\; f(x)=\frac{a}{2}f(a)$		
b)	**(i)** $y_1'=-\sin x \qquad y_2'=\sec^2 x$ At $x=\alpha,\quad m_1=-\sin\alpha,\qquad m_2=\frac{1}{\cos^2\alpha}$ $m_1\times m_2=-\sin\alpha\times\sec^2\alpha$ $=-\frac{\tan\alpha}{\cos\alpha}$ $=-1$ Since $\tan\alpha=\cos\alpha$ at point of intersection ($x=\alpha$) **(ii)** Solving for point of intersection $\cos x=\tan x$ $\cos^2 x=\sin x$ $1-\sin^2 x=\sin x$ $\sin^2 x+\sin x-1=0$ $\therefore\;\sin x=\frac{-1\pm\sqrt{1+4}}{2}$ $=\frac{-1\pm\sqrt{5}}{2}$ Since $\cos^2 x=\sin x,\quad \sin x=\frac{\sqrt{5}-1}{2}$ $\sec^2 x=\frac{4}{2(\sqrt{5}-1)}$ $=\frac{2}{\sqrt{5}-1}=\frac{2(\sqrt{5}+1)}{(\sqrt{5}-1)(\sqrt{5}-1)}$ $=\frac{\sqrt{5}+1}{2}$ **(i)**	b) 1 mark: finds gradients and attempts to multiply 1 mark: proves $m_1\times m_2=-1$ and hence the result (ii) 1 mark: finds the quadratic simultaneous equation and solves for sin*x*. 1 mark: explains why only the positive result is possible. 1 mark: uses the result in (i) $-\sin\alpha\times\sec^2\alpha=$ -1 to find $\sec^2 x$.	b) many students got 1 mark for this question (ii) many students did not attempt this question

c)	$I_n + I_{n+2} = \int_0^{\frac{\pi}{4}} \tan^n x + \tan^{n+2} x \, dx$ $= \int_0^{\frac{\pi}{4}} \tan^n x \, (1 + \tan^2 x) \, dx$ $= \int_0^{\frac{\pi}{4}} \tan^n x . \sec^2 x \, dx \quad 1\, mark$ $= \left[\frac{\tan^{n+1} x}{n+1} \right]_0^{\frac{\pi}{4}} = \frac{1}{n+1}$	1 mark: simplifies $I_n + I_{n+2}$. 1 mark: correct evaluation	Well done
	(ii) $J_n - J_{n-1} = (-1)^n I_{2n} - (-1)^{n-1} I_{2n-2}$ $= (-1)^n \left(I_{2n} + I_{2n-2} \right)$ $= \frac{(-1)^n}{2n-1}$	1 mark: proves the result	Some students did not realise the usefulness of (i) for this question and proved it using Q(i) method.
	(iii) $\sum_{n-1}^{n} \frac{(-1)^n}{2n-1}$ $= \sum_{n-1}^{n} (J_1 - J_0) + (J_2 - J_1) + ...(J_m - J_{m-1})$ $= J_m - J_0$ $= J_m - I_0$ $= J_m - \frac{\pi}{4}$ $\therefore \quad J_m = \sum_{n-1}^{n} \frac{(-1)^n}{2n-1} + \frac{\pi}{4}$	(iii) 1 mark: develops the sequence for J_m 1 mark: proves $J_0 = \frac{\pi}{4}$	Poorly done
	(iv) *Let u = tan x* $dx = \frac{du}{\sec^2 x} = \frac{du}{1+u^2}$ *When x = 0, u = 0* *x = 1, u = 1*	(iv) proves the result	Most students got this mark

	(v) $I_n = \int_0^1 \frac{u^n}{1+u^2}\,du$ $I_n \geq 0 \quad for\ 0 \leq u \leq 1$ **(1)** $I_n = \frac{1}{n+1} - I_{n-2}$ $\leq \frac{1}{n+1}$ **(2)** $\therefore\ 0 \leq I_n \leq \frac{1}{n+1}$ $As\ \ n \to \infty,\ \ \frac{1}{n+1} \to 0$ $Hence\ \ (-1)^n I_{2n} \to 0\ \ as\ \ n \to \infty$ **(3)** $\therefore\ J_n \to 0$	(v) 2 mark: all three aspects 1, 2, are proved 1 mark: 2 of the results proved	Poorly done