Killara HIGH SCHOOL

Student Number

Extension 2 Mathematics

HSC Assessment task 2
13th March 2015

General Instructions
- Reading time – 5 minutes
- Working tine 3 hours
- Write using blue or black pen
  Black pen is preferred
- Approved calculators may be used
- In Questions 11-16 show relevant mathematical reasoning and/or calculations
- Start a new booklet for each question

Total Marks –

Section A
10 marks
- Attempt Questions 1 – 10
- Allow about 10 minutes for this section

Section B
90 marks
- Attempt Questions 11 – 16
- Allow about 2 hour and 50 minutes for this section

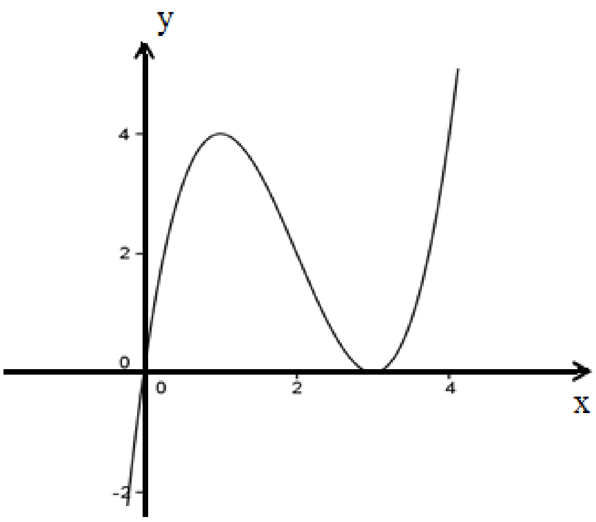
Question	Marks
1-10	/10
11	/15
12	/15
13	/15
14	/15
15	/15
16	/15

THIS QUESTION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM

This assessment task constitutes 10% of the Higher School Certificate Course Assessment

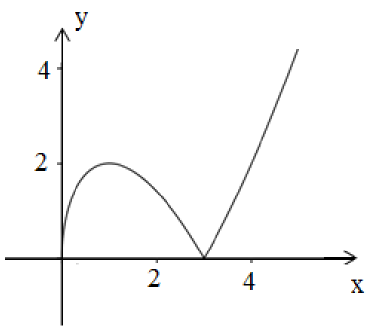
Section A: (10 marks)
Attempt Questions 1 – 10
Allow about 10 minutes for this section
Use the multiple-choice answer sheet for questions 1 – 10 (detach from paper)

1. The graph $y = f(x)$ is shown below.

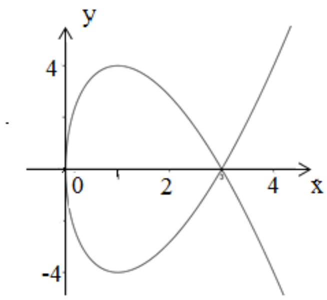


Which of the following graphs best represents $y^2 = f(x)$?

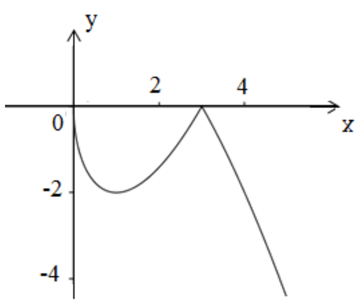
(A)



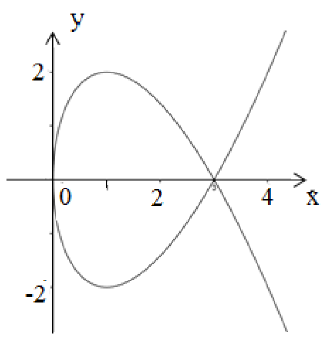
(B)



(C)



(D)



2. Let $z = a + ib$, where a and b are real and non-zero.

Which of the following statements is false?

A. $z - \bar{z} = 2bi$

B. $\arg(z) - \arg(\bar{z}) = 2\arg(z)$

C. $|z| + |\bar{z}| = |z + \bar{z}|$

D. $|z|^4 = |z|^2|\bar{z}|^2$

3. If $f'(x) = 2\cos^2(x) - 1$ and $f\left(\frac{\pi}{6}\right) = 0$, then $f(x)$ is equal to

A. $\frac{2sin2x - \sqrt{3}}{4}$

B. $\frac{sin2x}{2}$

C. $\frac{\sqrt{3}}{2} - sin\ 2x$

D. $\frac{sin2x}{2} + \frac{\sqrt{3}}{4}$

4. If $z = x + iy$, the locus of points that lie on a circle of radius 2 units centred at the origin on the Argand diagram can be represented by the equation,

A. $z\bar{z} = 2$

B. $(z + \bar{z})^2 - (z - \bar{z})^2 = 16$

C. $Re(z^2) + Im(z^2) = 4$

D. $\{Re(z)\}^2 + \{Im(z)\}^2 = 16$

5. What are the parametric equation of the ellipse $\frac{(x+1)^2}{4} + \frac{(y-3)^2}{2} = 1$?

(A) $x = 1 - 2cos\theta$, $y = 3 + \sqrt{2}sin\theta$.

(B) $x = -1 + 2cos\theta$, $y = 2 + \sqrt{3}sin\theta$.

(C) $x = 2 + \sqrt{2}cos\theta$, $y = 3 + \sqrt{2}sin\theta$.

(D) $x = -1 + 2cos\theta$, $y = 3 + \sqrt{2}sin\theta$.

6. z satisfies the equation $\left|z-\sqrt{2}-i\sqrt{2}\right|=1$. Find the minimum value of $\arg(z)$.

A. -75°

B. -15°

C. 15°

D. 75°

7. The equation of the directrices of the hyperbola $4x^2-9y^2-8x-18y-41=0$ are:

(A) $x=1\pm\frac{\sqrt{13}}{9}$

(B) $x=1\pm\frac{9}{\sqrt{13}}$

(C) $y=1\pm\frac{9}{13}$

(D) $y=1\pm\frac{\sqrt{13}}{3}$

8. α,β and γ are the zeros of the polynomial x^3+5x-3. Find the value of $\alpha(\alpha^2-\beta\gamma)+\beta(\beta^2-\gamma\alpha)+\gamma(\gamma^2-\alpha\beta)$.

A. -125

B. 9

C. 0

D. 116

9. The function $f(x)=\frac{x^2}{x^2-4}$ has

(A) A single vertical asymptote, two horizontal asymptotes and no turning points.

(B) A single horizontal asymptote, two vertical asymptotes and no turning points.

(C) A single vertical asymptote, two horizontal asymptotes and one turning point.

(D) A single horizontal asymptote, two vertical asymptotes and one turning point.

10. The polynomial $P(z)$ has the equation $P(z) = z^4 - 4z^3 + Az + 20$, where A is a real number. Given that $3 + i$ is a zero of $P(z)$, which of the following expressions is $P(z)$ as a product of two real quadratic factors in real field.

A. $(z^2 - 2z + 2)(z^2 - 6z + 10)$

B. $(z^2 + 2z + 2)(z^2 + 6z + 10)$

C. $(z^2 - 2z + 2)(z^2 + 6z + 10)$

D. $(z^2 + 2z + 2)(z^2 - 6z + 10)$

Section B (90 marks)
Attempt Questions 11 – 16
Allow about 1 hours and 50 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11 - 16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet

a) Simplify $\left|cos\frac{2\pi}{5} + isin\frac{2\pi}{5}\right|$ **1**

b)

i) Express $\frac{-1+i\sqrt{3}}{1-i}$ in modulus argument form **2**

ii) Hence find the exact value of $\tan\frac{11\pi}{12}$ **2**

c) Suppose $z = 1 + i\sqrt{3}$ and $w = z(cis\alpha)$ where $-\pi \leq \alpha \leq \pi$. If w is purely imaginary and $\text{Im}(w) > 0$, find the value of:

i) α **2**

ii) $arg\,(z + w)$ **2**

d)

i) Find the square roots of $24 + 10i$. **1**

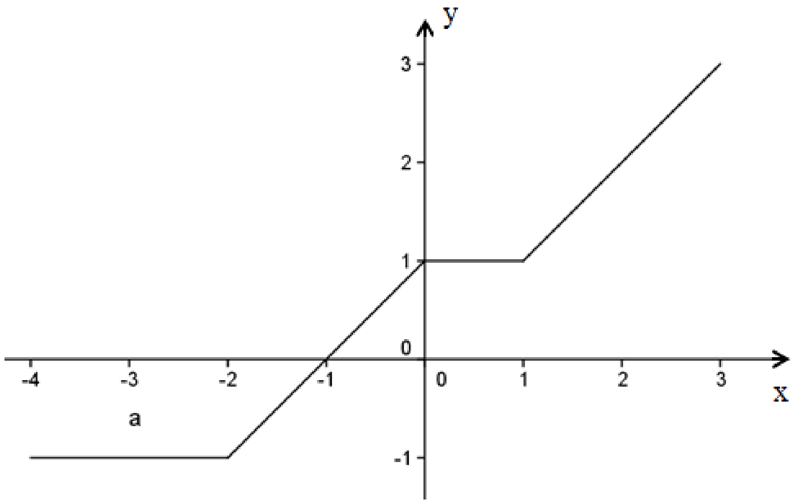
ii) Solve $z^2 + (1 + 3i)z - 8 - i = 0$ **2**

iii) Hence describe the locus of $|\ z - 2 + i\ | = \ |\ z^2 + (1 + 3i)z - 8 - i\ |$ **3**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet

a)



The diagram is a sketch of the function y = f(x). On separate diagrams sketch:

i) $y = |f(x)|$ **2**

ii) $y = f(|x|)$ **2**

iii) $y = \frac{1}{f(x)}$ **2**

iv) $y = sin^{-1}(f(x))$ **2**

The even function g is defined by

b)
$$g(x) = \begin{cases} 4e^{-x} - 6e^{-2x} & for\ x \geq 0 \\ g(-x) & for\ x \leq 0 \end{cases}$$

i) Show that the graph of *g(x)* has a maximum turning point at $\left(ln3,\ \frac{2}{3}\right)$. **2**

ii) Sketch the curve *y = g(x)* and label the turning points, point of inflexion, asymptotes, and points of intersection with the axes. **4**

iii) Discuss the behaviour of the curve *y = g(x)* at *x = 0.* **1**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet

a) α, β and γ are the roots of the equation $x^3 + 3px + q = 0$.

i) Find a polynomial equation of degree 3 with roots $\frac{\alpha\beta}{\gamma}, \frac{\beta\gamma}{\alpha}$ and $\frac{\gamma\alpha}{\beta}$. **3**

ii) Hence find a relationship between p and q if $\alpha\beta = \gamma$. **1**

b) $(-1 + i\sqrt{3})$ is a zero of the polynomial $P(x) = x^4 + x^3 - 3x^2 - 14x - 20$.
Factorise P(x) over the real field. **3**

c) The polynomial $P(x) = 16x^4 - 32x^3 + 16x^2 + kx - 5$, where k is an integer, has two rational roots which are equal but opposite signs and two non-real roots.

i) If α is a non-real root of $P(x)$, show that $Re(\alpha) = 1$ and $|\alpha| > 1$. **2**

ii) If the rational roots are $\pm\beta$, deduce that $\beta^2 < \frac{5}{16}$. **2**

iii) Find the rational roots and the value of k. **2**

iv) Factorise $P(x)$ into irreducible factors with integer coefficients. **2**

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet

a. Consider the conic defined by the equation $\frac{x^2}{19-l}+\frac{y^2}{7-l}=1$

i) Determine the real values of *l* for which the equation defines an ellipse and a hyperbola. **2**

ii) Sketch the curve corresponding to the value *l* = 3. **2**

iii) Describe how the shape of this curve changes as *l* increases from 3 towards 7. What is the limiting position of the curve as is approached? **2**

b. T is a point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with centre O.

A line drawn through O, parallel to the tangent to the ellipse at T, meets the ellipse at M and N. Find:

i) the equation of MN. **2**

ii) the perpendicular distance from T to MN. **4**

iii) Hence prove that the area of the triangle TMN is independent of the position of T **3**

End of Question 14

Question 15 (15 marks) Use a SEPARATE writing booklet

a) (i) Use De Moivre's theorem to express $\sin 3\theta$ in terms of $\sin\theta$. **2**

(ii) Use this result to solve $8x^3 - 6x + \sqrt{3} = 0$. **2**

(iii)Hence deduce that $\sin\frac{\pi}{9} + sin\frac{7\pi}{9} = sin\frac{5\pi}{9}$. **2**

b) z_1 and z_2 are the zeros of the equation $z^2 - (1+i)z + \left(1 + i\sqrt{3}\right) = 0$.

Find the value of : (i) $argz_1 + argz_2$ **1**

(ii) $|z_1 + z_2|$ **1**

c) Given that $z = cos\theta + isin\theta$, show that $cos^4\theta + sin^4\theta = \frac{1}{4}\ (cos4\theta + 3)$ **2**

d) Given that $P \equiv e^{i\theta} \equiv cos\theta + isin\theta$ where $-\pi \le \theta \le \pi$,

(i) Evaluate $\sqrt[i]{-1}$ correct to 4 decimal places. **2**

(ii) By considering the sum of the series 1 + P + P^2 + P^3 +…. to n terms, show that: **3**

$$1 + cos\theta + cos2\theta + cos3\theta + \cdots + \cos(n-1)\,\theta =$$

$$\frac{[(1 - cos\theta)(1 - \cos(n\theta))sin\theta\sin(n\theta)]}{2 - 2cos\theta}$$

End of Question 15

Question 16 (15 marks) Use a SEPARATE writing booklet

a) i) Show that $\sin x + \sin 3x = 2\sin 2x \cos x$. **2**

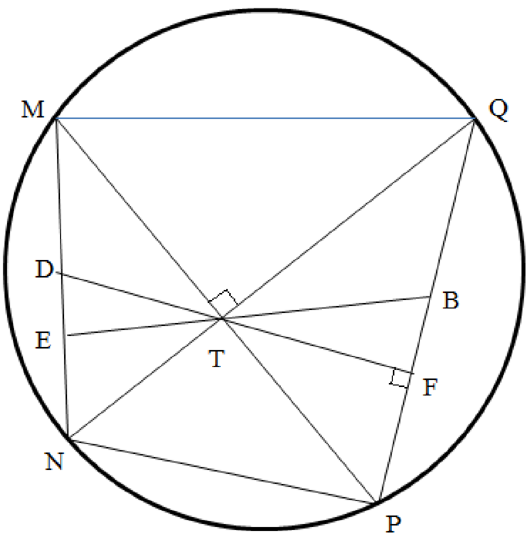
ii) Hence otherwise find all the solutions of
$\sin x + \sin 2x + \sin 3x = 0$ for $0 \le x \le 2\pi$. **2**

b) Given $f(x) = x - \log_e (1 + x^2)$

i) Show that $f'(x) \ge 0$ for all values of x. **2**

ii) Hence deduce that $e^x > 1 + x^2$ for all positive values of x. **2**

c)



In the diagram above, the diagonals MP and NQ of the cyclic quadrilateral MNPQ intersect at a right angle at the point T. The perpendicular TF to QP meets MN at D. The median from T to the midpoint of QP meets MN at E,

i) Show that the point D is the midpoint of MN. **3**

ii) Show that $\angle TEN = 90°$. **2**

iii) Let A and C be the midpoints of NP and MQ respectively. Show that the quadrilateral ABCD is a rectangle. **2**

END OF EXAMINATION

Killara HIGH SCHOOL

Student Number

Mathematics

Section I – Multiple Choice Answer Sheet

Use this multiple-choice answer sheet for questions 1 – 10. Detach this sheet.

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample: $2 + 4 =$ (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ⊗ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word *correct* and drawing an arrow as follows.

correct

A ⊗ B ⊗ C ○ D ○

Start Here →

1. A ○ B ○ C ○ D ○
2. A ○ B ○ C ○ D ○
3. A ○ B ○ C ○ D ○
4. A ○ B ○ C ○ D ○
5. A ○ B ○ C ○ D ○
6. A ○ B ○ C ○ D ○
7. A ○ B ○ C ○ D ○
8. A ○ B ○ C ○ D ○
9. A ○ B ○ C ○ D ○
10. A ○ B ○ C ○ D ○

Q11 (a)

$$\left|\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right|$$

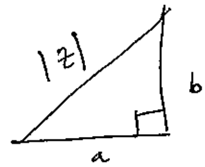
$$= \sqrt{\cos^2\frac{2\pi}{5} + \sin^2\frac{2\pi}{5}}$$

$$= \sqrt{1}$$

$$= \underline{\underline{1}}$$

$z = a + bi$

$|z| = a^2 + b^2$



correct answer → 1 mark

(b) (i) $\frac{-1 + i\sqrt{3}}{1 - i}$ → Mod arg. → Mod - arg.

Many people opting for the long process.

$$= \frac{2\,\text{Cis}\left(\frac{2\pi}{3}\right)}{\sqrt{2}\,\text{Cis}\left(-\frac{\pi}{4}\right)}$$

$\frac{z_1}{z_2} = \frac{r_1}{r_2}\,\text{Cis}(\alpha - \beta)$ → 1 mark for expressing both in mod-arg form

$$= \sqrt{2}\,\text{Cis}\left(\frac{2\pi}{3} + \frac{\pi}{4}\right)$$

$$= \sqrt{2}\,\text{Cis}\left(\frac{11\pi}{12}\right)$$ → 1 mark for correct answer

(ii) $$\frac{-1 + i\sqrt{3}}{1 - i} = \frac{(-1 + i\sqrt{3})(1 + i)}{2}$$

$$= \left(\frac{-\sqrt{3} - 1}{2}\right) + i\frac{(\sqrt{3} - 1)}{2}$$ → 1 mark for $a + ib$ form

$$\Longrightarrow \sqrt{2}\,\text{Cis}\left(\frac{11\pi}{12}\right) = \left(\frac{-\sqrt{3} - 1}{2}\right) + i\left(\frac{\sqrt{3} - 1}{2}\right)$$

Equating the imaginary parts

$$\sqrt{2}\sin\left(\frac{11\pi}{12}\right) = \frac{\sqrt{3} - 1}{2} \quad ①$$

Equating the ~~real~~ real parts

$$\sqrt{2}\cos\left(\frac{11\pi}{12}\right) = \frac{-\sqrt{3} - 1}{2} \quad ②$$

① ÷ ②

$$\tan\left(\frac{11\pi}{12}\right) = -\frac{(\sqrt{3} - 1)}{(\sqrt{3} + 1)}$$

$$= \underline{\underline{\sqrt{3} - 2}}$$

→ 1 mark the working & correct answer

Q4

(c) $Z = 1 + i\sqrt{3}$
$= 2 \operatorname{cis} \pi/3$

$W = Z \operatorname{cis}\alpha = 2 \operatorname{cis}(\pi/3 + \alpha)$

(i) W is purely imaginary

$\rightarrow Re(W) = 0$
$\rightarrow 2\cos(\pi/3 + \alpha) = 0$ → 1 mark for correctly equating

$\pi/3 + \alpha = \pi/2$
$\alpha = \pi/6$ → 1 mark for correct answer

(ii) $Z = 1 + i\sqrt{3}$, $W = 2i$ → 1 mark for evaluating $(z+w)$ correctly

$\therefore$ ~~$z + w = 3i$~~

$\therefore z + w = 1 + (\sqrt{3} + 2)i$

$\therefore \arg(z + w) = \tan^{-1}(2 + \sqrt{3})$
≈ 1.31 radians
$= \frac{5\pi}{12}$ → 1 mark for correct answer

Q11 (a)

(i) $\sqrt{24+10i} = \sqrt{(5+i)^2}$

$= \pm(5+i)$

1 mark for correct working + answer

Must learn an easier method to do this question.

(ii) $z^2+(1+3i)z-8-i=0$

$z = \frac{-(1+3i) \pm \sqrt{(1+3i)^2+4(8+i)}}{2}$

$= \frac{-(1+3i) \pm \sqrt{24+10i}}{2}$

1 mark for using quadratic formula correctly

$= \frac{-(1+3i) \pm (5+i)}{2}$

$= (2-i)$ or $(-3-2i)$

1 mark for correct answer

(iii)

$|z-2+i| = |z^2+(1+3i)z-8-i|$

$= |(z-2+i)(z+3+2i)|$

1 mark for identifying this

$\Rightarrow |z+3+2i| = 1$

1 mark

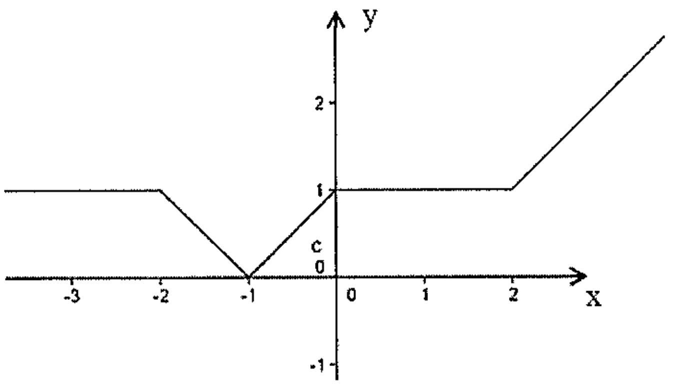
Many students failed to ~~see~~ make the connection between Q(ii) and (iii)

$\Rightarrow$ locus of z on an Argand diagram is a circle with centre at $(-3,-2)$ and radius 1 unit.

1 mark for correctly describing.

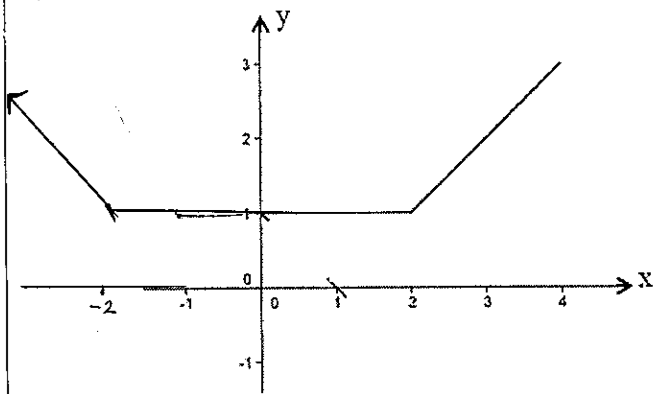
Q12(a)

i)



2 - each correct graph with correct label of values

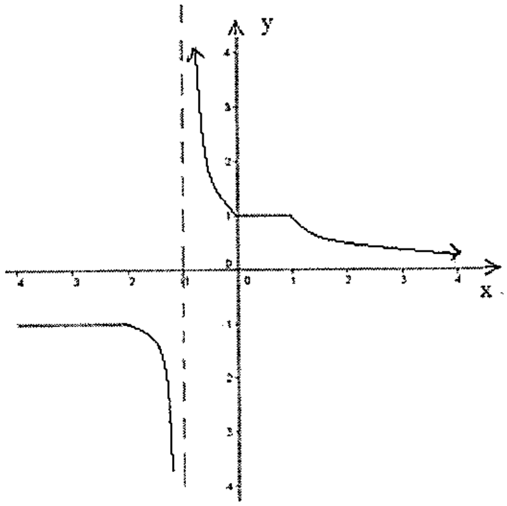
ii)



• Discards the $x<0$ ~~Left~~.

• reflects to right

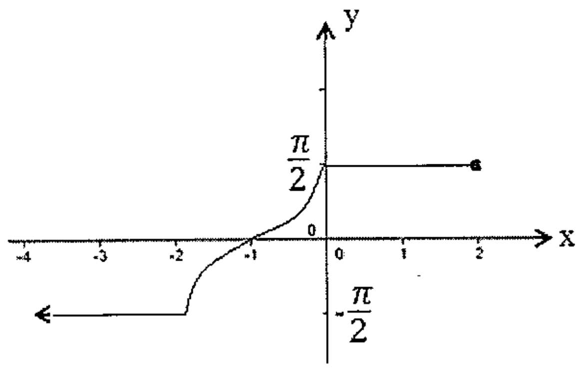
iii)



$x \geq 0$ Correct Shape.

$x < 0$. Correct Shape.

iv)



• Capped between $y = -1$ to 1.

• Correct Shape 1.

The shape of $\sin^{-1}(x)$ between $-2 \leq x \leq 1$ was a problem

Many students forgot that the original graph was nothing but $y = x$ in this region and hence $\sin^{-1} x$ should be the shape of the graph

(b)

$$g'(x) = \frac{12}{e^{2x}} - \frac{4}{e^{x}} \qquad g''(x) = \frac{4}{e^{x}} - \frac{24}{e^{2x}}$$

1- correctly differentiate g'(x)

For stationary point $g'(x) = 0$

$$\frac{12}{e^{2x}} - \frac{4}{e^{x}} = 0$$

$12 - 4e^{x} = 0$

$e^{x} = 3$

$x = \ln 3$

$y = 4e^{-\ln 3} - 6e^{-2\ln 3}$

$= \frac{4}{3} - \frac{6}{9} = \frac{2}{3}$

1-correct answer

$g''(\ln 3) = \frac{4}{3} - \frac{24}{9}$

$= -\frac{12}{9} < 0$

$\therefore$ $(\ln 3, \frac{2}{3})$ is a maximum point.

ii) For point of inflection $g''(x) = 0$

2-correct answer with all working

$4e^{-x}(1 - 6e^{-x}) = 0$

$4e^{x} = 24$

$e^{x} = 6$

$x = \ln 6$

$y = 4e^{-\ln 6} - 6e^{-2\ln 6}$

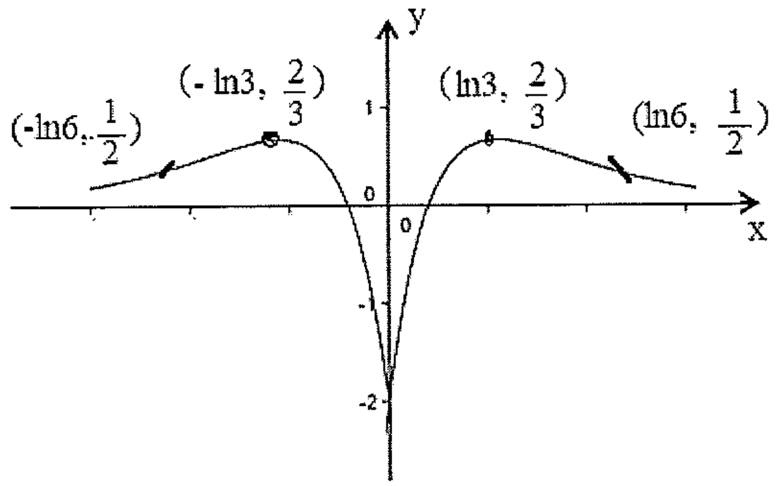
$= \frac{4}{6} - \frac{6}{36} = \frac{1}{2}$

x	ln5	ln6	ln7
$g''(x)$	$-\frac{8}{5}$	0	0.004

$\therefore$ $(\ln 6, \frac{1}{2})$ is the point of inflexion.

Not proving Maximum

point of inflection → not testing concavity



2-correct graphs and labelling

Must Note that this is an even function. So doing $x \geq 0$ is enough and for $x < 0$ the graph can be reflected

iii) At $x = 0$, $g(x) = -2$, $g'(x) = 8$ and $g'(x) = -8$

The curve has a cusp at (0, - 2)

1-correct answer

Must say why the function is undifferentiable at $x = 0$.

Q13 (a)

$x^3 + 3px + q = 0$

roots α, β, γ

(i) $\alpha\beta\gamma = -q \Rightarrow \frac{\alpha\beta}{\gamma} = -\frac{q}{\gamma^2}$

$\therefore \frac{\alpha\beta}{\gamma}, \frac{\beta\gamma}{\alpha}$ and $\frac{\gamma\alpha}{\beta}$ are roots

$\Rightarrow -\frac{q}{\alpha^2}, -\frac{q}{\beta^2}$ and $-\frac{q}{\gamma^2}$ are the roots

1 mark getting the roots in this form

$-\frac{q}{\alpha^2} = x$

$\therefore \alpha = \sqrt{\frac{-q}{x}}$

$P(\alpha) = 0$

$\Rightarrow P\left(\sqrt{\frac{-q}{x}}\right) = 0$

$\Rightarrow \left(\sqrt{\frac{-q}{x}}\right)^3 + 3p\sqrt{\frac{-q}{x}} + q = 0$

1 mark for correct substitution.

$\sqrt{\frac{-q}{x}}\left[-\frac{q}{x} + 3p\right] = -q$

squaring both sides,

$-\frac{q}{x}\left[-\frac{q}{x} + 3p\right]^2 = q^2$

$\Rightarrow \frac{q^2}{x^2} - \frac{6pq}{x} + 9p^2 = -qx$

$\Rightarrow qx^3 + 9p^2x^2 - 6pqx + q^2 = 0$

1 mark for correct working and correct eq^n

(ii) $\alpha\beta = \gamma \Rightarrow \frac{\alpha\beta}{\gamma} = 1$

$\Rightarrow q + 9p^2 - 6pq + q^2 = 0$

$\Rightarrow (3p - q)^2 = -q$

1 mark the correct relationship

In fact, substitution is the easiest way to eliminate x.

This is a very routine question, but some students have not bothered to learn it.

In general done well.

Many students failed to see that $x = 1$ is a root, $\therefore$ satis the equation.

Q13

(b) $P(x) = x^4 + x^3 - 3x^2 - 14x - 20$

Since the coefficients ~~of~~ of $P(x)$ are all real,

$(-1+i\sqrt{3})$ is a zero $\Rightarrow$ it's complex conjugate $(-1-i\sqrt{3})$ is also a zero.

$\Rightarrow [x-(-1+i\sqrt{3})][x-(-1-i\sqrt{3})]$ is a factor of $P(x)$

i.e. (x^2+2x+4) is a factor of $P(x)$.

$x^4 + x^3 - 3x^2 - 14x - 20$

$= (x^2+2x+4)(x^2-x-5)$

$= (x^2+2x+4)\left[x - \frac{1-\sqrt{21}}{2}\right]\left[x - \frac{1+\sqrt{21}}{2}\right]$

Many students still have problems to write the quadratic with given roots

→ 1 mark for using these

Many choose to guess it, rather than working it out.

→ 1 mark for arriving at this

→ 1 mark for correct answer

Q13 (c)

~~(b)~~ $P(x) = 16x^4 - 32x^3 + 16x^2 + kx - 5$

(i) Since the coefficients are all real,

α is a non-real root $\Rightarrow$ it's conjugate $\bar{\alpha}$ is also a root

$\therefore$ let the roots be $\beta, -\beta, \alpha$ and $\bar{\alpha}$

$$\text{sum of roots} = -\frac{(-32)}{16}$$

$$\Rightarrow \beta + -\beta + \alpha + \bar{\alpha} = 2$$

$$\Rightarrow 2\,Re(\alpha) = 2$$

$$\Rightarrow Re(\alpha) = 1$$

$$\Rightarrow |\alpha| = \sqrt{1 + [Im(\alpha)]^2} > 1$$

(ii) Products of roots $= \frac{-5}{16}$

$$\Rightarrow \beta \cdot -\beta \cdot \alpha \cdot \bar{\alpha} = \frac{-5}{16}$$

$$\Rightarrow -\beta^2 \cdot |\alpha|^2 = -\frac{5}{16}$$

$$\Rightarrow \beta^2 = \frac{5}{16|\alpha|^2} < \frac{5}{16} \quad \left(\because \frac{1}{|\alpha|^2} < 1\right)$$

(i) working up to $Re(\alpha) = 1$	1 mark correct working	Generally well done.
(i) $\lvert\alpha\rvert > 1$	1 mark for this reasoning	Those students who wrote $\alpha = x + iy$ got this question right.
(ii) working up to $-\beta^2 \lvert\alpha\rvert^2 = -\frac{5}{16}$	1 mark for this working	Well done! This is a "Show that" question. Every line of logical progression must be shown.
(ii) $\beta^2 < \frac{5}{16}$	1 mark for correct reasoning	

Q13

(iii) $\beta, -\beta$ are roots

$\therefore P(\beta) = 0, \ P(-\beta) = 0$

$\Rightarrow P(\beta) + P(-\beta) = 0$ → Hardly anybody could see this logic.

$\Rightarrow 16\beta^4 + 16\beta^2 - 5 = 0$

$\Rightarrow (4\beta^2 - 1)(4\beta^2 + 5) = 0$

Since β is real $4\beta^2 + 5 \neq 0$

$\Rightarrow 4\beta^2 = 1$

$\Rightarrow \beta = \frac{1}{2}$ or $\beta = -\frac{1}{2}$

1 mark working & correct answer

$\therefore$ The rational roots are $\frac{1}{2}$ and $-\frac{1}{2}$.

$P(\frac{1}{2}) = 0$

$\Rightarrow 16(\frac{1}{2})^4 - 32(\frac{1}{2})^3 + 16(\frac{1}{2})^2 + k(\frac{1}{2}) - 5 = 0$

1 mark correct answer

$\Rightarrow \underline{k = 8}$

(iv) $P(x) = 16x^4 - 32x^3 + 16x^2 + 8x - 5$

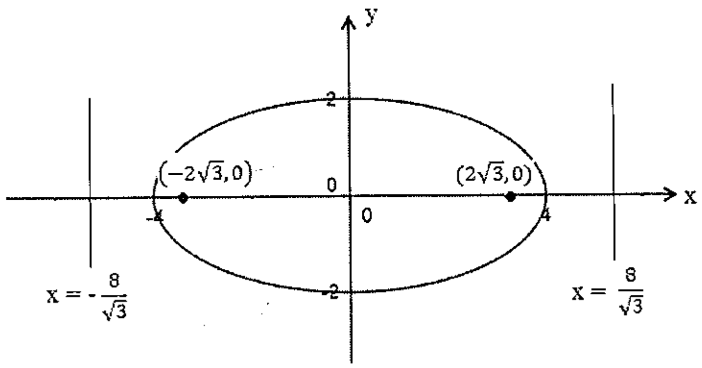
$= (2x-1)(2x+1)(4x^2+8x+5)$ → 1 mark

$= (2x-1)(2x+1)(2x-2-i)(2x-2+i)$ → 1 mark.

Very poorly done.

In general, Q (iii) & (iv) were N/A.

EX2 – Half Yearly – 2015

Section A 1. D 2. C 3. A 4. B 5. D 6. C 7. B 8. C 9. D 10. D **Section B**		
Q14(a) (i) For the ellipse $19 - l > 0$ and $7 - l > 0$ $l < 19$ and $l < 7$ $\therefore\ l < 7$	1 - correct answer	Using $0<e<1$ for ellipse and $e>1$ for hyperbola and trying to solve for l makes this question, dense in working.
For the hyperbola $19 - l > 0$ and $7 - l < 0$ $\therefore 7 < l < 19$	1- correct answer	
(ii) 	2-correct graphs and label of valu.	
(iii) As l increases from 3 towards 7 the shape of the curve approaches an interval where $-2\sqrt{3} \le x \le 2\sqrt{3}$	1 - correct answer	If you said y- you got 1 mark, but needed to sa the limiting value of x to get the full marks.
As l approaches 7 $a^2 \Rightarrow 19-7$, $a \Rightarrow 2\sqrt{3}$ and $b^2 \Rightarrow 7\text{-}7 \Rightarrow 0$ $\therefore$ The limiting position of the curve is SS' of the above ellipse.	1 - correct answer	
(b) $\frac{dx}{d\theta} = -a\sin\theta$ $\frac{dy}{d\theta} = b\cos\theta$ $\frac{dy}{dx} = \frac{b\cos\theta}{-a\sin\theta}$ MN // tangent at T	1- correctly differentiate & find m	Many students wasted time trying to find the equation of the tangent. But it has nothing to do with this question.

$\therefore \quad m_{\text{of MN}} = -\frac{b\cos\theta}{a\sin\theta}$ The equation of MN is: $y = -\frac{b\cos\theta}{a\sin\theta}x \quad (1)$	1- correct answer	Also, if it is passing through the origin, it' general form is $y=mx$. No need to use point-gradient formula and make it Complicated.
ii) Sub (1) into $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $\Rightarrow \quad \frac{x^2}{a^2}+\frac{x^2b^2\cos^2\theta}{b^2a^2\sin^2\theta}=1$ $\Rightarrow \quad \frac{x^2(\cancel{\sin^2\theta+\cos^2\theta})}{a^2\sin^2\theta}=1$ $\therefore$ The coordinates of MN are $x = \pm\, a\sin\theta$ $y = \frac{-b\cos\theta}{a\cancel{\sin\theta}} \times a\cancel{\sin\theta}$ $y = \pm\, b\cos\theta$	1- for each correct coordinates of M & N	
The perpendicular distance TP to MN at P is : $\left\lvert \frac{\frac{b\cos\theta \times a\cos\theta}{a\sin\theta}+b\sin\theta}{\sqrt{1+\frac{b^2\cos^2\theta}{a^2\sin\ \ \theta}}} \right\rvert = \left\lvert \frac{ab\cos^2\theta+ab\sin^2\theta}{\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}} \right\rvert$	1-correct formula & substitution of perpendicular distance	Again, those who used parametric form, got it much faster.
$= \frac{ab}{\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}}$	1-correct answer of TP	
iii) The distance MN is $\sqrt{(2a\sin\theta)^2+(2b\cos\theta)^2} = \sqrt{4(a^2\sin^2\theta+b^2\cos^2\theta)}$ $= 2\sqrt{a^2\sin^2\theta+b^2\cos\theta}$	1- correct answer of MN	
Area of Δ TMN is	1 - Finding area of triangle	
$= \frac{1}{2}\times 2\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}\times\frac{ab}{\sqrt{a^2\sin^2\theta+b\ \ \cos^2\theta}}$ $=$ ab which is independent of the position of T	1 – correct answer & conclusion	

Q15 (a)

(i) $(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$

(ie) $(\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta)$
$= \cos 3\theta + i\sin 3\theta$

1 mark for correct expansion & using De Moivre Theorem.

Matching the imaginary parts

$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$
$= 3(1-\sin^2\theta)\sin\theta - \sin^3\theta$
$= \underline{3\sin\theta - 4\sin^3\theta}$

1 mark for equating imag parts & simplifying

(ii) $8x^3 - 6x + \sqrt{3} = 0$

let $x = \sin\theta$

$8\sin^3\theta - 6\sin\theta = -\sqrt{3}$
$\Rightarrow 2\sin 3\theta = \sqrt{3}$
$\sin 3\theta = \frac{\sqrt{3}}{2}$

1 mark using the 1st part to get this

$\Rightarrow \theta = \frac{n\pi}{3} + (-1)^n \pi/9$

$n=0 \quad \theta = \pi/9 \Rightarrow x = \sin \pi/9$

$n=-1 \quad \theta = -\frac{\pi}{3} - \pi/9 \Rightarrow x = \sin(-4\pi/9) = -\sin 5\pi/9$

$n=1 \quad \theta = \pi/3 - \pi/9 \Rightarrow x = \sin 2\pi/9 = \sin 7\pi/9$

$\rightarrow$ 1 mark

$\therefore$ The roots are
$x = \sin\pi/9, -\sin 5\pi/9, \sin 7\pi/9$

(iii) Sum of roots $= 0$ $\longrightarrow$ 1 mark

$\Rightarrow \sin\pi/9 - \sin 5\pi/9 + \sin 7\pi/9 = 0$

$\Rightarrow \underline{\sin\pi/9 + \sin 7\pi/9 = \sin 5\pi/9}$

A large number of students stopped with $n=0,1,2$ yielding $\frac{2\pi}{9}, \frac{7\pi}{9}$. But they did not realise $\sin\frac{2\pi}{9} = \sin\frac{7\pi}{9}$.

I think, Q(iii) should give you the clue.

Q15 (b)

$$z^2 - (1+i)z + (1+i\sqrt{3}) = 0$$

(i) Product of roots $= 1 + i\sqrt{3}$ → 1 mark correct answer

$\Rightarrow \quad z_1 z_2 = 1 + i\sqrt{3}$

$\Rightarrow \quad \arg(z_1 z_2) = \arg(1 + i\sqrt{3})$

$= \pi/3$

$\Rightarrow \quad \arg(z_1) + \arg(z_2) = \underline{\pi/3}$

1 mark correct answer

(ii) Sum of roots $= 1 + i$

$\Rightarrow \quad z_1 + z_2 = 1 + i$

$\Rightarrow \quad |z_1 + z_2| = |1 + i|$

$= \underline{\sqrt{2}}$ → 1 mark correct answer

(d)

(i) $\cos\theta + i\sin\theta = e^{i\theta}$

when $\theta = \pi$

$-1 = e^{i\pi}$ → 1 mark for correct subs to get this

$\Rightarrow \quad \sqrt[i]{-1} = \sqrt[i]{e^{i\pi}}$

$= e^{\pi}$ → 1 mark correct answer

$= \underline{23.1407}$

~~16~~ 15 b(c) $z = \cos\theta + i\sin\theta$

$z^n = \cos(n\theta) + i\sin(n\theta)$ — ①

$\frac{1}{z^n} = \cos(n\theta) - i\sin(n\theta)$ — ②

① + ② $\Rightarrow z^n + \frac{1}{z^n} = 2\cos(n\theta)$

① − ② $\Rightarrow z^n - \frac{1}{z^n} = 2i\sin(n\theta)$

$\therefore 16\cos^4\theta + 16\sin^4\theta$

$= (z + \frac{1}{z})^4 + (z - \frac{1}{z})^4$

1 mark for getting this

$= 2[z^4 + 6 + \frac{1}{z^4}]$

$= 2[2\cos 4\theta + 6]$

$\therefore \cos^4\theta + \sin^4\theta = \frac{1}{4}[\cos 4\theta + 3]$

1 mark ~~for~~ correctly simplifying to get the answer

When you need to find powers of $\cos\theta$ and $\sin\theta$, you use the ~~fo~~ result

$\boxed{z + z^{-1} = 2\cos\theta}$, ~~$z \neq z$~~ $\boxed{z - z^{-1} = 2i\sin\theta}$

Then, break it down to

$z^n + z^{-n} = 2\cos n\theta$

$z^n - z^{-n} = 2i\sin n\theta$

$Q_{\cancel{16}}^{15}$ ~~d~~ (ii)

$$1+\rho+\rho^2+\rho^3+\cdots+\rho^{n-1} = \frac{1-\rho^n}{1-\rho}$$

~~Sum of GP with 1st~~
sum of the first n terms of a GP with first term 1 and common ratio ρ

Matching the real parts

$$1+\cos\theta+\cos 2\theta+\cos 3\theta+\cdots+\cos(n-1)\theta = Re\left[\frac{1-\rho^n}{1-\rho}\right]$$

1 mark using sum of series to get Term

$$\frac{1-\rho^n}{1-\rho} = \frac{[(1-\cos n\theta)-i\sin n\theta]}{[(1-\cos\theta)-i\sin\theta]}\times\frac{(1-\cos\theta)+i\sin\theta}{(1-\cos\theta)+i\sin\theta} \rightarrow \text{1 mark}$$

~~1 mark~~

$$= \frac{[(1-\cos\theta)(1-\cos n\theta)+\sin\theta\sin n\theta]}{(1-\cos\theta)^2+\sin^2\theta}+i[\quad]$$

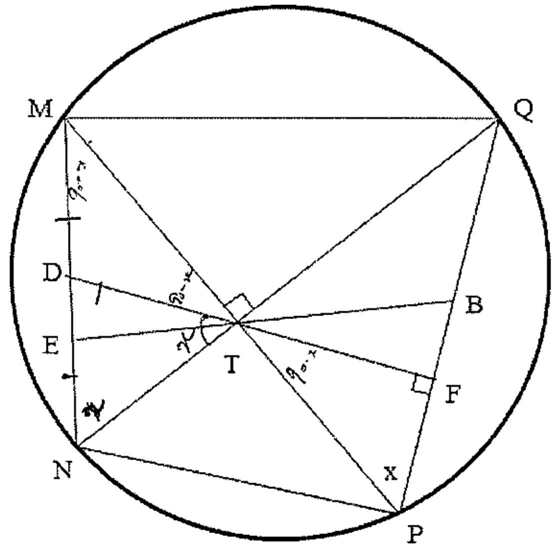
~~1 mark~~

$$\Rightarrow Re\left(\frac{1-\rho^n}{1-\rho}\right) = \frac{(1-\cos\theta)(1-\cos n\theta)+\sin\theta\sin n\theta}{(1-\cos\theta)^2+\sin^2\theta}$$

$$= \frac{(1-\cos\theta)(1-\cos n\theta)+\sin\theta\sin n\theta}{2-2\cos\theta}$$

1 mark for correct working

Solution	Marking criteria	Comments
Q16(a) (i) $L.H.S = \sin(2x+x) + \sin(2x-x)$ $= \sin 2x\cos x + \cos 2x\sin x + \sin 2x\cos x - \cos 2x\sin x$ $= 2\sin 2x\sin x$ $= R.H.S$	1 – correct method 1- correctly show L.HS = RHS	Must learn the "Sums as products" formulae. These are very useful for Simplifying expressions
(ii) $\sin x + \sin 2x + \sin 3x = 0$ Sub (i) $2\sin 2x\cos x + \sin 2x = 0$ $\sin 2x(2\cos x + 1) = 0$ $\sin 2x = 0$ $2x = 0, \pi, 2\pi, 3\pi, 4\pi$ $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$ and $2\cos x = -1$ $x = \frac{2\pi}{3}, \frac{4\pi}{3}$	1-Showing correct substitution 1 - correct answer	$\sin 2x = 0$. $0 \le x \le 2\pi$ $\therefore 0 \le 2x \le 4\pi$. Many students lost the solutions $\frac{3\pi}{2}$ and 2π as they mistook $0 \le 2x \le 2\pi$.
(b) (i) $f'(x) = 1 - \frac{2x}{1+x^2}$ $= \frac{1+x^2-2x}{1+x^2}$ $= \frac{(1+x)^2}{1+x^2}$ $(1+x)^2$ and $1+x^2 > 0$ for all x $\therefore$ *f(x)* is monotonic increasing.	1- correctly differentiation 1- correct explanation	
(ii) When $x = 0$, $f(x) = -\log_e 1$ $= 0$ $\therefore$ $f(x) > 0$ for all $x > 0$ Hence $x - \log_e(1+x^2) > 0$ for all $x > 0$ $\log_e(1+x^2) < x$ for all $x > 0$ $\therefore e^x > 1 + x^2$	1- finding the value of f(x) when x = 0 1- correct reasoning	$f'(x) > 0$ does not mean $f(x) > 0$. of Course, it is an increasing function. You must find the minimum value of this function and show that it is > 0.

(b) 		
(i) Let $\angle MPQ = x$ $\angle MNQ = \angle MPQ$ ($\angle$s in the same segment) **(1)** $= x$ $\angle PTF = 90^0 - x$ ($\angle$ sum in right angled triangle) $= \angle MTD$ (vertically opposite angles)	1- first three steps shown	
$\angle DMT = 90^0 - \angle MNQ$ ($\angle$ sum in right angled triangle) $= 90^0 - x$ $= \angle MTD$ **(2)** $\therefore$ ΔDMT is isosceles (2 equal angles) Hence $DT = DM$ **(3)**	1- Proving the triangle is isosceles	
$\angle DT\cancel{M}^{N} = 90^0 - (\angle MTD) = 90 - (90 - x) = x.$ $\cancel{= 90^0 - x}$ from **(2)** $= \angle DNT$ from **(1)** $\therefore$ ΔDNT is isosceles (2 equal angles) Hence $DT = DN$ $\therefore$ $DM = DN$, hence D is the mid-point of MN	1- showing the result	
(ii) $\angle QTP = 90^0$ and B is the mid-point of PQ $\therefore \Delta$ TQP is in a semicircle with TB, BQ and BP are radii. Hence ΔQBT is isosceles.	1	
$\therefore$ $\angle QTB =$ (vertically opposite angles)TQP $= 90^0 - x$ ($\angle$ sum in ΔTPQ) $= \angle ETN$ (vertically opposite angles)		

$\angle TEN = 180^0 - x - (90^0 - x)$ ($\angle$ sum in a triangle) $= 90^0$	1-showing the result	
(iii) In ΔDMC and ΔMNQ $\angle M$ is common $\frac{MD}{MN} = \frac{MC}{MQ} = \frac{1}{2}$ (D & C are mid-points of MN & MQ) $\therefore$ $\Delta DMC /// \Delta MNQ$ (two pairs of corresponding lengths are in same ratio and one pair of equal corresponding angles) Hence corresponding angles are equal $\therefore$ $\angle MCD = \angle MQN$ $\therefore$ DC// NQ (corresponding angles are equal on // lines) Similarly AB // NQ, BC //MP and AD // MP $\therefore$ DC // AB and AD // BC Hence ABCD is a parallelogram (opposite sides are parallel)	1-showing the quad. Is a parallelogram	
MP $\perp$ NQ (Given) But BC // MP and DC // NQ $\therefore$ DC $\perp$ BC Hence ABCD is a rectangle (the parallelogram ABCD has one right angle).	1-showing the parallelogram is a rectangle	